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A sixth order Cahn-Hilliard type equation

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Outline:

– The model and its reduction
– Local in time weak solutions
– Regularity of weak solutions
– Uniqueness
– Global in time solutions
– Asymptotics

1. The model and its reduction

We study a model of thin films, where the surface diffusion plays a major role. Crystal surface is represented by a height function $h : \Omega \times [0, T) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, $d = 1, 2$. Actually, we take $\Omega = \mathbb{T}^d$. The basic equation is

$$h_t = \sqrt{1 + |\nabla h|^2}(\mathcal{D}\Delta_S \mu - f \cdot \mathbf{n}), \text{ on} \tag{1}$$

where

$\mathcal{D}$ – diffusion constant;

$f \cdot \mathbf{n}$ – atomic flux;

μ – variational derivative of chemical potential depending on surface energy.

The surface energy density is $\gamma(\nabla h) + \frac{1}{2}\nu\kappa^2$, here $\nu > 0$ is a Willmore regularization, and κ is the mean curvature.

After reductions the surface energy functional takes the form

$$\mathcal{L}(h) = \int_\Omega (\frac{1}{2}|\Delta h|^2 + \Phi(\nabla h))\, dx \tag{2}$$

here Φ is a quartic potential.
If $d = 2$, then:

$$\Phi(F_1, F_2) = \frac{\alpha}{12}\left(F_1^4 + F_2^4\right) + \frac{\beta}{2}F_1^2 F_2^2 - \frac{1}{2}(F_1^2 + F_2^2),$$

where α, $\beta > 0$. This function has four wells.

$d = 1$:

$$\Phi(F) = \frac{1}{2}(1 - |F|^2)^2.$$

This function has two wells.

The L^2-derivative of $\mathcal{L}$ is

$$\begin{aligned}(\frac{\delta \mathcal{L}}{\delta h}(h), \varphi) &= \int_\Omega (\Delta^2 h - \text{div} \nabla_F \Phi) \varphi \, dx \\ &= \int_\Omega (\Delta^2 h + \Delta h - \Psi) \varphi \, dx,\end{aligned}$$

where

$$\Psi = \beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4 h_x h_y h_{xy}) + \alpha(h_x^2 h_{xx} + h_y^2 h_{yy})$$

Thus, the model equation is

$$h_t = \frac{D}{2} |\nabla h|^2 + \Delta \frac{\delta \mathcal{L}}{\delta h}(h) \quad \text{in } \mathbb{T}^d \tag{3}$$

augmented with initial conditions.

History:

Savina et all (2003),
Korzec, Evans, Wagner, Münch (2008),
Korzec (2010),
Wise, J.Kim, Lowengrub (2007),
Pawłow–Zajączkowski (2011),
Vougalter, Volpert (2012).

2. Local in time weak solutions.

Equations

Finally, the system takes the following forms, if $d = 1$,

$$h_t = \frac{D}{2} h_x^2 + h_x^{(6)} + [h_x - (h_x)^3]^{(4)}; \tag{4}$$

if $d = 2$,

$$\begin{aligned} h_t \ &= \ \frac{D}{2} |\nabla h|^2 + \Delta^2 h + \Delta^3 h - \Delta[\beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4 h_x h_y h_{xy})] \\ &\quad + \alpha \Delta (h_x^2 h_{xx} + h_y^2 h_{yy}). \end{aligned} \tag{5}$$

Both systems are gradient flows perturbed by a destabilizing quadratic term. There is a difference between $d = 1$ and $d = 2$.

After differentiating (4) wrt x and substituting $u = h_x$ we obtain an equation for the slope u,

$$u_t = Duu_x + u_x^{(6)} - (\Phi'_u(u))_x^{(4)} \tag{6}$$

for a conserved quantity, u. Moreover,

$$\int_{\mathbb{T}^1} u(t,x)\,dx = \int_{\mathbb{T}^1} u_0(x)\,dx = 0.$$

Weak solutions
Local existence of weak solutions is not difficult, once we properly define this notion.

