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Parabolic Variational Inequalities for a Class of Weakly Time-dependent Constraints and Applications

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Plan of the talk

0. Prof. Gianni Gilardi

1. Parabolic variational inequalities

2. Weak solvability of (E)

3. Activation of bacteria with environmental constrant

Congratulations on
Gianni's 65th Birthday

Research Subjects:

Dam problems, Phase field models, Doubly non-linear problems

Long-time behaviour of solutions, Optimal control problems

Profound knowledge on the theory of functional analysis

Many collaborations:

H.W. Alt, V. Barbu, M.L. Bernardi, E. Bonetti, L.A. Caffarelli,
J. Carrillo, P. Colli, J.I. Diaz, W. Dreyer, M. Fabrizio,
G. Geymonat, M. Grasselli, P. Fernandes, P. Krejci,
P. Laurencot, S. Luckhaus, A. Marson, A. Miranville,
A. Novick-Cohen, P. Podio-Guidugli, E. Rocca, G. Schimperna,
J. Sprekels, U. Stefanelli, ...

5 Grandchildren: Luca, Lorenzo, Matteo, Margheritta, Giacomo

1. Parabolic variational inequalities

H : Hilbert space, norm $|\cdot|_H$, inner product $(\cdot,\cdot)_H$

φ : proper, l.s.c., convex, ≥ 0, on H, $\varphi(0) = 0$,

any level set $\{z \in H \mid \varphi(z) + |z|_H \leq r\}$ is compact in H for $\forall r \geq 0$

$K(t) : 0 \leq t \leq T, \neq \emptyset$, closed convex subset of H, $K(t) = \overline{K(t) \cap D(\varphi)}$,

$\varphi^t(z) := \varphi(z) + I_{K(t)}(z), \quad \forall z \in H,$

Strong Class (K_S) (cf.K.[7], Yamada[10])

$$(K_S) := \left\{ \{K(t)\} \;\middle|\; \begin{array}{l} \exists\alpha(\cdot) \in W^{1,2}(0,T), \exists\beta(\cdot) \in W^{1,1}(0,T) \text{ such that} \\ \quad \forall s,t \in [0,T], \forall z \in K(s) \cap D(\varphi), \exists\tilde{z} \in K(t); \\ \qquad |\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)|(1 + \varphi(z)^{\frac{1}{2}}), \\ \qquad |\varphi(\tilde{z}) - \varphi(z)| \leq |\beta(t) - \beta(s)|(1 + \varphi(z)) \end{array} \right\}$$

For $\{K(t)\} \in (K_S)$, $f \in L^2(0,T;H)$ and $u_0 \in K(0) \cap D(\varphi)$,

$(E) \quad u'(t) + \partial\varphi^t(u(t)) \ni f(t), \; u(0) = u_0,$

has a unique (strong) solution u in $W^{1,2}(0,T;H)$ such that

$u(t) \in K(t)$, $\varphi(u(\cdot))$ is absolutely continuous on $[0,T]$.

Let us consider (E) for a more general class of $\{K(t)\}$ such as $t \to K(t)$ is continuous.

Weak Class (K_W) (A model case)

$$
(K_W) := \left\{ \{K(t)\} \;\middle|\; \begin{array}{l} \exists \{K_n(t)\} \in (K_S), n = 1, 2, \cdots, \text{ such that} \\ \forall \varepsilon \in (0,1], \exists A_\varepsilon \in BL(H), A_\varepsilon = A_\varepsilon^*, \ |A_\varepsilon| \le \varepsilon, \ \exists n_\varepsilon; \\ (I + A_\varepsilon) K_n(t) \subset K(t), \ (I + A_\varepsilon) K(t) \subset K_n(t), \\ \qquad \forall t \in [0,T], \forall n \ge n_\varepsilon \\ \varphi((I + A_\varepsilon) v) \le (1 + C\varepsilon) \varphi(v), \quad \forall v \in D(\varphi). \end{array} \right\}
$$

Example (Basic idea).

