A Quasistatic Model for Perfectly Plastic Plates Derived by Γ-convergence

Maria Giovanna MORA

Università di Pavia, Italy

Joint work with E. Davoli (CMU, Pittsburgh)

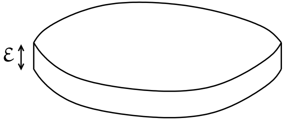
To appear on *Ann. I. H. Poincaré Anal. Non Linéaire*

ADMAT2012 – PDEs for multiphase advanced materials

Cortona, 17–21 September 2012

Introduction

Problem: rigorous derivation of a reduced model for a thin plastic plate starting from 3d plasticity



Our framework:

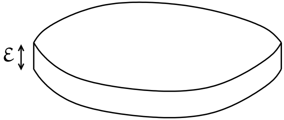
- small-strain perfect plasticity
- quasistatic evolutionary setting

Result: a quasistatic evolution model for a thin plate

- coupling of the stretching and bending components of the stress through the stress constraint and the plastic flow rule
- genuinely three-dimensional

Introduction

Problem: rigorous derivation of a reduced model for a thin plastic plate starting from 3d plasticity



Our framework:

- small-strain perfect plasticity
- quasistatic evolutionary setting

Result: a quasistatic evolution model for a thin plate

- coupling of the stretching and bending components of the stress through the stress constraint and the plastic flow rule
- genuinely three-dimensional

$\rightsquigarrow$ new model, different from the classical 2d linearly plastic plate model

Small-Strain Perfect Plasticity

$\Omega \subset \mathbb{R}^3$ reference configuration, $T > 0$

$u\colon [0,T] \times \Omega \to \mathbb{R}^3$ displacement, $Eu := \mathrm{sym}\,\nabla u$ linearized strain

additive decomposition

$$Eu = e + p$$

$e\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ elastic strain

$p\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{D}$ plastic strain

Small-Strain Perfect Plasticity

$\Omega \subset \mathbb{R}^3$ reference configuration, $T > 0$

$u\colon [0,T] \times \Omega \to \mathbb{R}^3$ displacement, $Eu := \mathrm{sym}\,\nabla u$ linearized strain

additive decomposition
$Eu = e + p$

$e\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ elastic strain

$p\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{D}$ plastic strain

Here $\mathbb{M}^{3\times 3}_{D} := \{A \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} : \mathrm{tr}\, A = 0\}$ $\rightsquigarrow$ plastic incompressibility

$\mathrm{tr}\, p = 0$

Small-Strain Perfect Plasticity

$\Omega \subset \mathbb{R}^3$ reference configuration, $T > 0$

$u \colon [0, T] \times \Omega \to \mathbb{R}^3$ displacement, $Eu := \mathrm{sym}\, \nabla u$ linearized strain

additive decomposition
$Eu = e + p$

$e \colon [0, T] \times \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ elastic strain

$p \colon [0, T] \times \Omega \to \mathbb{M}^{3\times 3}_{D}$ plastic strain

Here $\mathbb{M}^{3\times 3}_{D} := \{A \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} : \mathrm{tr}\, A = 0\}$ $\rightsquigarrow$ plastic incompressibility
$\mathrm{tr}\, p = 0$

The elastic behaviour is linear: $\sigma := \mathbb{C}e$ stress

Small-Strain Perfect Plasticity

$\Omega \subset \mathbb{R}^3$ reference configuration, $T > 0$

$u\colon [0,T] \times \Omega \to \mathbb{R}^3$ displacement, $Eu := \mathrm{sym}\, \nabla u$ linearized strain

additive decomposition
$Eu = e + p$

$e\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{sym}$ elastic strain

$p\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{D}$ plastic strain

Here $\mathbb{M}^{3\times 3}_{D} := \{A \in \mathbb{M}^{3\times 3}_{sym} : \mathrm{tr}\, A = 0\}$ $\rightsquigarrow$ plastic incompressibility
$\mathrm{tr}\, p = 0$

The elastic behaviour is linear: $\sigma := \mathbb{C}e$ stress

stress constraint
$\sigma_D(t,x) \in K$

$K \subset \mathbb{M}^{3\times 3}_{D}$ convex compact, $0 \in \mathrm{int}\, K$

∂K yield surface

Small-Strain Perfect Plasticity

$\Omega \subset \mathbb{R}^3$ reference configuration, $T > 0$

$u\colon [0,T] \times \Omega \to \mathbb{R}^3$ displacement, $Eu := \mathrm{sym}\,\nabla u$ linearized strain

additive decomposition
$Eu = e + p$

$e\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{sym}$ elastic strain

$p\colon [0,T] \times \Omega \to \mathbb{M}^{3\times 3}_{D}$ plastic strain

Here $\mathbb{M}^{3\times 3}_{D} := \{A \in \mathbb{M}^{3\times 3}_{sym} : \mathrm{tr}\,A = 0\}$ $\rightsquigarrow$ plastic incompressibility
$\mathrm{tr}\,p = 0$

