Weierstrass Institute for
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Representation of hysteresis operators for vector-valued inputs by string functions

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- Many PDEs, including also some modeling multiphase advanced materials, contain hysteresis operator with vectorial inputs or are connected to operators of this type, see, e.g. Krejčí 1996; Krejčí–Sprekels 2000, 2001, 2002a, 2002b; Krejčí–Sprekels–Stefanelli 2002; Klein 2004, Seidman-Klein 2012; Visintin 1994, 2001; Mielke + Coauthors

- **Brokate–Sprekels 1996:**
    - Representation of hysteresis operators with scalar, piecewise monotone inputs and scalar outputs by functionals acting on alternating strings
    - Investigation of operators by considering their representations
    - Representation result also used in Brokate 1994,Brokate 2000, Ekanayake–Iyer 2008, Gasiński 2004, 2008, Jais-2008, Kaltenbacher–Kaltenbacher-2007, Löschner–Brokate 2008, Löschner-Greenberg 2008, Miettinen–Panagiotopoulos 1998, Tan–Baras–Krishnaprasad 2005, Visone 2008
- **K. 2012a = Phys. B. Vol 407, K. 2012b = WIAS-Preprint 1697:**
  Introduction of appropriate functions space for input functions and of appropriate set for strings allowing a similar representation result for hysteresis operators with vectorial inputs

- Let $T > 0$ denote some final time.
- Let X be some topological vector space.
- Let Y be some nonempty set, and let $\mathrm{Map}\left([0,T],Y\right) := \{v : [0,T] \to Y\}$.
- Let $\mathrm{C}\left([0,T];X\right)$ denote the set of all continuous functions $u : [0,T] \to X$.
- $\alpha : [0,T] \to [0,T]$ is an *admissible time transformation* $:\Longleftrightarrow$ α is continuous and increasing (not necessary strictly increasing), $\alpha(0) = 0$ and $\alpha(T) = T$.

Let $\mathcal{H} : D(\mathcal{H})(\subseteq \mathrm{Map}\,([0,T],X)) \to \mathrm{Map}\,([0,T],Y)$ with $D(\mathcal{H}) \neq \emptyset$ be given.

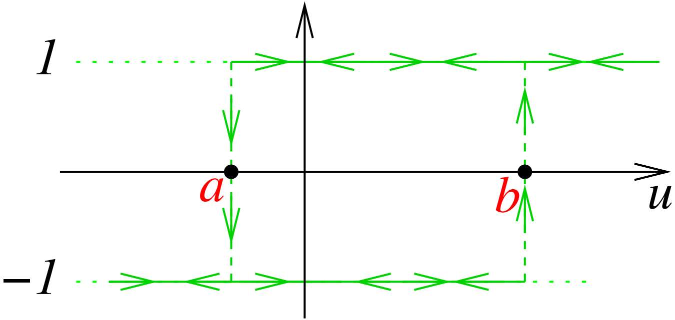
- $\mathcal{H}$ a *hysteresis operator* $:\Longleftrightarrow$ $\mathcal{H}$ is rate-independent and causal.
- $\mathcal{H}$ is *rate-independent* $:\Longleftrightarrow \forall\, v \in D(\mathcal{H}), \forall$ admissible time transformation $\alpha : [0,T] \to [0,T]$ with $v \circ \alpha \in D(\mathcal{H}), \forall\, t \in [0,T]$:

$$\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t)).$$

- $\mathcal{H}$ is *causal* $:\Longleftrightarrow \forall\, v_1, v_2 \in D(\mathcal{H}), \forall\, t \in [0,T]$:
  If $v_1(\tau) = v_2(\tau) \quad \forall\, \tau \in [0,t]$ then $\mathcal{H}[v_1](t) = \mathcal{H}[v_2](t)$.

- Let $D_0 \subset \mathrm{Map}\,([0,T],X)$ be nonempty. Let Z be a nonempty set of *initial states*. Let $\mathcal{G} : Z \times D_0 \to \mathrm{Map}\,([0,T],Y)$ be given.
  $\mathcal{G}$ is a *hysteresis operator* $:\Longleftrightarrow$ for all $z_0 \in Z$ it holds that $\mathcal{G}[z_0,\cdot]$ is a hysteresis operator

For thresholds values $a < b$ the *standard relay* $\mathcal{R}_{a,b}$ is defined by



- The scalar relay operator is a hysteresis operator.

