Non-isothermal cyclic fatigue in an oscillating elastoplatic material with phase transition

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Introduction

Elastoplastic materials subject to cyclic loading exhibit increasing fatigue, which is manifested by

material softening, heat release and material failure in finite time.

In the uniaxial processes there is a qualitative and quantitative relationship between

- accumulated fatigue (by the rainflow algorithm, which counts closed hysteresis loops in the loading history and with each closed loop associates a number depending on its amplitude – the contribution of the loop to the total damage)
- dissipated energy (the number associated with a closed loop is its area).

Introduction

In multiaxial loading processes

- the concept of closed loop is meaningless
- reliable counterpart of the rainflow algorithm ?
- the notion of energy dissipation is a purely thermodynamic one – independent of the experimental setting

We propose a thermodynamic model for material fatigue accumulation based on the hypothesis that there exists a qualitative and quantitative relation between accumulated fatigue and dissipated energy.

We demonstrate our model on the example of a transversally oscillating elastoplastic beam.

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- Constitutive laws of elastoplaticity, stop operator, Prandtl-Ishlinskii energy balance

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- Model with fatigue, temperature and phase transition

Constitutive laws of elastoplasticity

A classical hysteresis-type model for one-dimensional elastoplasticity by L. Prandtl and A. Yu. Ishlinskii - the relation between strain ε and stress σ given by the formula

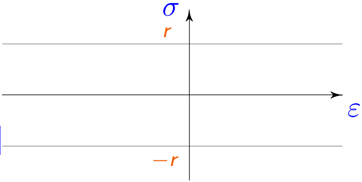
$$\sigma = \mathcal{P}[\varepsilon](t) = \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r \qquad (1)$$

for $\varepsilon \in W^{1,1}(0, T; \mathbb{R})$. Here, $\varphi(r) > 0$ is a weight function, and $\mathfrak{s}_r[\varepsilon](t)$ represents the elastic-ideally plastic element or stop operator with the threshold $r > 0$.

The stop operator

Given a parameter $r > 0$, a function $\varepsilon : [0, T] \to \mathbb{R}$, and an initial condition $\sigma^0 \in [-r, r]$, we look for functions $\sigma, \xi : [0, T] \to \mathbb{R}$ such that $\sigma(0) = \sigma^0$, and

$$\begin{aligned} \sigma(t) + \xi(t) &= \varepsilon(t) \\ |\sigma(t)| &\leq r \\ \dot{\xi}(t)\,(\sigma(t) - \tilde{\sigma}) &\geq 0 \quad \forall \tilde{\sigma} \in [-r, r] \end{aligned}$$



For every $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the problem has a unique solution $\sigma \in W^{1,1}(0, T)$. The solution mapping

$$\boxed{\mathfrak{s}_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T)\,, \;\; \sigma = \mathfrak{s}_r[\sigma^0, \varepsilon]}\,,$$

is called the stop (or elastoplastic element).

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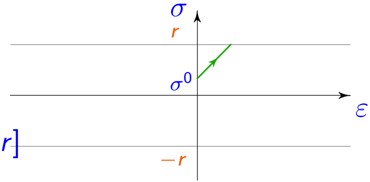

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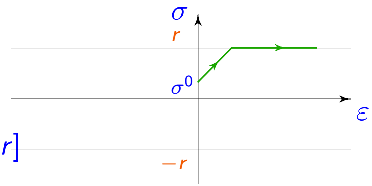

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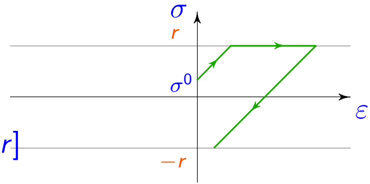

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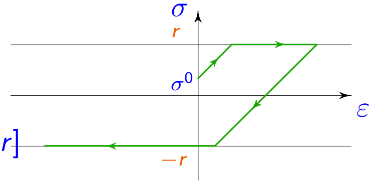

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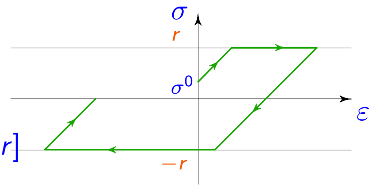

For every $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the problem has a unique solution $\sigma \in W^{1,1}(0, T)$. The solution mapping is Lipschitz continuous and admits Lipschitz continuous extension to $\mathfrak{s}_r : [-r, r] \times C[0, T] \to C[0, T]$.

