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PDEs for Multiphase Advanced Materials

Homogenization of Laminate Single-Negative Metamaterials

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Outline

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What is a Metamaterial?

A Metamaterial is an artificial composite with a periodic structure exhibiting extraordinary electromagnetic properties which cannot be found in natural composites.

Metamaterials may basically be divided into two categories:

- the electromagnetic (or photonic) crystals,
- the effective media.

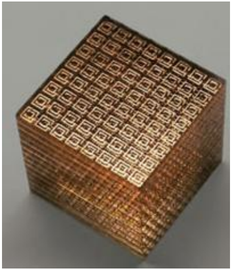

Figure: Metamaterial cube. (John Pendry and David Smith, *Physics Today* **57**, June 2004.)

The electric permittivity ε and the magnetic permeability μ are two parameters used to characterize, respectively, the electric and magnetic properties of materials interacting with electromagnetic fields.

According to the sign of these parameters we may classify the materials as:

- doble-positive (DPS):
  $\varepsilon > 0$ and $\mu > 0$;
- single-negative (SNG):
  $\varepsilon < 0$ or $\mu < 0$;
- doble-negative (DNG):
  $\varepsilon < 0$ and $\mu < 0$.

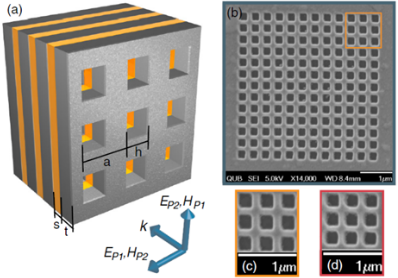

Figure: Fishnet multilayered metamaterial. (García-Meca, Hurtado, Martí, Martínez, Dickson and Zayats, *Physical Review Letters* **106**, February 2011.)

Consider a laminate composite material occupying a region Ω in $\mathbb{R}^3$ made of $2h$ alternate layers (with relative thickness $\alpha/2h$ and $(1-\alpha)/2h$) of two linear inhomogeneous materials *I* and *II* with

- different constant positive electric permittivity ε_I and ε_{II},
- same spatial dependent negative magnetic permeability μ.

The electric permittivity ε_h of this composite is no longer constant since it depends on the position of each layer, and may be defined as

$$\varepsilon_h(x) \;=\; \varepsilon_I\, \chi_{(0,\alpha)}(hx\cdot\mathbf{e}) + \varepsilon_{II}\,\left(1-\chi_{(0,\alpha)}(hx\cdot\mathbf{e})\right)$$

for $x\in\Omega$, where $\chi_{(0,\alpha)}$ stands for the characteristic function of interval $(0,\alpha)$ over $(0,1)$, and $\mathbf{e}$ is the unit vector normal to each layer.

The magnetic permeability is the negative function $\mu:\Omega\to(-\infty,0)$.

The electromagnetic properties of this laminate composite made of $2h$ layers may be described by the nonstationary Maxwell equations:

$$\begin{cases} \operatorname{curl} H &= J + \partial_t D & (\textit{Ampère Law}) \\ \operatorname{curl} E &= -\partial_t B & (\textit{Faraday Law}) \\ \operatorname{div} B &= 0 & (\textit{Gauss Law}) \\ \operatorname{div} D &= \rho & (\textit{Coulomb Law}) \end{cases} \tag{1}$$

where D, E, B, H, J are fields depending on position $x \in \Omega$ and time $t \in (0, T)$

- D and B stand for the electric and magnetic induction, respectively,
- E and H stand for the electric and magnetic field, respectively,
- J is a given current density while ρ denotes the charge density.