The elastic behaviour is linear: $\sigma := \mathbb{C}e$ stress

stress constraint
$\sigma_D(t,x) \in K$

$K \subset \mathbb{M}^{3\times 3}_{D}$ convex compact, $0 \in \mathrm{int}\,K$

∂K yield surface

- $\sigma_D \in \mathrm{int}\,K \quad \Rightarrow$ perfectly elastic behaviour
- $\sigma_D \in \partial K \quad \Rightarrow$ plasticity occurs

The Quasistatic Evolution Problem

Datum

a time-dependent boundary displacement $w : [0, T] \times \Gamma \to \mathbb{R}^3$, $\Gamma \subset \partial\Omega$
(no applied forces for simplicity)

Problem

Find $(t, x) \mapsto (u(t, x), e(t, x), p(t, x))$ such that for every $t \in [0, T]$

- *kinematic admissibility:* $Eu(t, x) = e(t, x) + p(t, x)$ for $x \in \Omega$
  $u(t, x) = w(t, x)$ for $x \in \Gamma$
- *constitutive equation:* $\sigma(t, x) := \mathbb{C}e(t, x)$
- *equilibrium:* $\mathrm{div}_x \sigma(t, x) = 0$ in Ω, $\sigma(t, x)\nu_{\partial\Omega}(x) = 0$ for $x \in \partial\Omega \setminus \Gamma$
- *stress constraint:* $\sigma_D(t, x) \in K$
- *flow rule:* $\dot{p}(t, x) \in N_K(\sigma_D(t, x))$,
  where $N_K(\tau)$ is the normal cone to K at τ

The Flow Rule

$$\dot{p}(t,x) \in N_K(\sigma_D(t,x))$$

- If $\sigma_D(t,x) \in \operatorname{int} K$, then $\dot{p}(t,x) = 0 \;\Rightarrow\;$ no plastic evolution
  If $\sigma_D(t,x) \in \partial K$, then $\dot{p}(t,x) \perp \partial K$ at $\sigma_D(t,x)$

The Flow Rule

$$\dot{p}(t,x) \in N_K(\sigma_D(t,x))$$

- If $\sigma_D(t,x) \in \operatorname{int} K$, then $\dot{p}(t,x) = 0 \;\Rightarrow\;$ no plastic evolution
  If $\sigma_D(t,x) \in \partial K$, then $\dot{p}(t,x) \perp \partial K$ at $\sigma_D(t,x)$
- Rate-independent process

The Flow Rule

$$\dot{p}(t,x) \in N_K(\sigma_D(t,x))$$

- If $\sigma_D(t,x) \in \text{int}\, K$, then $\dot{p}(t,x) = 0 \;\Rightarrow\;$ no plastic evolution
  If $\sigma_D(t,x) \in \partial K$, then $\dot{p}(t,x) \perp \partial K$ at $\sigma_D(t,x)$
- Rate-independent process
- Equivalent formulations:

$$\text{flow rule} \quad \Leftrightarrow \quad \sigma_D(t,x) \in \partial H(\dot{p}(t,x))$$

where $H\colon \mathbb{M}^{3\times 3}_D \to [0,+\infty)$ is the support function of K

$$H(q) := \sup_{\tau \in K} \tau : q$$

H is convex, positively one-homogeneous and $\alpha|q| \leqslant H(q) \leqslant \beta|q|$

The Flow Rule

$$\dot{p}(t,x) \in N_K(\sigma_D(t,x))$$

- If $\sigma_D(t,x) \in \operatorname{int} K$, then $\dot{p}(t,x) = 0 \;\Rightarrow\;$ no plastic evolution
  If $\sigma_D(t,x) \in \partial K$, then $\dot{p}(t,x) \perp \partial K$ at $\sigma_D(t,x)$
- Rate-independent process
- Equivalent formulations:

$$\text{flow rule} \quad \Leftrightarrow \quad \sigma_D(t,x) \in \partial H(\dot{p}(t,x))$$

$$\Leftrightarrow \quad H(\dot{p}(t,x)) = \sigma_D(t,x) : \dot{p}(t,x)$$

where $H \colon \mathbb{M}^{3\times 3}_D \to [0,+\infty)$ is the support function of K

$$H(q) := \sup_{\tau \in K} \tau : q$$

H is convex, positively one-homogeneous and $\alpha|q| \leqslant H(q) \leqslant \beta|q|$

The Flow Rule

$$\dot{p}(t,x) \in N_K(\sigma_D(t,x))$$

- If $\sigma_D(t,x) \in \operatorname{int} K$, then $\dot{p}(t,x) = 0 \;\Rightarrow\;$ no plastic evolution

  If $\sigma_D(t,x) \in \partial K$, then $\dot{p}(t,x) \perp \partial K$ at $\sigma_D(t,x)$
- Rate-independent process
- Equivalent formulations:

$$\text{flow rule} \quad \Leftrightarrow \quad \sigma_D(t,x) \in \partial H(\dot{p}(t,x))$$

$$\Leftrightarrow \quad H(\dot{p}(t,x)) = \sigma_D(t,x) : \dot{p}(t,x)$$

where $H\colon \mathbb{M}^{3\times 3}_D \to [0,+\infty)$ is the support function of K

$$H(q) := \sup_{\tau \in K} \tau : q$$

H is convex, positively one-homogeneous and $\alpha|q| \leqslant H(q) \leqslant \beta|q|$

Existence results: Suquet 1981, Dal Maso-DeSimone-Mora 2006

The Incremental Formulation

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ and let (u_0, e_0, p_0) be a stable and kinematically admissible initial datum.