Many vectorial relays considered in the literature (see, e.g., Della Torre-Pinzaglia-Cardelli 2006a,b, Löschner-Greenberg 2008, (Mayergoyz 2003), Visintin-1994) can be rewritten as an operator of the following form (see K. 2012b):

For a nonempty, open subset O of X, a nonempty set Y, and a function $\zeta : X \setminus O \to Y$ the generalized vectorial relay operator is defined by

$$\mathcal{R}_{O,\zeta} : Y \times \mathrm{C}\left([0,T];X\right) \to \mathrm{Map}\left([0,T],Y\right),$$

$$\mathcal{R}_{O,\zeta}[\eta^0,u](t) := \begin{cases} \zeta(u(t)), & \text{if} \quad u(t) \notin O, \\ \eta^0, & \text{if} \quad u([0,t]) \subset O, \\ \zeta\left(u\left(\max\left\{s \in [0,t] \,\middle|\, u(s) \notin O\right\}\right)\right), & \text{otherwise.} \end{cases}$$

$$\mathcal{R}_{O,\zeta} : Y \times \mathrm{C}\left([0,T];X\right) \to \mathrm{Map}\left([0,T],Y\right),$$

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- The generalized vectorial relay operator is a hysteresis operator.
- For thresholds values $a < b$, we obtain the standard relay operator $\mathcal{R}_{a,b}$. for $X = Y = \mathbb{R}$, $O :=]a,b[$, and $\zeta := \zeta_{a,b}$ with

$$\zeta_{a,b}(w) = \begin{cases} 1, & \text{if} \quad w \geq b, \\ -1, & \text{if} \quad w \leq a. \end{cases}$$

Brokate–Sprekels 1996: Hysteresis operators act on $\mathrm{C}_{\mathrm{pm}}([0,T])$ = the set of all piecewise monotone and continuous functions from $[0,T]$ to $\mathbb{R}$.

K. 2012a, K. 2012b: Hysteresis operators acting on piecewise **monotaffine** (=composition of **monot**one with **affine** function) and continuous function form $[0,T] \to X$.

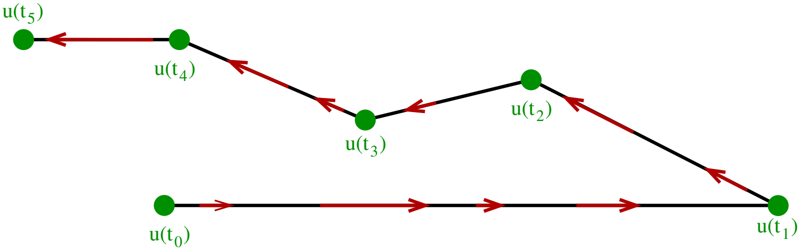
Monotaffine function precise: K. 2012a, K. 2012b:

Let some function $u : [0,T] \to X$ be given Let some $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ be given.

u is *monotaffine on* $[t_1,t_2]$ $:\Longleftrightarrow$

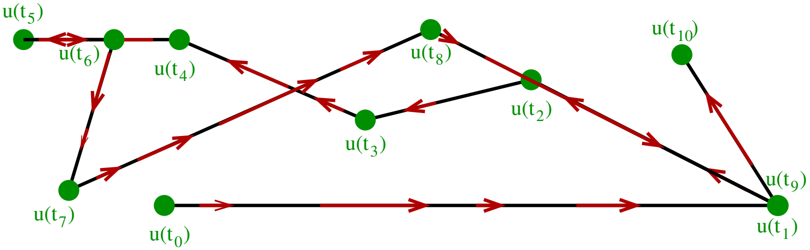
$\exists\, \beta : [t_1,t_2] \to [0,1]$ monotone increasing (not necessary strictly increasing) such that

$$u(t) = (1-\beta(t))u(t_1) + \beta(t)u(t_2), \quad \forall\, t \in [t_1,t_2].$$



K. 2012a, K. 2012b:

- u is denoted as *piecewise monotaffine* $:\Longleftrightarrow$
  there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$
  such that for $\forall\, 1 \le i \le n$: u is monotaffine on $[t_{i-1}, t_i]$.
- Let $\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ be the set of all piecewise monotaffine
  functions in $C([0,T];X)$.