For linear media, the constitutive relations between these fields are the following:

$$D = \varepsilon_h(x)E, \qquad B = \mu(x)H. \tag{2}$$

Since μ is a negative function, we define the positive function $\beta \equiv -\frac{1}{\mu}$. Thus, if we replace the field D by $\varepsilon_h E$, and H by $-\beta B$, then the nonstationary Maxwell equations in (1) may be written as

$$\begin{cases} -\mathrm{curl}\,(\beta B) & = & J + \partial_t\,(\varepsilon_h E) \\ \mathrm{curl}\,E & = & -\partial_t B \\ \mathrm{div}\,B & = & 0 \\ \mathrm{div}\,(\varepsilon_h E) & = & \rho. \end{cases} \tag{3}$$

Assuming that $\partial\Omega$ is a perfectly conducting boundary, we add to the previous system the following boundary conditions

$$E \times \mathbf{n} = 0 \quad \text{and} \quad B \cdot \mathbf{n} \; = \; 0,$$

where $\mathbf{n}$ denotes the outward normal vector to $\partial\Omega$.

Now, multiply the first equation by ε_h^{-1}, consider its curl, and then take the derivative in time in the second one, and replace $\partial_t E$ in the first one, so that the previous system in (3) may be written as a system which depends only on the magnetic induction B:

$$\begin{cases} \partial_t^2 B - \mathrm{curl}\left(\varepsilon_h^{-1}\mathrm{curl}\left(\beta B\right)\right) = \mathrm{curl}\left(\varepsilon_h^{-1} J\right) & \text{in } \Omega \times (0, T) \\ \mathrm{div}\ B = 0 & \text{in } \Omega \times (0, T) \\ B \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T) \\ \varepsilon_h^{-1}\mathrm{curl}\left(\beta B\right) \times \mathbf{n} = \varepsilon_h^{-1} J \times \mathbf{n} & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Regarding the initial and final data, we shall assume that

$$\begin{cases} B(x,0) = B_0(x) & \text{in } \Omega \\ \partial_t B(x,0) = \partial_t B(x,T) = 0 & \text{in } \Omega, \end{cases}$$

given a divergence-free field B_0.

Therefore, the electromagnetic problem we are interested in reduces to the initial-boundary value problem

$$\begin{cases} -\partial_t^2 B + \mathrm{curl}\left(\varepsilon_h^{-1}\,\mathrm{curl}\,(\beta B)\right) & = & -\mathrm{curl}\left(\varepsilon_h^{-1} J\right) & \text{in } \Omega \times (0,T) \\ \mathrm{div}\ B & = & 0 & \text{in } \Omega \times (0,T) \\ B \cdot \mathbf{n} & = & 0 & \text{on } \partial\Omega \times (0,T) \\ \varepsilon_h^{-1}\mathrm{curl}\,(\beta B) \times \mathbf{n} & = & -\varepsilon_h^{-1} J \times \mathbf{n} & \text{on } \partial\Omega \times (0,T) \\ B(x,0) & = & B_0(x) & \text{in } \Omega \\ \partial_t B(x,0) & = & \partial_t B(x,T) = 0 & \text{in } \Omega, \end{cases}$$

where

- B_0 is a given divergence-free field in Ω,
- $\beta : \Omega \to (0,+\infty)$ is a bounded function defined by $\beta(x) = -\frac{1}{\mu(x)}$,
- $\varepsilon_h^{-1} : \Omega \to (0,+\infty)$ is given by

$$\varepsilon_h^{-1}(x) = \frac{1}{\varepsilon_I}\chi_{(0,\alpha)}(hx \cdot \mathbf{e}) + \frac{1}{\varepsilon_{II}}\left(1 - \chi_{(0,\alpha)}(hx \cdot \mathbf{e})\right).$$

We are interested in study the homogenization, i.e. the asymptotic behaviour as the parameter h goes to ∞ of the sequence of solutions B_h, of the initial-boundary value problem

$$\begin{cases} -\partial_t^2 B + \mathrm{curl}\left(\varepsilon_h^{-1}\,\mathrm{curl}\,(\beta B)\right) & = & -\mathrm{curl}\left(\varepsilon_h^{-1} J\right) & \text{in } \Omega \times (0,T) \\ \mathrm{div}\; B & = & 0 & \text{in } \Omega \times (0,T) \\ B \cdot \mathbf{n} & = & 0 & \text{on } \partial\Omega \times (0,T) \\ \varepsilon_h^{-1}\mathrm{curl}\,(\beta B) \times \mathbf{n} & = & -\varepsilon_h^{-1} J \times \mathbf{n} & \text{on } \partial\Omega \times (0,T) \\ B(x,0) & = & B_0(x) & \text{in } \Omega \\ \partial_t B(x,0) & = & \partial_t B(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{4}$$