Let $\{t_0, t_1, \dots, t_N\}$ be a partition of $[0,T]$.

By induction define (u_i, e_i, p_i) as a minimizer of

$$\int_\Omega Q(e)\,dx + \int_\Omega H(p - p_{i-1})\,dx$$

among all $(u, e, p) \in \dots$

such that $Eu = e + p$ in Ω and $u = w(t_i)$ on Γ

$$Q(e) := \tfrac{1}{2}\mathbb{C}e : e, \qquad H(p) = \sup_{\tau \in K} \tau : p$$

The Incremental Formulation

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ and let (u_0, e_0, p_0) be a stable and kinematically admissible initial datum.

Let $\{t_0, t_1, \dots, t_N\}$ be a partition of $[0,T]$.

By induction define (u_i, e_i, p_i) as a minimizer of

$$\int_\Omega Q(e)\,dx + \int_\Omega H(p - p_{i-1})\,dx$$

among all $(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}) \times M_b(\Omega;\mathbb{M}^{3\times 3}_D)$

such that $Eu = e + p$ in Ω and $u = w(t_i)$ on Γ

$$Q(e) := \tfrac{1}{2}\mathbb{C}e : e, \qquad H(p) = \sup_{\tau \in K} \tau : p$$

$$BD(\Omega) := \left\{u \in L^1(\Omega;\mathbb{R}^3) : \; Eu \in M_b(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})\right\}$$

The Incremental Formulation

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ and let (u_0, e_0, p_0) be a stable and kinematically admissible initial datum.

Let $\{t_0, t_1, \ldots, t_N\}$ be a partition of $[0,T]$.

By induction define (u_i, e_i, p_i) as a minimizer of

$$\int_\Omega Q(e)\,dx \;+\; \mathcal{H}(p - p_{i-1})$$

among all $(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times M_b(\Omega;\mathbb{M}^{3\times3}_D)$

such that $Eu = e + p$ in Ω and $u = w(t_i)$ on Γ

$$Q(e) := \tfrac{1}{2}\mathbb{C}e : e, \qquad H(p) = \sup_{\tau \in K} \tau : p$$

$$BD(\Omega) := \left\{ u \in L^1(\Omega;\mathbb{R}^3) :\ Eu \in M_b(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \right\}$$

$$\mathcal{H}(p) := \int_\Omega H\Big(\frac{dp}{d|p|}\Big)\,d|p|$$

The Incremental Formulation

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ and let (u_0, e_0, p_0) be a stable and kinematically admissible initial datum.

Let $\{t_0, t_1, \dots, t_N\}$ be a partition of $[0,T]$.

By induction define (u_i, e_i, p_i) as a minimizer of

$$\int_\Omega Q(e)\,dx \;+\; \mathcal{H}(p - p_{i-1})$$

among all $(u, e, p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}) \times M_b(\Omega \cup \Gamma; \mathbb{M}^{3\times 3}_D)$
such that $Eu = e + p$ in Ω and $p = (w(t_i) - u) \odot \nu_{\partial\Omega}\mathcal{H}^2$ on Γ

$$Q(e) := \tfrac{1}{2}\mathbb{C}e : e, \qquad H(p) = \sup_{\tau \in K} \tau : p$$

$$BD(\Omega) := \left\{u \in L^1(\Omega;\mathbb{R}^3) :\; Eu \in M_b(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})\right\}$$

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma} H\Big(\frac{dp}{d|p|}\Big)\, d|p|$$

Existence of a Quasistatic Evolution

Theorem (Dal Maso-DeSimone-Mora *ARMA* 2006)

If $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$, then there exists a quasistatic evolution

$$(u,e,p) \in AC\big([0,T]; BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\text{sym}}) \times M_b(\Omega\cup\Gamma;\mathbb{M}^{3\times 3}_{D})\big)$$

with prescribed initial data (u_0, e_0, p_0), in the following sense:

- global minimality: for every $t \in [0,T]$

$$\int_\Omega Q(e(t))\,dx \leqslant \int_\Omega Q(\tilde e)\,dx + \mathcal{H}(\tilde p - p(t))$$

  for every $(\tilde u, \tilde e, \tilde p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\text{sym}}) \times M_b(\Omega\cup\Gamma;\mathbb{M}^{3\times 3}_{D})$ with $E\tilde u = \tilde e + \tilde p$ in Ω and $\tilde p = (w(t) - \tilde u)\odot \nu_{\partial\Omega}\mathcal{H}^2$ on Γ