K. 2012a, K. 2012b:

- For $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ and every admissible time transformation $\alpha : [0,T] \to [0,T]$ is holds that $u \circ \alpha \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$.
- $\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];\mathbb{R}) = \mathrm{C}_{\mathrm{pm}}([0,T])$.

- **Brokate–Sprekels 1996:** Let $u \in \mathrm{C}_{\mathrm{pm}}([0,T])$ be given.
  The *standard monotonicity partition of* $[0,T]$ *for* u := unique defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that for $1 \leq i \leq n$ holds:
  t_i is the maximal $t \in]t_{i-1}, T]$ such that u is monotone on $[t_{i-1}, t]$.
- **K. 2012a, K. 2012b:** Let $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T]; X)$ be given.
  The *standard monotaffinicity partition of* $[0,T]$ *for* u := uniquely defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that for $1 \leq i \leq n$ holds:
  t_i is the maximal $t \in]t_{i-1}, T]$ such that u is monotaffine on $[t_{i-1}, t]$.

Brokate–Sprekels: Set of *alternating strings:*

$$S_A = \{(v_0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} \,|\, n \geq 1,$$
$$(v_i - v_{i-1})(v_{i+1} - v_i) < 0, \quad \forall\, 1 \leq i < n\}.$$

K. 2012a, K. 2012b:

- A *convexity triple free string of elements of X* is any $(v_0, \ldots, v_n) \in X^{n+1}$ with $n \in \mathbb{N}$ and $v_i \notin \operatorname{conv}(v_{i-1}, v_{i+1})$ for all $1 \leq i < n$.
- For $v, w \in X$:
  $\operatorname{conv}(v, w) := \{(1 - \lambda)v + \lambda w \,|\, \lambda \in [0, 1]\}$.
- Let $S_F(X) := \{V \in X^{n+1} \,|\, n \in \mathbb{N}$ and V is a convexity triple free string of elements of $X\}$.
- It holds $S_F(\mathbb{R}) = S_A$.

- **Brokate–Sprekels 1996:**
  - For $V = (v_0, \ldots, v_n) \in S_A$ let *the piecewise linear representation of V* be the function $\pi[V] : [0,T] \to \mathbb{R}$ satisfying $\pi[V](t_0) = v_0$,

$$\pi[V](t_i) = v_i, \quad \pi[V] \text{ is linear on } [t_{i-1}, t_i] \quad \forall\, i = 0, \ldots, n$$

    with $t_i := \frac{i}{n} T$.
  - Let $\mathcal{H} : \mathrm{C}_{\mathrm{pm}}([0,T]) \to \mathrm{Map}\,([0,T], \mathbb{R})$ be an operator. Now, the *string functional $F_{\mathcal{H}} : S_A \to \mathbb{R}$ generated by $\mathcal{H}$* is defined by

$$F_{\mathcal{H}}(V) := \mathcal{H}\left[\pi[V]\right](T), \quad \forall\, V \in S_A.$$

- **K. 2012a, K. 2012b:**
  - For $V = (v_0, \ldots, v_n) \in S_F(X)$ let *the piecewise affine representation of V* be defined analogously to the piecewise linear one, just with “linear” replaced by “affine”.

- **K. 2012a, K. 2012b:**
  - Let $\mathcal{H} : \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T]; \to)\mathrm{Map}\,([0,T], Y)$ be an operator. The *string function $F_{\mathcal{H}} : S_F(X) \to Y$ generated by $\mathcal{H}$* is defined by

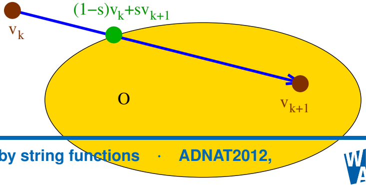
$$F_{\mathcal{H}}(V) := \mathcal{H}[\pi_{\mathrm{pw.af.}}[V]](T), \quad \forall\, V \in S_F(X)$$