Definition. A function $h \in C([0,T); H^3(\mathbb{T}^2))$, (resp. $u \in L^2(0,T; \dot{H}^3(\mathbb{T}^1)))$, $h(0) = h_0$, (resp. $u(0) = u_0$), such that $h_t \in L^\infty(0,T; H^{-3}(\mathbb{T}^2))$, (resp. $u_t \in L^2(0,T; H^{-3}(\mathbb{T}^1)))$ such that

$$\langle h_t, \varphi \rangle = \int_{\mathbb{T}^2} (D|\nabla h|^2\varphi - \nabla\Delta h \nabla\Delta\varphi + \text{div}\nabla_F\Phi\Delta\varphi) \quad \forall\varphi \in C([0,T); H^3(\mathbb{T}^2)).$$

It is relatively straightforward to establish local-in-time existence.

Theorem 1. (a) For a given $h_0 \in H^3$ there is $T > 0$ such that there exists a weak solution to (5) on $[0, T)$.
(b) For given $T > 0$ and $u_0 \in \dot{H}^2$ there exists a weak solution to (6) on $[0, T)$.

Part (a) is proved by Banach contraction principle, (b) is shown by Galerkin approximation.

3. Regularity of weak solutions

Theorem 2. (a) $d = 2$; If $h_0 \in H^3$ and h is a corresponding weak solution to (5) on $[0, T]$ (hence, $h \in C([0, T]; H^3)$), then $h \in L_2(0, T; H^5)$ and $h_t \in L_2(0, T; H^{-1})$. Moreover, the bounds for the norms $\|h\|_{L_2(0,T;H^5)}$ and $\|h_t\|_{L_2(0,T;H^{-1})}$ depend only on $\|h_0\|_{H^3}$ and $\|h\|_{C([0,T];H^3)}$.
(b) $d = 1$; Let us suppose that u is a weak solution to (4) given by Theorem 1 (b), then

$$u \in L^2(0, T; \dot{H}^4_{per}) \quad \text{and} \quad u_x \in L^\infty(0, T; L^\infty).$$

In the case (a) we apply the variation of parameter formula and proceed by patient boot-strapping argument. Its advantage is it may be applied endlessly, until something goes wrong.

To prove (b) we proceed by establishing energy estimates.

4. Global in time solutions

The drawback of Theorem 2. is that $\|u\|_{L^\infty(0,T;H^5)}$ depends on unspecified norm $\|u\|_{L^\infty(0,T;H^3)}$ while we would be most happy with an estimate of the form

$$\|u\|_{L^\infty(0,T;H^5)} \le C(\|u_0\|_{H^3}, T).$$

Theorem 3. Let us assume than h is a weak solution to (5) with initial condition, which we constructed. Then,

$$\|h\|_{L^\infty(0,T;H^3)} \le C_3(1 + \|h_0\|_{H^3} + \mathcal{L}(h_0))e^{\lambda T}. \tag{7}$$

A similar estimate is valid also in the one-dimensional case.

Estimate (7) tell us that weak solutions **may not** blow up in **finite** time and in particular Theorem 2 is valid for all $t > 0$.

Remarks on the proof. We compute $\frac{d\mathcal{L}}{dt}$. One can seem that

$$\frac{d\mathcal{L}}{dt} = \int_\Omega \mathcal{H} h_t = -\int_\Omega |\nabla \mathcal{H}|^2 + \frac{D}{2}\int_\Omega \mathcal{H}|\nabla h|^2.$$

Sobolev inequality implies (note $\int \mathcal{H} = 0$).

$$\int_\Omega \mathcal{H}^2 \le \int_\Omega |\nabla \mathcal{H}|^2.$$

Moreover,

$$\frac{D}{2}\mathcal{H}|\nabla h|^2 \le \mathcal{H}^2 + \frac{D^2}{8}|\nabla h|^4.$$

Hence,

$$\frac{d\mathcal{L}}{dt} \leq C_1 + C_2\mathcal{L}.$$

As a result,

$$\|h\|_{H^2}^2 \leq C(1 + \mathcal{L}(h_0)).$$

Subsequently, we lifting the regularity by bootstrapping argument if $d = 2$.