defined for every field B in the Banach space V given by

$$V = \left\{ B \in L^2(0,T; X(\Omega)) \;:\; \partial_t B \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)) \right\}$$

where

$$X(\Omega) = \left\{ U \in L^2(\Omega;\mathbb{R}^3) : \mathrm{curl}\beta U \in L^2(\Omega;\mathbb{R}^3), \mathrm{div}\, U = 0 \,\text{in}\, \Omega,\, U\cdot \mathbf{n} = 0 \,\text{on}\, \partial\Omega \right\}.$$

Homogenization of initial-boundary value problems coming from electromagnetic problems has been studied in the last decades by many authors such as:

A. Bensoussan, J. L. Lions & G. Papanicolaou; G. Duvaut & J. L. Lions; V. Girault & P.-A. Raviart; V. V. Jikov, S. M. Kozlov & O. A. Oleinik; L. Tartar & F. Murat; E. Sanchez-Palencia, . . .

Our aim is to study the homogenization of problem (4) from a variational point of view through the study of the Γ-convergence of the sequence of associated energies.

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The initial-boundary value problem in (4) may be consider as the Euler-Lagrange equation associated with the functional I_h defined in V by

$$I_h(B) = \int_0^T \int_\Omega \left(\frac{\beta}{2} |\partial_t B|^2 + \frac{\varepsilon_h^{-1}}{2} |\mathrm{curl}\,(\beta B)\,| + \varepsilon_h^{-1} J \cdot \mathrm{curl}\,(\beta B) \right) dx\ dt. \qquad (5)$$

This means that if B_h is a minimizer of I_h in V, then it turns out that B_h is a solution of the initial-boundary value problem in (4).

Thanks to a property of Γ-convergence, we know that if the sequence of energies $\{I_h\}$ Γ-converges to the functional I, then the sequence of minimizers $\{B_h\}$ converges to the minimizer of the Γ-limit I. The Euler-Lagrange equation associated with the Γ-limit I will be the homogenized problem for the family of initial-boundary value problems in (4).

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Γ-convergence is a variational convergence for sequences of functionals introduced by De Giorgi in the 70's in order to study the asymptotic behaviour of minima problems depending on a parameter. It has been developed and studied by many authors such as:

Braides, Buttazzo, Dal Maso, Marcellini, Sbordone, Spagnolo, ...

Definition of Γ-convergence

Let (V, d) be a metric space, and $\{I_h\}$ be a sequence of functionals $I_h : V \to [-\infty, +\infty]$. We say that the sequence $\{I_h\}$ is $\Gamma(d)$-convergent to the functional I if, for any B in V, it holds:

1. for every sequence $\{B_h\} \subset V$ such that $d(B_h, B) \to 0$ we have
$$\liminf_{h \searrow 0} I_h(B_h) \geq I(B);$$
2. there exists a sequence $\{B_h\} \subset V$ such that $d(B_h, B) \to 0$ and
$$\lim_{h \searrow 0} I_h(B_h) = I(B).$$