- energy balance: for every $t \in [0,T]$

$$\int_\Omega Q(e(t))\,dx + \int_0^t \mathcal{H}(\dot p(s))\,ds = \int_\Omega Q(e_0)\,dx + \int_0^t \langle \mathbb{C}e(s), E\dot w(s)\rangle_{L^2}\,ds$$

Properties of Quasistatic Evolutions

- Euler conditions: setting $\sigma(t) := \mathbb{C}e(t)$, for every $t \in [0, T]$

  $\operatorname{div} \sigma(t) = 0$ in Ω, $\quad \sigma(t)\nu_{\partial\Omega} = 0$ on $\partial\Omega \setminus \Gamma$

  $\sigma_D(t) \in K$ a.e. in Ω

Properties of Quasistatic Evolutions

- Euler conditions: setting $\sigma(t) := \mathbb{C}e(t)$, for every $t \in [0, T]$

  $\operatorname{div} \sigma(t) = 0$ in Ω, $\sigma(t)\nu_{\partial\Omega} = 0$ on $\partial\Omega \setminus \Gamma$

  $\sigma_D(t) \in K$ a.e. in Ω

- balance of powers: for a.e. $t \in [0, T]$

$$\langle \sigma(t), \dot{e}(t) \rangle_{L^2} + \mathcal{H}(\dot{p}(t)) = \langle \sigma(t), E\dot{w}(t) \rangle_{L^2}$$

  or equivalently, by the integration by parts formula

$$\mathcal{H}(\dot{p}(t)) = \langle \sigma_D(t), \dot{p}(t) \rangle$$

  where the last duality is in the sense of Kohn-Temam.

Properties of Quasistatic Evolutions

- Euler conditions: setting $\sigma(t) := \mathbb{C}e(t)$, for every $t \in [0, T]$

  $\operatorname{div} \sigma(t) = 0$ in Ω, $\quad \sigma(t)\nu_{\partial\Omega} = 0$ on $\partial\Omega \setminus \Gamma$

  $\sigma_D(t) \in K$ a.e. in Ω

- balance of powers: for a.e. $t \in [0, T]$

$$\langle \sigma(t), \dot{e}(t) \rangle_{L^2} + \mathcal{H}(\dot{p}(t)) = \langle \sigma(t), E\dot{w}(t) \rangle_{L^2}$$

  or equivalently, by the integration by parts formula

$$\mathcal{H}(\dot{p}(t)) = \langle \sigma_D(t), \dot{p}(t) \rangle$$

  where the last duality is in the sense of Kohn-Temam.

  This can be interpreted as a maximum plastic work condition owing to the duality formula

Stress-Strain Duality

Kohn-Temam Stress-Strain Duality (*Appl. Math. Optim.* 1983)

If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ with $\operatorname{div}\sigma \in L^n(\Omega; \mathbb{R}^n)$, $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, and $u \in BD(\Omega)$ with $\operatorname{div} u \in L^2(\Omega)$, then

$$\langle \sigma_D, E_D u \rangle \text{ is well defined}$$

Stress-Strain Duality

Kohn-Temam Stress-Strain Duality (*Appl. Math. Optim.* 1983)

If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ with $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, $\sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, and $u \in BD(\Omega)$ with $\operatorname{div} u \in L^2(\Omega)$, then

$$\langle \sigma_D, E_D u\rangle \text{ is well defined}$$

Using that $p = E_D u - e_D + (w-u)\odot \nu_{\partial\Omega}\mathcal{H}^2\lfloor\Gamma$ and $\operatorname{tr} p = 0$, we can define

$$\langle \sigma_D, p\rangle := \langle \sigma_D, E_D u\rangle - \langle \sigma_D, e_D\rangle_{L^2} + \langle(\sigma\nu_{\partial\Omega})_{\tan}, w-u\rangle_\Gamma$$

- integration by parts formula:

$$\langle \sigma_D, p\rangle + \langle \sigma, e - Ew\rangle_{L^2} = -\langle \operatorname{div}\sigma, u-w\rangle_{L^n, L^{n/(n-1)}} + \langle \sigma\nu_{\partial\Omega}, u-w\rangle_{\partial\Omega\setminus\Gamma}$$

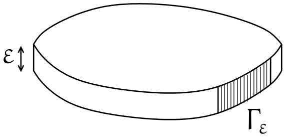
- duality formula:

$$\mathcal{H}(p) = \sup\left\{\langle \tau_D, p\rangle \;:\tau\in L^2,\ \operatorname{div}\tau\in L^n,\ \tau_D\in K,\ \tau\nu_{\partial\Omega}=0 \text{ on } \partial\Omega\setminus\Gamma\right\}$$