An additional effort is needed if $d = 1$, becuase we test equation (6) with suitable test functions. It is summarized in the following result.

Lemma 1. (Folland)
Consider a domain $\Omega \subset \mathbb{R}^n$, let $s > 0, t > s + n/2$ and $u \in H^s(\Omega), \phi \in H^t(\Omega) \cap L^\infty(\Omega)$. Then $\phi u \in H^s(\Omega)$ and it holds for some constant $C > 0$ that

$$\|\phi u\|_{H^s} \leq \|\phi\|_\infty \|u\|_{H^s} + C\|\phi\|_{H^t}\|u\|_{H^{s-1}}. \tag{8}$$

5. Uniqueness

This depends upon uniform H^3 bounds established earlier, see (7).

Theorem 4. If $h_0 \in H^3$ and h^i, $i = 1, 2$ are weak solutions to (5) with initial condition h_0, then $h^1 = h^2$. The same is true if $d = 1$.

After straightforward estimates we obtain, for $h = h_1 - h_2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\nabla\Delta h\|^2 \;\le\; & \|\Delta h\|^2 + \frac{D}{4}\|h\|^2 + C_2(K)\frac{D}{2}\|\nabla h\|^2 \\ & + \frac{C_3(K)}{\epsilon}\|\nabla h\|^2 + \epsilon(\frac{\alpha}{3} + \beta)\|\nabla\Delta h\|^2. \end{aligned} \tag{9}$$

We so choose ϵ that $(\frac{\alpha}{3} + \beta)\epsilon = 1/2$.

Combining this with the interpolation inequality below

$$\|\Delta u\|_{L^2} \leq C_\epsilon \|u\|_{L^2} + \epsilon \|\nabla \Delta u\|_{L^2},$$

yields

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 \leq K_\epsilon \|h\|^2.$$

Thus,

$$h \equiv 0.$$

6. Asymptotics

We will study only the slope systems in $d = 1, 2$.

We will show that there is a compact absorbing set in H^2 topology. This will imply existence of a **global** attractor. The choice of the norm is related to uniqueness theorems.

We begin with the $d = 1$ case, which explains the idea of the calculations.

Proposition 1. $(d = 1)$ There is an absorbing ball in H^1. More precisely, there is C_U such that for any set B, bounded in the H^2, if $u(0) \in B$, then

$$\|u(t)\|_{L^2} \leq C_U,\ \|u(t)\|_{L^4} \leq C_U,\ \|u_x(t)\|_{L^2} \leq C_U$$

for $t \geq t(B)$.

This is done in two steps.

Lemma 2.

$$\frac{d}{dt}\left[\int_{\mathbb{T}^1}(\Phi(u) + \frac{1}{2}\|u_x\|^2) + \frac{1}{2}\|(-\Delta)^{-1}u_t\|^2\right] \leq C\|u\|_{L^4}^4. \tag{10}$$

Lemma 3.

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{-1}u\| + \frac{1}{2}\|u\|_{L^4}^4 + \|u_x\|^2 \leq C_2. \tag{11}$$

We define

$$\mathcal{E}_1 = \int_{\mathbb{T}^1} \Phi(u)\,dx + \frac{1}{2}\|u_x\|_{L^2}^2 + 2C_1\|(-\Delta)^{-1}u\|_{L^2}^2 \tag{12}$$

adding $4C_1$ times (10) to (11) yields (after some work)

$$\begin{aligned} \tfrac{d}{dt}\mathcal{E}_1(t) + \epsilon\mathcal{E}_1(t) + (C_1-\epsilon)\|u\|_{L^4}^4 + (4C_1-\epsilon/2)\|u_x\|^2 \\ \leq C_6 = 4C_1C_2 + \tfrac{\epsilon}{4}L + C_5. \end{aligned} \tag{13}$$

By Gronwall inequality, for sufficiently small ϵ we obtain

$$\mathcal{E}_1 \leq (\mathcal{E}_1(0) - \frac{C_6}{\epsilon})e^{-\epsilon t} + C_6/\epsilon.$$

Remarks.