Prandtl-Ishlinskii energy balance

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For a single stop, the energy balance reads

$$\dot{\varepsilon}\,\mathfrak{s}_r[\varepsilon] - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\mathfrak{s}_r^2[\varepsilon]\Big) = r\Big|\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon - \mathfrak{s}_r[\varepsilon])\Big|.$$

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$$\mathcal{P}[\varepsilon] = \int_0^\infty \varphi(r)\,\mathfrak{s}_r[\varepsilon]\mathrm{d}r$$

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$$V[\varepsilon] = \frac{1}{2}\int_0^\infty \varphi(r)\, \mathfrak{s}_r^2[\varepsilon]\mathrm{d}r$$

and the dissipation operator

$$\mathcal{D}[\varepsilon] = \int_0^\infty r\varphi(r)\, (\varepsilon - \mathfrak{s}_r[\varepsilon])\mathrm{d}r\,.$$

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Prandtl-Ishlinskii energy balance

$$\dot{\varepsilon}\,\mathcal{P}[\varepsilon] - V[\varepsilon]_t = |\mathcal{D}[\varepsilon]_t|.$$

Kinematic hardening:

$$\sigma = B\varepsilon + \mathcal{P}[\varepsilon] \tag{2}$$

with B positive.
The momentum balance equation:

$$\varrho u_{tt} - \operatorname{div} \sigma = f,$$

in $\Omega \times (0, T)$ and with suitable initial and boundary conditions. Here u is the displacement, $\varepsilon = u_x$, f is a given volume force and ϱ is the mass density.

First idea of the model

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Disadvanteges:

- The density function φ is a priori unknown and must be identified
- other 3D plasticity models like von Mises or Tresca models are available.

In P. Krejčí, J. Sprekels: Elastic-ideally plastic beams and Prandtl-Ishlinskii hysteresis operators, Math. Meth. Appl. Sci. 30 (2007), 2371–2393 they showed that in the modeling of the one-dimensional transversal vibrations of an elastoplastic beam the tree-dimensional von Mises model leads to a scalar Prandtl-Ishlinskii model whose density function is a priori given.

The model

is based on the idea that the Euler-Bernoulli dimensional reduction applied to transversal oscillations of an elastoplastic beam leads, as a result of averaging over the thickness of the beam, to a Prandtl-Ishlinskii constitutive law:

$$\sigma = \mathcal{P}[\varepsilon](t) = \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r$$

for $\varepsilon \in W^{1,1}(0, T; \mathbb{R})$ and to the momentum balance equation: (after rescaling all constants to 1)

$$w_{tt} - w_{xxtt} + \sigma_{xx} = f,$$

where w is the transversal displacement, $\varepsilon = w_{xx}$.

PDE system

The resulting prototypical system of partial differential equations is of the form

$$w_{tt} - w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} = f$$

with boundary conditions

$$w(0,t) = w(L,t) = \mathcal{P}[w_{xx}](0,t) = \mathcal{P}[w_{xx}](L,t) = 0\,.$$

Prandtl-Ishlinskii operators are not differentiable in general; hence, for the existence and uniqueness analysis, we rewrite the PDE as a system

$$\begin{aligned} u_t &= \mathcal{P}[w_{xx}] \\ w_t - w_{xxt} &= -u_{xx} + g \end{aligned}$$

with boundary conditions

$$u(0,t) = w(0,t) = u(L,t) = w(L,t) = 0.$$

The fatigue model

Our basic modeling assumption consists in replacing the elastoplastic constitutive law (2) by

$$\sigma = B(m)\varepsilon + \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\,\varphi(r)\,\mathrm{d}r \qquad (3)$$

where

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where m is the fatigue parameter
and the momentum balance equation becomes

$$w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]]_{xx} = f.$$

The fatique equation

We complete the system by an evolution equation for the fatigue parameter m:

$$\left(\frac{1}{C}+\frac{1}{2}B'(m)\varepsilon^2\right)m_t = \int_0^\infty \partial_t(\varepsilon-\mathfrak{s}_r[\varepsilon])\mathfrak{s}_r[\varepsilon]\varphi(r)\,\mathrm{d}r$$
$$= \ |\mathcal{D}[\varepsilon]_t|,$$

assuming that the rate of fatigue m_t is proportional to the dissipation rate D

$$D = -B'(m)\varepsilon^2 m_t + |\mathcal{D}[\varepsilon]_t|$$

with a proportionality factor C.

The associated system

$$
\begin{aligned}
u_t &= B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}] && \text{in } Q_T\,, \\
w_t - w_{xxt} &= -u_{xx} + g(x,t) && \text{in } Q_T\,, \\
u(1,t) = u_x(1,t) &= 0 && 0 \le t \le T\,, \\
w(0,t) = w_x(0,t) &= 0 && 0 \le t \le T\,, \\
u(x,0) &= w^1(x) && 0 \le x \le 1\,, \\
w(x,0) &= w^0(x) && 0 \le x \le 1\,,
\end{aligned}
$$

where we put

$$
\begin{aligned}
u(x,t) &= w^1(x) + \int_0^t [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]](x,s)\,ds\,, \\
g(x,t) &= w^1(x) + \int_0^t f(x,s)\,ds
\end{aligned}
$$

is well posed on some a priori unknown time interval $[0, T^*]$.