A Perfectly Plastic Thin Plate

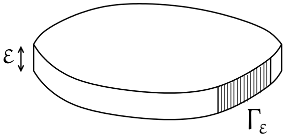
$\Omega_\varepsilon = \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \;\; \omega \subset \mathbb{R}^2$

$\Gamma_\varepsilon = \gamma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \;\; \gamma \subset \partial\omega$



$\Omega_\varepsilon = \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \ \ \omega \subset \mathbb{R}^2$

$\Gamma_\varepsilon = \gamma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \ \ \gamma \subset \partial\omega$



Perform a change of variable:

$$\psi^\varepsilon : \ \Omega := \omega \times \big(-\tfrac{1}{2}, \tfrac{1}{2} \big) \to \Omega_\varepsilon \ : \ (x', x_3) \mapsto (x', \varepsilon x_3)$$

For (u, e, p) kinematically admissible in Ω_ε we set

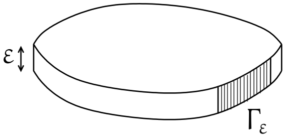
$y := (u_1 \circ \psi^\varepsilon, u_2 \circ \psi^\varepsilon, \varepsilon u_3 \circ \psi^\varepsilon)$

$f := \Lambda_\varepsilon^{-1} e \circ \psi^\varepsilon, \quad$ "$q := \Lambda_\varepsilon^{-1} p \circ \psi^\varepsilon$"

where $\Lambda_\varepsilon \xi := \left(\begin{array}{c|c} \xi_{\alpha\beta} & \frac{1}{\varepsilon}\xi_{\alpha 3} \\ \hline \frac{1}{\varepsilon}\xi_{3\beta} & \frac{1}{\varepsilon^2}\xi_{33} \end{array} \right)$

$\Omega_\varepsilon = \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \;\; \omega \subset \mathbb{R}^2$

$\Gamma_\varepsilon = \gamma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \;\; \gamma \subset \partial\omega$



Perform a change of variable:

$$\psi^\varepsilon\colon\; \Omega := \omega \times \big(-\tfrac{1}{2}, \tfrac{1}{2} \big) \to \Omega_\varepsilon \;:\; (x', x_3) \mapsto (x', \varepsilon x_3)$$

For (u, e, p) kinematically admissible in Ω_ε we set

$y := (u_1 \circ \psi^\varepsilon, u_2 \circ \psi^\varepsilon, \varepsilon u_3 \circ \psi^\varepsilon)$

$f := \Lambda_\varepsilon^{-1} e \circ \psi^\varepsilon$, "$q := \Lambda_\varepsilon^{-1} p \circ \psi^\varepsilon$"

where $\Lambda_\varepsilon \xi := \left(\begin{array}{c|c} \xi_{\alpha\beta} & \frac{1}{\varepsilon}\xi_{\alpha 3} \\ \hline \frac{1}{\varepsilon}\xi_{3\beta} & \frac{1}{\varepsilon^2}\xi_{33} \end{array} \right)$

Then $(y, f, q) \in A_\varepsilon(\hat{w}^\varepsilon)$, that is:

- $(y, f, q) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \times M_b(\Omega \cup \Gamma; \mathbb{M}^{3\times 3}_{sym})$
- $Ey = f + q$ in Ω and $q = (\hat{w}^\varepsilon - y) \odot \nu_{\partial\Omega} \mathcal{H}^2$ on $\Gamma := \gamma \times (-\frac{1}{2}, \frac{1}{2})$
- $\mathrm{tr}\,(\Lambda_\varepsilon q) = 0$ in $\Omega \cup \Gamma$

Convergence of Rescaled Quasistatic Evolutions

Theorem (Davoli-Mora 2012)

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ be such that $w_3 \in AC([0,T]; H^2(\omega))$ and

$$w_\alpha(t,x) = \bar{w}_\alpha(t,x') - x_3 \partial_\alpha w_3(t,x'), \quad \alpha = 1,2.$$

Let $(y^\varepsilon, f^\varepsilon, q^\varepsilon)$ be a quasistatic evolution in Ω_ε, rescaled to Ω, with boundary value w and initial value $(y_0^\varepsilon, f_0^\varepsilon, q_0^\varepsilon)$.

Convergence of Rescaled Quasistatic Evolutions

Theorem (Davoli-Mora 2012)

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ be such that $w_3 \in AC([0,T]; H^2(\omega))$ and

$$w_\alpha(t,x) = \bar{w}_\alpha(t,x') - x_3 \partial_\alpha w_3(t,x'), \quad \alpha = 1,2.$$

Let $(y^\varepsilon, f^\varepsilon, q^\varepsilon)$ be a quasistatic evolution in Ω_ε, rescaled to Ω, with boundary value w and initial value $(y_0^\varepsilon, f_0^\varepsilon, q_0^\varepsilon)$. Assume that

$\Lambda_\varepsilon f_0^\varepsilon \to f_0$ strongly in $L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$,

$\|\Lambda_\varepsilon p_0^\varepsilon\|_{\mathcal{M}_b} \leqslant C$.