This type of argument is borrowed from Eden-Kalantarov (2007).

We can continue in the same spirit, to conclude that,

Proposition 2. There is $\rho > 0$ such that for any bounded $B \subset H^2$ we have

$$\|u(t)\|_{H^3} \leq \rho \quad \forall t \geq t'(B), \tag{14}$$

if $u(0) \in B$.

The calculations are more complex, than in 1-d case.

We conclude existence of a global attractor.

Theorem 5. There is a global attractor for equation (4) in H^2-topology.

We have to show that our absorbing set is compact.

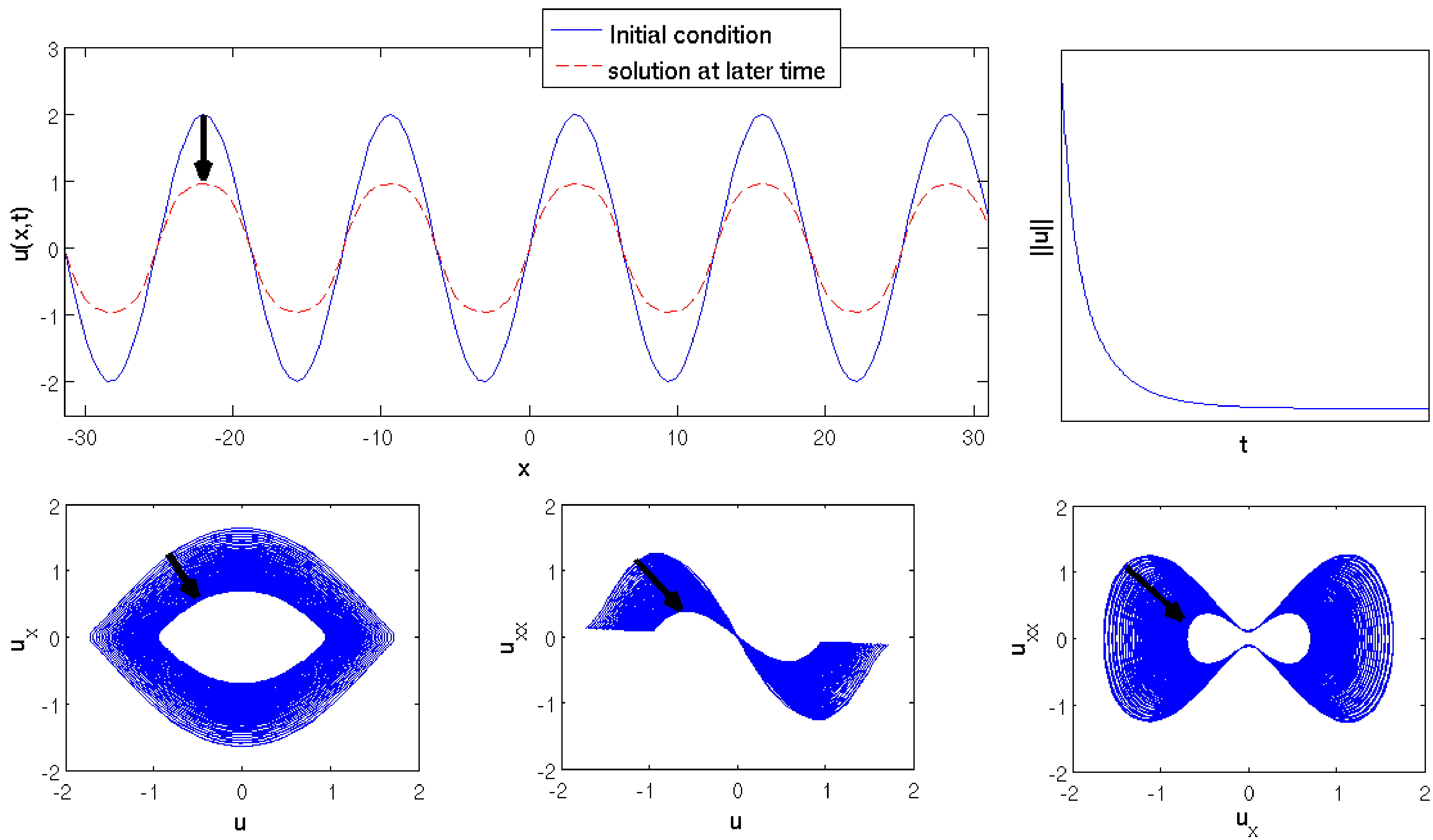
Case $d = 2$.
We have to impose the same structure as in in the case of $d = 1$. For this purpose we take gradient of (4). This yields