The fatigue model with temperature

Our basic modeling assumption consists in replacing the elastoplastic constitutive law (5) by

$$\sigma \; = \; B(m)\varepsilon + \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\, \varphi(r)\, \mathrm{d}r - \beta(\theta - \theta_c) + \nu\varepsilon_t, \qquad (4)$$

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where m is the fatigue parameter, θ is the absolute temperature, $\beta > 0$ is the thermal dilation coefficient, $\theta_c > 0$ is a fixed reference temperature and ν a viscosity parameter.

The momentum balance equation becomes

$$w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]]_{xx} - \beta\theta_{xx} + \nu w_{xxxxt} = f.$$

Thermodynamics

With the constitutive law (4) we associate the specific entropy

$$\mathcal{S}[\theta, \varepsilon] \;=\; c_V \log(\theta/\theta_c) + \beta\varepsilon$$

and the specific internal energy

$$\mathcal{U}[\theta, \varepsilon] \;=\; c_V\theta + \frac{1}{2}B(m)\varepsilon^2 + \frac{1}{2}\int_0^\infty \varphi(r)\mathfrak{s}_r^2[\varepsilon](t)\mathrm{d}r + \beta\theta_c\varepsilon\,.$$

We have the energy balance

$$\mathcal{U}_t + q_x = \sigma\varepsilon_t,$$

and again we assume that the fatigue rate m_t is proportional to the dissipation rate D with a proportionality factor $C(\theta)$:

$$\begin{aligned}\left(\frac{1}{C(\theta)} + \frac{1}{2}B'(m)\varepsilon^2\right)m_t &= \int_0^\infty \partial_t(\varepsilon - \mathfrak{s}_r[\varepsilon])\mathfrak{s}_r[\varepsilon]\varphi(r)\,\mathrm{d}r \\ &= \;|\mathcal{D}[\varepsilon]_t|\end{aligned}$$

Thermodynamics

The second Principle of Thermodynamics (Claudius-Duhem inequality) states for the entropy production ψ

$$\psi := \mathcal{S}[\theta, \varepsilon]_t + \left(\frac{q}{\theta}\right)_x \geq 0 .$$

We rewrite it in the form

$$\theta\psi := \sigma\varepsilon_t + \theta\mathcal{S}[\theta, \varepsilon]_t - \mathcal{U}[\theta, \varepsilon]_t - \frac{q\theta_x}{\theta} \geq 0 ,$$

use the Fourier law $q = -k\theta_x$ and get that the dissipation rate

$$D = -B'(m)\varepsilon^2 m_t + |\mathcal{D}[\varepsilon]_t|$$

has to be nonnegative. The fatigue accumulation rate m_t should be nonnegative, so we need to assume that $B'(m)$ is negative.

Oscillating elastoplastic beam

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For the oscillating elastoplastic beam problem

$$
\begin{aligned}
u_t &= B(m)w_{xx} + \mathcal{P}[w_{xx}] + w_{xxt} - (\theta - \theta_c)\,, \\
w_t - w_{xxt} &= -u_{xx} + g \\
\theta_t &= \theta_{xx} - \frac{1}{2}B'(m)w_{xx}^2 m_t + |\mathcal{D}[w_{xx}]_t| + w_{xxt}^2 - \theta w_{xxt}\,, \\
m_t &= \int_0^1 \lambda(x-y)\left(-\frac{1}{2}B'(m)w_{xx}^2 m_t + |\mathcal{D}[w_{xx}]_t|\right)(y,t)\mathrm{d}y
\end{aligned}
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with a spatially regularized fatigue equation, and with zero initial and boundary conditions for w and u, and homogeneous Neumann boundary conditions for θ, we find and efficient lower bound for the existence and uniqueness time T^*.

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Extension to temperature and fatigue dependent plasticity is straightforward.

Hypotheses

(i) $\mathcal{P}$ is a Prandtl-Ishlinskii operator and $\mathcal{D}$ is its associated dissipation operator. We assume that its distribution function $\varphi \in L^1(0, \infty)$ is such that $\varphi \geq 0$ a.e., and $\int_0^\infty r\varphi(r)\mathrm{d}r < \infty$.

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(iv) $g \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $g_{tt}, g_{xx} \in L^2(\Omega_T)$.

