On the omega-limit set for a nonlocal evolution problem

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Introduction

First, we consider a problem involving a partial differential equation

$$(PDE)\begin{cases} v_t = \Delta v + f(v) - \fint_\Omega f(v) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ v(x,0) = v_0(x) & x \in \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded connected open set with smooth boundary, ∂_ν is the outer normal derivative to $\partial\Omega$ and

$$\fint_\Omega f(v) := \frac{1}{|\Omega|} \int_\Omega f(v(x))\, dx.$$

Introduction

- Problem (PDE) was proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture.
- We assume that the function f is of the form

$$f(s) = \sum_{i=1}^{n} a_i s^i \quad \text{where } n \geq 3 \text{ is an odd number, } a_n < 0.$$

Introduction

- Mass conservation property

$$\int_{\Omega} v(x,t)\,dx = \int_{\Omega} v_0(x)\,dx.$$

- Lyapunov functional

$$\mathcal{E}(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2\,dx - \int_{\Omega} F(v)\,dx,$$

where $F(s) = \int_0^s f(\tau)\,d\tau.$

Introduction

- BOUSSAÏD, HILHORST and NGUYEN gave a version of Lojasiewicz inequality and used it to prove that as $t \to \infty$

$$v(t) \text{ converges to a stationary solution } \varphi \text{ in } H^1(\Omega).$$

In other words, the omega-limit set of Problem (PDE) is a singleton.

Introduction

The stabilization and the existence of a global attractor in the case that f is singular will be studied in the doctoral thesis of Samira Boussaid

Introduction

Next, we consider a nonlocal differential equation on $I := (-L, L)$

$$(ODE) \begin{cases} u_t = f(u) - \fint_I f(u) & \text{in } I \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in I, \end{cases}$$

where $L > 0$, and

$$\fint_I f(u) := \frac{1}{2L} \int_I f(u(x))\, dx.$$

Our aim is to study the omega-limit set

$$\omega(u_0) := \{\varphi \in L^1(I) : \exists t_n \to \infty \text{ such that } u(t_n) \to \varphi \text{ in } L^1(I) \text{ as } n \to \infty\}.$$

Introduction

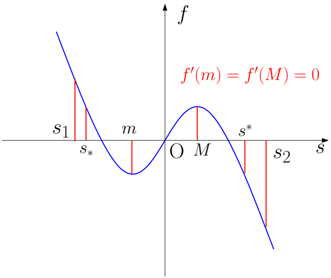
Problem (ODE) has the following properties:

- Mass conservation
- Lyapunov functional

$$E(u) = -\int_{\Omega} F(u)\, dx, \text{ where } F(s) = \int_0^s f(\tau)\, d\tau.$$

- Howerver, the technique used to study Problem (PDE) can not be used for Problem (ODE). In the following, we give a different method, which is based on studying the profile of $u(t)$ for each time t.

The function f



We choose s_1 (large enough) and s_2 (small enough) such that

$$f(s_2) < f(s) < f(s_1) \quad \text{for all} \quad s \in (s_1, s_2).$$

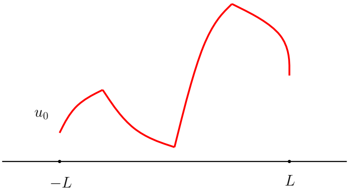
s_* and s^* satisfy $f(s_*) = f(M), \quad f(s^*) = f(m).$

Hypothesis

We assume that the initial function satisfies the hypothesis:

$$(\mathbf{H}): \quad u_0 \text{ is piecewise monotone, continuous on } [-L, L],$$

$$\text{and } \operatorname{lap}(u_0) \text{ is finite.}$$



Lap-number

Lap-number

Let w be a piecewise monotone continuous function from $\overline{I}$ into $\mathbb{R}$. Then $\overline{I}$ can be divided into a finite number of non-overlapping sub-intervals $J_1, \ldots, J_m(\cup_{i=1}^{m} J_m = \overline{I})$, where w is monotone.

Such a division of $\overline{I}$ is not unique, but there exists a minimum value m for which we can find a division $\{J_i\}$ as above. This value is called the lap-number of w and we shall denote it by $\mathrm{lap}(w)$.