Convergence of Rescaled Quasistatic Evolutions

Theorem (Davoli-Mora 2012)

Let $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^3))$ be such that $w_3 \in AC([0,T]; H^2(\omega))$ and

$$w_\alpha(t,x) = \bar{w}_\alpha(t,x') - x_3 \partial_\alpha w_3(t,x'), \quad \alpha = 1,2.$$

Let $(y^\varepsilon, f^\varepsilon, q^\varepsilon)$ be a quasistatic evolution in Ω_ε, rescaled to Ω, with boundary value w and initial value $(y_0^\varepsilon, f_0^\varepsilon, q_0^\varepsilon)$. Assume that

$\Lambda_\varepsilon f_0^\varepsilon \to f_0$ strongly in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$,

$\|\Lambda_\varepsilon p_0^\varepsilon\|_{M_b} \leqslant C$.

Then for every $t \in [0,T]$

- $y^\varepsilon(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$
- $f^\varepsilon(t) \to e(t)$ strongly in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$
- $q^\varepsilon(t) \rightharpoonup p(t)$ weakly* in $M_b(\Omega\cup\Gamma;\mathbb{M}^{3\times3}_{sym})$

where (u,e,p) is a “reduced” quasistatic evolution.

Characterization of the Limit Evolution – I

1) Kinematic admissibility: $(u(t), e(t), p(t)) \in A_{KL}(w(t))$, that is,

$$u_\alpha(t,x) = \bar{u}_\alpha(t,x') - x_3 \partial_\alpha u_3(t,x') \quad (\alpha = 1,2)$$

with $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$

$$BH(\omega) := \{z \in W^{1,1}(\omega) : D^2 z \in M_b(\omega; \mathbb{M}^{2\times 2}_{sym})\} .$$

In particular, $E_{\alpha\beta} u(t) = E_{\alpha\beta}\bar{u}(t) - x_3 \partial^2_{\alpha\beta} u_3(t)$, $E_{i3} u(t) = 0$.

Characterization of the Limit Evolution – I

1) Kinematic admissibility: $(u(t), e(t), p(t)) \in A_{KL}(w(t))$, that is,

$$u_\alpha(t,x) = \bar{u}_\alpha(t,x') - x_3 \partial_\alpha u_3(t,x') \quad (\alpha = 1,2)$$

with $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$

$$BH(\omega) := \{z \in W^{1,1}(\omega) : D^2 z \in M_b(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})\}.$$

In particular, $E_{\alpha\beta} u(t) = E_{\alpha\beta}\bar{u}(t) - x_3 \partial^2_{\alpha\beta} u_3(t)$, $E_{i3} u(t) = 0$.
Moreover,

$$e(t,x) = \bar{e}(t,x') + x_3 \hat{e}(t,x') + e_\perp(t,x)$$

$$p(t) = \bar{p}(t) \otimes \mathcal{L}^1 + x_3 \hat{p}(t) \otimes \mathcal{L}^1 - e_\perp(t)$$

where $\bar{e} := \int_{-\frac{1}{2}}^{\frac{1}{2}} e \, dx_3$, $\hat{e} := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 e \, dx_3$

Characterization of the Limit Evolution – I

1) Kinematic admissibility: $(u(t), e(t), p(t)) \in A_{KL}(w(t))$, that is,

$$u_\alpha(t,x) = \bar{u}_\alpha(t,x') - x_3 \partial_\alpha u_3(t,x') \quad (\alpha = 1,2)$$

with $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$

$$BH(\omega) := \{z \in W^{1,1}(\omega) : D^2 z \in M_b(\omega; \mathbb{M}^{2\times 2}_{\text{sym}})\}\,.$$

In particular, $E_{\alpha\beta}u(t) = E_{\alpha\beta}\bar{u}(t) - x_3 \partial^2_{\alpha\beta} u_3(t)$, $E_{i3}u(t) = 0$.
Moreover,

$$e(t,x) = \bar{e}(t,x') + x_3 \hat{e}(t,x') + e_\perp(t,x)$$

$$p(t) = \bar{p}(t) \otimes \mathcal{L}^1 + x_3 \hat{p}(t) \otimes \mathcal{L}^1 - e_\perp(t)$$

where $\bar{e} := \int_{-\frac{1}{2}}^{\frac{1}{2}} e \, dx_3$, $\hat{e} := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 e \, dx_3$ and

- $E\bar{u}(t) = \bar{e}(t) + \bar{p}(t)$ in ω, $\bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega} \mathcal{H}^1$ on γ
- $-D^2 u_3(t) = \hat{e}(t) + \hat{p}(t)$ in ω, $u_3(t) = w_3(t)$ on γ,
  $\hat{p}(t) = (\nabla u_3(t) - \nabla w_3(t)) \odot \nu_{\partial\omega} \mathcal{H}^1$ on γ