Hypotheses

(i) $\mathcal{P}$ is a Prandtl-Ishlinskii operator and $\mathcal{D}$ is its associated dissipation operator. We assume that its distribution function $\varphi \in L^1(0,\infty)$ is such that $\varphi \geq 0$ a.e., and $\int_0^\infty r\varphi(r)\mathrm{d}r < \infty$.

(ii) $B : [0,\infty) \to (0,\infty)$ is a C^2 function, $B'(0) = 0$, $-1 \leq B''(m) \leq 0$ for all $m > 0$.

(iii) $\lambda : \mathbb{R} \to [0,\infty)$ is a C^1 function with compact support, $L := \max\{\lambda(x) + |\lambda'(x)|\,,\ x \in \mathbb{R}\}$.

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- On the interval $[0, T^R)$, we derive estimates for the approximate solutions which enable us to show that the truncation never becomes active if R is sufficiently large, and can be removed
- We select a convergent subsequence indexed by n and pass to the limit as $n \to \infty$ to obtain the solution

Energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\theta+\frac{1}{2}\left(w_t^2+w_{xt}^2+B(m)w_{xx}^2\right)+V[w_{xx}]+\theta_c w_{xx}\right)\mathrm{d}x$$

$$=\int_\Omega g_t w_t \mathrm{d}x\,.$$

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$$w_{xxxt}, w_{xxtt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(0, T^*; L^2(\Omega)).$$

Model with temperature and phase transition

The idea of this model is to consider also decreasing fatigue. Our present model takes into account the possibility to repair a partially damaged material by the effects of partial melting, so that fatigue can also decrease in time.
The constitute law will be the same as before

$$\sigma = B(m)\varepsilon + \int_0^\infty \mathfrak{s}_r[\varepsilon](t)\,\varphi(m, r)\,\mathrm{d}r - \beta(\theta - \theta_c) + \nu\varepsilon_t, \quad (5)$$

and also the momentum balance equation stayes the same:

$$w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]]_{xx} = f.$$

With the constitutive law (5) we associate the specific entropy

$$\mathcal{S}[\theta,\varepsilon,\chi] \;=\; c_V \log(\theta/\theta_c) + \beta\varepsilon + \frac{L}{\theta_c}\chi$$

and the specific internal energy

$$\mathcal{U}[\theta,\varepsilon,\chi] \;=\; c_V\theta + \frac{1}{2}B(m)\varepsilon^2 + \frac{1}{2}\int_0^\infty \varphi(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r + \beta\theta_c\varepsilon + L\chi + \mathrm{I}_{[0,1]}(\chi),$$

where L is the constant latent heat, χ is the space and time dependent phase variable and I_A is the indicator function of the set A. We have the energy balance

$$\mathcal{U}_t + q_x = \sigma\varepsilon_t,$$

and the second Principle of Thermodynamics (Claudius-Duhem inequality) for the entropy production ψ

$$\psi := \mathcal{S}[\theta,\varepsilon]_t + \left(\frac{q}{\theta}\right)_x \geq 0\,.$$

Thermodynamics

We assume as before that $B'(m)$ is negative and we have an equation for the phase variable χ:

$$-\gamma\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_c}(\theta - \theta_c),$$

and the evolution equation for the fatigue rate we assume in the form

$$\frac{1}{C(\theta)} m_t + \frac{1}{2} B'(m)\varepsilon^2 m_t^- = -h(m)\chi_t|\chi_t| + \int_0^\infty \partial_t(\varepsilon - \mathfrak{s}_r[\varepsilon])\mathfrak{s}_r[\varepsilon]\varphi(r)\,\mathrm{d}r\,.$$

The system

The system

For the problem

$$u_t = B(m)w_{xx} + \mathcal{P}[w_{xx}] + w_{xxt} - (\theta - \theta_c)\,,$$

$$w_t - w_{xxt} = -u_{xx} + g,$$

$$\theta_t = \theta_{xx} - \frac{1}{2}B'(m)w_{xx}^2 m_t + |\mathcal{D}[w_{xx}]_t| + w_{xxt}^2 - \theta w_{xxt}\,, - L\chi_t$$

$$-\frac{1}{2}m_t \int_0^\infty \varphi(m,r)\mathfrak{s}_r[\varepsilon]dr,$$

$$-\gamma\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_c}(\theta - \theta_c),$$

$$\frac{1}{C(\theta)}m_t + \frac{1}{2}B'(m)\varepsilon^2 m_t^- = -h(m)\chi_t|\chi_t| + \int_0^\infty \partial_t(\varepsilon - \mathfrak{s}_r[\varepsilon])\mathfrak{s}_r[\varepsilon]\varphi(r)\,\mathrm{d}r.$$

with zero initial and boundary conditions for w and u, and homogeneous Neumann boundary conditions for θ, we expect existence of a global solution.

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