$$u_t = D\nabla|u|^2 + \Delta^3 u + \Delta\nabla \text{div} \nabla_F \Phi. \tag{15}$$

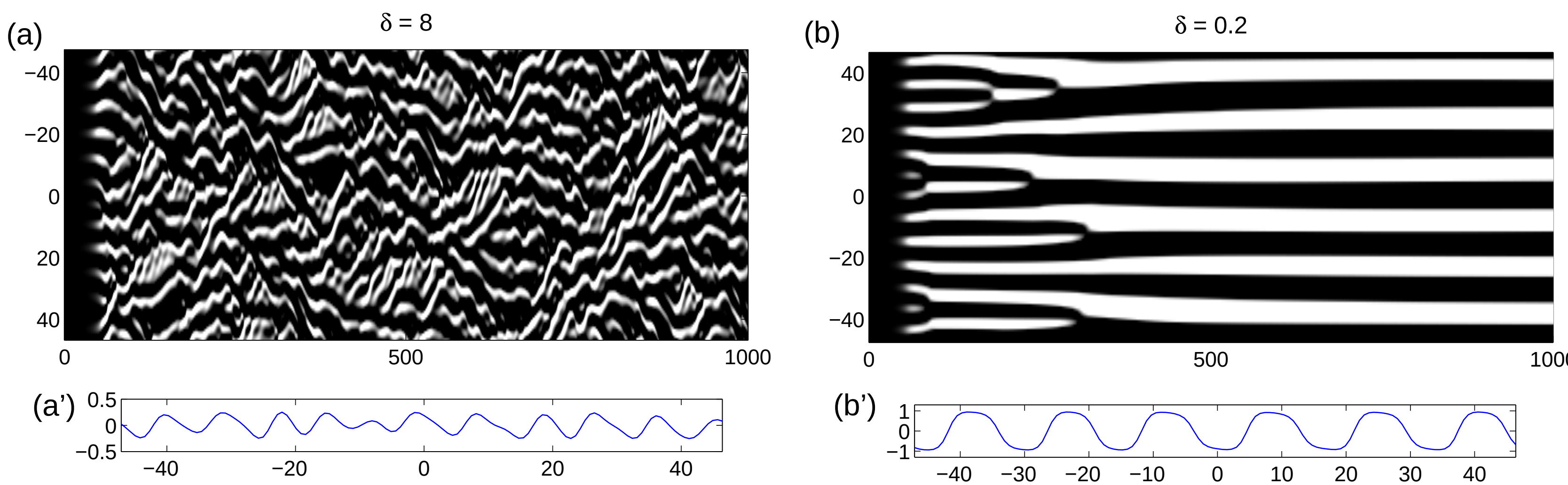
for $u = \nabla h$. Then, we proceed as in the proof of Theorem 5. We first claim existence of an absorbing set for (15) in the H^1 topology. Next, this fact and the constant variation formula yield existence of a compact (in the H^2 topology) absorbing set. for h.

Theorem 6. ($d = 2$) There is a global attractor for equation (15) in H^2-topology.

Note, these Theorems are concerned with the 'slope systems' for u. At the moment we do not have tools to control the L^2 norm of h.



Initial condition
solution at later time
u(x,t)
x
||u||
t
u
u_x
u_{xx}


(a)
δ = 8
−40
−20
0
20
40
0
500
1000
(b)
δ = 0.2
40
20
0
−20
−40
0
500
1000
(a')
0.5
0
−0.5
−40
−20
0
20
40
(b')
1
0
−1
−40
−30
−20
−10
0
10
20
30
40