For a nonempty, open and convex subset O of X, a nonempty set Y, a function $\zeta : X \setminus O \to Y$ and $\eta_0 \in Y$ it hold that: The string function generated by generalized relay $\mathcal{R}_{O,\zeta}[\eta_0, \cdot]$ in K. 2012b satisfies:

$$F_{\mathcal{R}_{O,\zeta}[\eta_0,\cdot]}(v_0, v_1, \ldots, v_n)$$

$$= \begin{cases} \zeta(v_n), \quad \text{if} \quad v_n \notin O, \\ \eta^0, \quad \text{if} \quad v_i \in O \quad \forall i \in \{0, \ldots, n\}, \\ \zeta\left((1-s)v_k + sv_{k+1}\right) \quad \text{with} \quad s \in [0,1] \text{ such that} \\ \qquad (1-s)v_k + sv_{k+1} \in \partial O\big\}, \\ \qquad k := \max\big\{i \in \{0, \ldots, n-1\} \bigm| v_i \notin O.\big\}, \quad \text{otherwise,} \end{cases}$$

(Formulation for O not convex: see K. 2012b)



K. 2012a, K. 2012b: (Result in Brokate-Sprekels 1996 corresponds to $X = \mathbb{R}$, $Y := \mathbb{R}$, "monotaffinicity" $\mapsto$ "monotonicity"), $S_F(X) \mapsto S_A$, $\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X) \mapsto \mathrm{C}_{\mathrm{pm}}([0,T])$)

Theorem

a.) *Every function* $G : S_F(X) \to Y$ *generates a hysteresis operator* $\mathcal{H}_G^{\mathrm{gen}} : \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X) \to$ *Map*$([0,T];Y)$ *by mapping* $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ *to the function* $\mathcal{H}_G^{\mathrm{gen}}[u] : [0,T] \to Y$ *by defined by considering standard monotaffinicity partition* $0 = t_0 < t_1 < \cdots < t_n = T$ *of* $[0,T]$ *for* u *and defining*

$$\mathcal{H}_G^{\mathrm{gen}}[u](t) = G\left(u(t_0), u(t)\right), \forall\, t \in [t_0, t_1],$$

$$\mathcal{H}_G^{\mathrm{gen}}[u](t) = G\left(u(t_0), \ldots, u(t_{i-1}), u(t)\right), \forall \quad t \in]t_{i-1}, t_i], \quad \forall \quad 2 \leq i \leq n.$$

Theorem

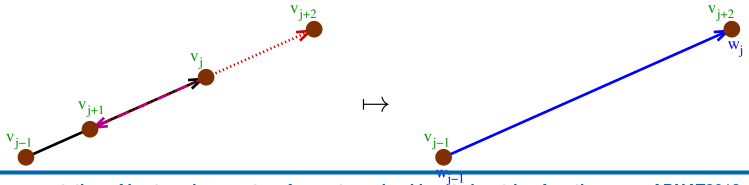
b.) *For **every hysteresis operator** $\mathcal{B} : \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X) \to$ **Map**$([0,T];Y)$ there **exists a unique function** $G : S_F(X) \to Y$ such that $\mathcal{B} = \mathcal{H}_G^{\mathrm{gen}}$.*
It is the string function $F_{\mathcal{B}}$ generated by $\mathcal{B}$

- If one is evaluating a hysteresis operator acting on continuous, piecewise monotaffine functions, then it is sufficient to keep track of the positions of the changes of direction of the input function. (in the scalar case, one needs to keep track of local extrema of function)
- Many properties of this hysteresis operators can be conveniently formulated and investigated by considering the functional generated by the operator. (scalar case: Brokate-Sprekels 1996)

Generalization of Madelung deletion for scalar strings in Brokate-Sprekels 1996, Sec. 2.6.1 for vector valued inputs:

- Let $V = (v_0, \dots, v_n) \in X^{n+1}$ and $W(w_0, \dots, w_{n-2}) \in X^{n-1}$ with $n \in \mathbb{N}$ and $n > 2$ be given.
  W is denoted as *the result of a Madelung deletion in V*, if there is some $j \in \{1, \dots, n-2\}$ such that

$$
\begin{aligned}
&W = (v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_n), \\
&\mathrm{conv}\,(v_j, v_{j+1}) \subseteq \mathrm{conv}\,(v_{j-1}, v_{j+2}), \\
&v_j \notin \mathrm{conv}\,(v_{j-1}, v_{j+1}), \quad v_{j+1} \notin \mathrm{conv}\,(v_j, v_{j+2}).
\end{aligned}
$$