Heuristics

- For every $t > 0$, we have

$$\mathrm{lap}(u(t)) = \mathrm{lap}(u(0)).$$

- A comparison result for the nonlocal problem (ODE):

$$s_1 \leq u(0) \leq s_2 \quad \Longrightarrow \quad s_1 \leq u(t) \leq s_2 \text{ for all } t \geq 0.$$

We prove that $\{u(t), t \geq 0\}$ is bounded in $BV(I)$ so that

$$\{u(t), t \geq 0\} \text{ is relatively compact in } L^1(I).$$

First result

Theorem 1

Let $\varphi \in \omega(u_0)$, then φ is a step function. More precisely,

$$\varphi = a_- \mathcal{X}_{A_-} + a_0 \mathcal{X}_{A_0} + a_+ \mathcal{X}_{A_+},$$

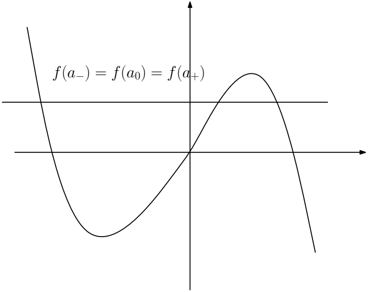
where A_-, A_0, A_+ (which depend on φ, and may not exist) are pairwise disjoint subsets of I such that

$$A_- \cup A_0 \cup A_+ = I.$$

a_-, a_0, a_+ satisfy

$$f(a_-) = f(a_0) = f(a^+) = \eta_\varphi (\text{some constant})$$

First result



First Idea

We have the following constraints

The constraints

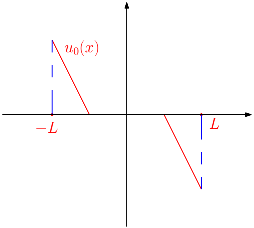
- The functional Lyapunov is constant on omega-limit set,
- $\int_I \varphi = \int_I u_0,$
- $f(a_-) = f(a_0) = f(a_+) = \eta_\varphi.$

and we have six unknowns: $a_-, a_0, a_+, A_-, A_0, A_+$. Therefore, we need more conditions to find these unknows.

- The first idea is to prove that $|A_0| = 0$, since a_0 is an unstable point.

Counterexample

Let u_0 be an odd function on $\bar{I}$.



Assume that $f(s) = s - s^3$. Then $\omega(u_0)$ possesses a unique element φ which is given by

$$\varphi(x) = \begin{cases} -1 & \text{if } u_0(x) < 0 \\ 0 & \text{if } u_0(x) = 0 \\ 1 & \text{if } u_0(x) > 0. \end{cases}$$

Consequently, if $|u_0^{-1}(\{0\})| \neq 0$, then $|A_0| \neq 0$.

Theorem 1

We shall use the notations for each $t \geq 0$,

$$I_-(t) := \{x \in \overline{I}, u(x,t) \leq m\},$$
$$I_0(t) := \{x \in \overline{I}, m < u(x,t) < M\},$$
$$I_+(t) := \{x \in \overline{I}, u(x,t) \geq M\}.$$

Key lemma

Assume that $s_* \leq u_0 \leq s^*$, then for each $t \geq 0$ and for every $t' > t$,

$$I_-(t) \subset I_-(t'),\ \ I_+(t) \subset I_+(t') \text{ and } I_0(t) \supset I_0(t').$$

On the other words, $I_-(t), I_+(t)$ are monotonically expanding in t and $I_0(t)$ is monotonically shrinking in t.

Arguments for the key lemma

Arguments

For all $t' > t \geq 0$ and $x \in \bar{I}$, we have

- if $u(x,t) \leq m$ then $u(x,t') \leq m$,
- if $u(x,t) \geq M$ then $u(x,t') \geq M$.

Theorem 1

Theorem

Assume that $s_* \leq u_0 \leq s^*$. *There exists* α *such that* **for all** $\varphi \in \omega(\mathbf{u_0})$ *with* $\eta_\varphi \in (f(m), f(M))$.

$$A_- = u_0^{-1}((-\infty, \alpha)), A_0 = u_0^{-1}(\{\alpha\}), A_+ = u_0^{-1}(\alpha, +\infty).$$

Corollary

Assume that u_0 *is strictly monotone on every connected components of* $u_0^{-1}((m, M))$, *then* $|A_0| = 0$. *Moreover,* $\omega(u_0)$ *possesses a unique element.*

Theorem 2

We note that if $u_0(x) \in [s_*, s^*]$ for all $x \in \overline{I}$, then $\fint_I u_0 \in [s_*, s^*]$.

Now, we consider the case that

$$\fint_I u_0 \notin [s_*, s^*].$$

Theorem

Assume that

$$\fint_I u_0 \notin [s_*, s^*];$$

then $\omega(u_0)$ possesses a unique element φ. Moreover,

$$\varphi(x) \equiv \fint_I u_0(y)\, dy.$$

Theorem 3

Theorem

Assume that for all $x \in \overline{I}$,

$$\text{either } u_0(x) \leq m \text{ or } u_0(x) \geq M.$$

Then $\omega(u_0)$ possesses a unique element φ. Moreover,

$$\varphi(x) \equiv \fint_I u_0(y)\,dy.$$

Future work

Hilhorst, Matano, and Nguyen are planning to study a generation of interface property for the equation

$$u_t = u_{xx} + \frac{1}{\varepsilon^2}\left(f(u) - \fint_I f(u)\right).$$

Thank you for your attention!