Characterization of the Limit Evolution – II

2) Regularity:

$$(u, e, p) \in AC([0,T]; BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \Gamma; \mathbb{M}^{3\times3}_{sym}))$$

3) "Reduced" quasistatic evolution: (u, e, p) satisfies

- "reduced" global minimality: for every $t \in [0,T]$

$$\int_\Omega Q_r(e(t))\,dx \leqslant \int_\Omega Q_r(\tilde{e})\,dx + \mathcal{H}_r(\tilde{p} - p(t))$$

for every $(\tilde{u}, \tilde{e}, \tilde{p}) \in A_{KL}(w(t))$

- "reduced" energy balance: for every $t \in [0,T]$

$$\int_\Omega Q_r(e(t))dx + \int_0^t \mathcal{H}_r(\dot{p}(s))ds = \int_\Omega Q_r(e_0)dx + \int_0^t \langle \mathbb{C}_r e(s), E\dot{w}(s)\rangle_{L^2} ds$$

where $Q_r(e) := \min\left\{Q(\xi) : \xi \in \mathbb{M}^{3\times3}_{sym}, \xi_{\alpha\beta} = e_{\alpha\beta}\ (\alpha, \beta = 1, 2)\right\}$
and $H_r(p) := \min\left\{H(q) : q \in \mathbb{M}^{3\times3}_{D}, q_{\alpha\beta} = p_{\alpha\beta}\ (\alpha, \beta = 1, 2)\right\}$

Characterization in Rate Form

- Euler conditions: setting $\sigma(t) := \mathbb{C}_r e(t)$, for every $t \in [0, T]$

  $\mathrm{div}_{x'} \bar{\sigma}(t) = 0$ and $\mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\sigma}(t) = 0$ in ω

  with Neumann boundary conditions on $\partial\omega \setminus \gamma$

  $\sigma(t) \in K_r$ a.e. in Ω, where $K_r := \partial H_r(0)$

Characterization in Rate Form

- Euler conditions: setting $\sigma(t) := \mathbb{C}_r e(t)$, for every $t \in [0, T]$

  $\mathrm{div}_{x'} \bar{\sigma}(t) = 0$ and $\mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\sigma}(t) = 0$ in ω

  with Neumann boundary conditions on $\partial\omega \setminus \gamma$

  $\sigma(t) \in K_r$ a.e. in Ω, where $K_r := \partial H_r(0)$

- balance of powers: for a.e. $t \in [0, T]$

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle$$

where the stress-strain duality is now defined as

$$\langle \sigma(t), \dot{p}(t) \rangle := \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle + \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \langle \sigma_\perp(t), \dot{e}_\perp(t) \rangle_{L^2}$$

The first duality is in the sense of Kohn-Temam 1983, while the second one is an adaptation of Demengel 1983. In particular, we show that

$$\mathcal{H}_r(p) = \sup \Big\{ \langle \tau, p \rangle : \tau \in L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}}),\ \tau(x) \in K_r \text{ for a.e. } x \in \Omega, \\ \mathrm{div}_{x'} \bar{\tau} \in L^2,\ \mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\tau} \in L^2 \text{ satisfying Neumann} \Big\}$$

Convergence of Rate-Independent Processes

Γ-limits of rate-independent evolutions (Mielke-Roubíček-Stefanelli 2008):

$\mathcal{X}$ state space

$\mathcal{E}_\varepsilon : [0, T] \times \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ stored-energy functionals

$\mathcal{D}_\varepsilon : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ dissipation distances

If $\mathcal{E} := \Gamma\text{-}\lim \mathcal{E}_\varepsilon$, $\mathcal{D} := \Gamma\text{-}\lim \mathcal{D}_\varepsilon$, and $\exists$ a "joint recovery sequence", then quasistatic evolutions associated with $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ converge to a quasistatic evolution associated with $(\mathcal{E}, \mathcal{D})$.

Applications: linearized plasticity with hardening (Liero-Mielke 2011)

Convergence of Rate-Independent Processes

Γ-limits of rate-independent evolutions (Mielke-Roubíček-Stefanelli 2008):

$\mathcal{X}$ state space

$\mathcal{E}_\varepsilon : [0,T] \times \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ stored-energy functionals

$\mathcal{D}_\varepsilon : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ dissipation distances

If $\mathcal{E} := \Gamma\text{-}\lim \mathcal{E}_\varepsilon$, $\mathcal{D} := \Gamma\text{-}\lim \mathcal{D}_\varepsilon$, and $\exists$ a "joint recovery sequence", then quasistatic evolutions associated with $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ converge to a quasistatic evolution associated with $(\mathcal{E}, \mathcal{D})$.