- Let a function $G : S_F(X) \to Y$ be given. The *function G forgets according to the Madelung deletion*, if for all $V, W \in S_F(X)$ holds: If W is the result of a Madelung deletion in V, then $G(V) = G(W)$ is satisfied.
- Let some hysteresis operator $\mathcal{H} : \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X) \to \mathrm{Map}\left([0,T],Y\right)$ be given. The *operator $\mathcal{H}$ forgets according to the Madelung deletion*, if the string function $F_{\mathcal{H}} : S_F(X) \to Y$ generated by $\mathcal{H}$ forgets according to the Madelung deletion.

Operators forgetting according to the Madelung deletion

- scalar relay, scalar play, scalar stop, scalar Preisach and scalar Prandtl-Ishlinskii operator
- generalized vectorial relay as in Kl. 2012b, corresponding vectorial Preisach operators, vectorial play/stop with very special characteristic sets

Operators not forgetting according to the Madelung deletion

- scalar Duhem operator
- most vectorial play/stop operators; generalized vectorial relays using switching sets (Seidman 1983, 1989, 1990; Hante–Leugering–Seidman 2010; Belbas 2004; Seidman–Klein 2012), except or special switching sets

- Using the monotaffine functions and convexity triple free strings the representation result in Brokate-Sprekels 1996 has been extended to to the vectorial case in K. 2012 a,b
- If one is considering piecewise affine or piecewise monotaffine and continuous inputs, one needs not the memorizes the whole evolution of the input to be able to evaluate the value of the hysteresis operator, one needs only to keep track of the positions of the directions changes of the input.
- The string representation of hysteresis operators can be used to formulated and investigated properties of these operators.
  For example: Forgetting with respect to the Madelung deletion

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Thank you for your attention.

The above construction can also be applied for piecewise monotaffine inputs that are not continuous, leading to an hysteresis operator, corresponding to the arclen extension, see Recupero 2011. But, not all hysteresis operators for inputs of this type can be constructed in this way.

For example: The *glue operator*: For $x_* \in X$ let $\mathcal{G}_{x_*} : \mathrm{Map}\,([0,T], X) \to \mathrm{Map}\,([0,T], X)$ be defined by

$$\mathcal{G}_{x_*}[u](t) := \begin{cases} x_*, & \text{if there exits } s \in [0,t] \quad x_* = u(s), \\ u(t), & \text{otherwise,} \end{cases}$$

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it holds

$$F_{\mathcal{G}_{x_*}}(v_0, \ldots, v_n) = \begin{cases} x_*, & \text{if there exists} \quad i \in \{1, \ldots, n\} : \\ & \qquad x_* \in \operatorname{conv}(v_{i-1}, v_i), \\ v_n, & \text{otherwise,} \end{cases}$$

$$\mathcal{H}^{\text{gen}}_{F_{\mathcal{G}_{x_*}}}[u](t) = \begin{cases} x_*, & \text{if there exits} \quad s \in (0,t] \quad x_* \operatorname{conv}(u(s-), u(s)), \\ x_*, & \text{if there exits} \quad s \in [0,t) \quad x_* \operatorname{conv}(u(s), u(s+)), \\ u(t), & \text{otherwise.} \end{cases}$$

Brokate–Sprekels 1996: Let $u \in \mathrm{C}_{\mathrm{pm}}([0,T])$ be given.

For the standard monotonicity partition

$0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ for u it holds

$$(u(t_0), u(t_1), \ldots, u(t_{i-1}), u(t)) \in S_A,$$
$$\forall t \in]t_{i-1}, t_i], \quad 1 \leq i \leq n.$$

K. 2012a, K. 2012b: Let $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ be given.

For the standard monotaffinicity partition

$0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ for u it holds

$$(u(t_0), u(t_1), \ldots, u(t_{i-1}), u(t)) \in S_F(X),$$
$$\forall t \in]t_{i-1}, t_i], \quad 1 \leq i \leq n.$$