Outline

Theorem

The sequence of functionals

$$I_h(B) = \int_0^T \int_\Omega \left(\frac{\beta}{2} |\partial_t B|^2 + \frac{\varepsilon_h^{-1}}{2} |\mathrm{curl}\,(\beta B)|^2 + \varepsilon_h^{-1} J \cdot \mathrm{curl}\,(\beta B) \right) dx\, dt$$

is Γ-convergent to the functional I defined by

$$I(B) = \int_0^T \int_\Omega \left(\frac{\beta}{2} |\partial_t B|^2 + \Psi(\mathrm{curl}\,(\beta B)) \right) dx\, dt,$$

where the function $\Psi : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\Psi(\Lambda) = \inf_{A_1, A_2 \in \mathbb{R}^3} \left\{ \frac{\alpha}{\varepsilon_I} \left(\frac{1}{2} |A_1|^2 + J \cdot A_1 \right) + \frac{(1-\alpha)}{\varepsilon_{II}} \left(\frac{1}{2} |A_2|^2 + J \cdot A_2 \right) : \Lambda = \alpha A_1 + (1-\alpha) A_2,\ \ (A_1 - A_2) \cdot \mathbf{e} = 0 \right\}.$$

Outline

In the particular case of a laminate composite formed by two materials I and II whose electric permittivities are $\varepsilon_I = 7$ and $\varepsilon_{II} = 3.4$, respectively, and the magnetic permeability is constant $\mu \equiv -1$ so that $\beta \equiv 1$, the Γ-limit functional I is defined as

$$I(B) = \int_0^T \int_\Omega \left(\frac{1}{2} |\partial_t B|^2 + (\mathrm{curl}\, B)^T \, \frac{\varepsilon_{hom}^{-1}}{2} \, \mathrm{curl}\, B + \varepsilon_{hom}^{-1} J \cdot \mathrm{curl}\, B \right) dx\, dt,$$

where the homogenized matrix ε_{hom}^{-1} is given by

$$\varepsilon_{hom}^{-1} = \begin{pmatrix} \frac{29}{119} & 0 & 0 \\ 0 & \frac{5}{23} & 0 \\ 0 & 0 & \frac{5}{23} \end{pmatrix}.$$

In this case, the sequence of solutions $\{B_h\}$ of the family of initial-boundary value problems of type

$$\left\{\begin{array}{rcll} -\partial_t^2 B + \mathrm{curl}\left(\varepsilon_h^{-1}\,\mathrm{curl}\ B\right) & = & -\mathrm{curl}\left(\varepsilon_h^{-1} J\right) & \text{in } \Omega\times(0,T) \\ \mathrm{div}\ B & = & 0 & \text{in } \Omega\times(0,T) \\ B\cdot\mathbf{n} & = & 0 & \text{on } \partial\Omega\times(0,T) \\ \varepsilon_h^{-1}\mathrm{curl}\ B\times\mathbf{n} & = & -\varepsilon_h^{-1} J\times\mathbf{n} & \text{on } \partial\Omega\times(0,T) \\ B(x,0) & = & B_0(x) & \text{in } \Omega \\ \partial_t B(x,0) & = & \partial_t B(x,T) = 0 & \text{in } \Omega, \end{array}\right.$$

is such that the sequences $\{\mathrm{curl}\, B_\varepsilon\}$ and $\{\partial_t\, B\}$ weak converge, as h goes to ∞, to $\mathrm{curl}\, B$ and $\partial_t B$, respectively, in $L^2(\Omega\times(0,T);\mathbb{R}^3)$, where B is the solution of the homogenized problem

$$\left\{\begin{array}{rcll} -\partial_t^2 B + \mathrm{curl}\left(\varepsilon_{hom}^{-1}\,\mathrm{curl}\ B\right) & = & -\mathrm{curl}\left(\varepsilon_h^{-1} J\right) & \text{in } \Omega\times(0,T) \\ \mathrm{div}\ B & = & 0 & \text{in } \Omega\times(0,T) \\ B\cdot\mathbf{n} & = & 0 & \text{on } \partial\Omega\times(0,T) \\ \varepsilon_{hom}^{-1}\mathrm{curl}\ B\times\mathbf{n} & = & -\varepsilon_{hom}^{-1} J\times\mathbf{n} & \text{on } \partial\Omega\times(0,T) \\ B(x,0) & = & B_0(x) & \text{in } \Omega \\ \partial_t B(x,0) & = & \partial_t B(x,T) = 0 & \text{in } \Omega. \end{array}\right.$$

Thank you

for your attention !!