Applications: linearized plasticity with hardening (Liero-Mielke 2011)

This approach is not suited to our case:

$$\mathcal{E}_\varepsilon(u, e, p) := \int_\Omega Q(\Lambda_\varepsilon e)\, dx$$

$$\mathcal{D}_\varepsilon((u, e, p), (y, f, q)) := \mathcal{H}(\Lambda_\varepsilon p - \Lambda_\varepsilon q)$$

for $(u, e, p), (y, f, q) \in A_\varepsilon(w(t))$

Convergence of Rate-Independent Processes

Γ-limits of rate-independent evolutions (Mielke-Roubíček-Stefanelli 2008):

$\mathcal{X}$ state space

$\mathcal{E}_\varepsilon : [0,T] \times \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ stored-energy functionals

$\mathcal{D}_\varepsilon : \mathcal{X} \times \mathcal{X} \to [0,+\infty]$ dissipation distances

If $\mathcal{E} := \Gamma\text{-}\lim \mathcal{E}_\varepsilon$, $\mathcal{D} := \Gamma\text{-}\lim \mathcal{D}_\varepsilon$, and $\exists$ a "joint recovery sequence", then quasistatic evolutions associated with $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ converge to a quasistatic evolution associated with $(\mathcal{E}, \mathcal{D})$.

Applications: linearized plasticity with hardening (Liero-Mielke 2011)

This approach is not suited to our case:

$$\mathcal{E}_\varepsilon(u,e,p) := \int_\Omega Q(\Lambda_\varepsilon e)\,dx$$

$$\mathcal{D}_\varepsilon((u,e,p),(y,f,q)) := \mathcal{H}(\Lambda_\varepsilon p - \Lambda_\varepsilon q)$$

$\rightsquigarrow$ no coercivity

for $(u,e,p),(y,f,q) \in A_\varepsilon(w(t))$

Our Proof

Static case: we fix time and prove Γ-convergence of

$$\mathcal{J}_\varepsilon(u,e,p) := \int_\Omega Q(\Lambda_\varepsilon e)\,dx + \mathcal{H}(\Lambda_\varepsilon p) \quad \text{for } (u,e,p) \in A_\varepsilon(w)$$

to

$$\mathcal{J}(u,e,p) := \int_\Omega Q_r(e)\,dx + \mathcal{H}_r(p) \quad \text{for } (u,e,p) \in A_{KL}(w)$$

Quasistatic case: compactness via a priori estimates; passing to the limit in the minimality relies on the recovery sequence of the static case

Our Proof

Static case: we fix time and prove Γ-convergence of

$$\mathcal{J}_\varepsilon(u, e, p) := \int_\Omega Q(\Lambda_\varepsilon e)\, dx + \mathcal{H}(\Lambda_\varepsilon p) \quad \text{for } (u, e, p) \in A_\varepsilon(w)$$

to

$$\mathcal{J}(u, e, p) := \int_\Omega Q_r(e)\, dx + \mathcal{H}_r(p) \quad \text{for } (u, e, p) \in A_{KL}(w)$$

Quasistatic case: compactness via a priori estimates; passing to the limit in the minimality relies on the recovery sequence of the static case

Main ingredient: density of smooth enough triples in $A_{KL}(0)$ with respect to strong-L^2 for the elastic strains and strict-M_b for the plastic strains

Our Proof

Static case: we fix time and prove Γ-convergence of

$$\mathcal{J}_\varepsilon(u,e,p) := \int_\Omega Q(\Lambda_\varepsilon e)\,dx + \mathcal{H}(\Lambda_\varepsilon p) \quad \text{for } (u,e,p) \in A_\varepsilon(w)$$

to

$$\mathcal{J}(u,e,p) := \int_\Omega Q_r(e)\,dx + \mathcal{H}_r(p) \quad \text{for } (u,e,p) \in A_{KL}(w)$$

Quasistatic case: compactness via a priori estimates; passing to the limit in the minimality relies on the recovery sequence of the static case

Main ingredient: density of smooth enough triples in $A_{KL}(0)$ with respect to strong-L^2 for the elastic strains and strict-M_b for the plastic strains

- the Kirchhoff-Love structure has to be preserved
- the boundary conditions are imposed only on a portion of $\partial\omega$

Our Proof

Static case: we fix time and prove Γ-convergence of

$$\mathcal{J}_\varepsilon(u,e,p) := \int_\Omega Q(\Lambda_\varepsilon e)\,dx + \mathcal{H}(\Lambda_\varepsilon p) \quad \text{for } (u,e,p) \in A_\varepsilon(w)$$

to

$$\mathcal{J}(u,e,p) := \int_\Omega Q_r(e)\,dx + \mathcal{H}_r(p) \quad \text{for } (u,e,p) \in A_{KL}(w)$$

Quasistatic case: compactness via a priori estimates; passing to the limit in the minimality relies on the recovery sequence of the static case

Main ingredient: density of smooth enough triples in $A_{KL}(0)$ with respect to strong-L^2 for the elastic strains and strict-M_b for the plastic strains

- the Kirchhoff-Love structure has to be preserved
- the boundary conditions are imposed only on a portion of $\partial\omega$

  Here we assume $\partial\omega \in C^2$ and γ open with $\partial_{\partial\omega}\gamma = \{P_1, P_2\}$