$\Omega \subset \mathbf{R}^N$, bounded, smooth $\Gamma := \partial\Omega$, $Q := \Omega \times (0,T)$

$H := L^2(\Omega)$, $V := H^1(\Omega)$, $A_\varepsilon z = -\varepsilon z$

$$
\varphi(z) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla z(x)|^2 dx, & \text{for } z \in V, \\ \infty, & \text{otherwise.} \end{cases}
$$

Given a cotinuous function $\psi(x,t)$ on $\overline{Q}$, we put

$$
K(t) := \{ z \in H \mid z(x) \le \psi(x,t) \text{ for a.e. } x \in \Omega \}, \quad \forall t \in [0,T].
$$

If $\psi \in W^{1,2}(0,T;V)$, then $\{K(t)\} \in (K_S)$; in this case we can take

$$\alpha(t) = \beta(t) = \text{const.} \int_0^t |\psi'(\tau)|_V d\tau.$$

But, in general, $\{K(t)\} \in (K_W)$. In fact, assuming that

$$0 < c_* \le \psi \le c^* < \infty \text{ on } \overline{Q},$$

take a sequence of smooth functions $\{\psi_n\}, c_* \le \psi_n \le c^*$, on $\overline{Q}$ which converges to ψ uniformly on $\overline{Q}$, namely

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ s.t. } |\psi_n(x,t) - \psi(x,t)| \le \varepsilon c_*, \ \forall (x,t) \in \overline{Q}, \ \forall n \ge n_\varepsilon.$$

Given $\eta \in K_n(t) := \{z \in H \mid z(\cdot) \le \psi_n(\cdot,t) \text{ a.e. on } \Omega\}$, we have:

$$(1-\varepsilon)\eta(x) \le (1-\varepsilon)\psi_n(x,t) \le (1-\varepsilon)(\psi(x,t) + \varepsilon c_*) \le \psi(x,t).$$

Hence

$$(I + A_\varepsilon)K_n(t) \subset K(t) \text{ for all } t \in [0,T] \text{ and } n \ge n_\varepsilon.$$

Similarly,

$$(I + A_\varepsilon)K(t) \subset K_n(t) \text{ for all } t \in [0,T] \text{ and } n \ge n_\varepsilon.$$

Since $\{K_n(t)\} \in (K_S)$, it follows that $\{K(t)\} \in (K_W)$.

2. Weak solvability of (E)

Assume $\{K(t)\} \in (K_W)$, $f \in L^2(0,T;H)$, $u_0 \in K(0)$, and consider

$$(E_w) \qquad \begin{aligned} &\int_0^t (\eta'(\tau), u(\tau) - \eta(\tau))_H d\tau + \int_0^t \varphi(u(\tau))d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2 \\ &\le \int_0^t \varphi(\eta(\tau))d\tau + \int_0^t (f(\tau), u(\tau) - \eta(\tau))_H d\tau + \frac{1}{2}|u_0 - \eta(0)|_H^2, \\ &\qquad \forall t \in [0,T], \quad \forall \eta \in \mathcal{K}_0, \end{aligned}$$

$$\mathcal{K}_0 := \{\eta \in W^{1,2}(0,T;H) \mid \eta(t) \in K(t), \forall t \in [0,T], \varphi(\eta) \in L^1(0,T)\}$$

<u>**Theorem 1.**</u> Assume $\{K(t)\} \in (K_W)$. Then:

(i) $\forall f \in L^2(0,T;H)$, $\forall u_0 \in K(0)$, $\exists_1 u \in C([0,T];H)$ such that

$$u(0) = u_0,\ u(t) \in K(t),\ \forall t \in [0,T],\ \varphi(u) \in L^1(0,T),$$

and (E_w) holds.

(ii) Let $u_i, i = 1,2$, be solutions of $(E_w; f_i, u_{i0})$. Then

$$\frac{1}{2}|u_1(t) - u_2(t)|_H^2 \le \frac{1}{2}|u_1(s) - u_2(s)|_H^2 + \int_s^t (f_1 - f_2, u_1 - u_2)_H d\tau,$$

for all $s, t \in [0,T], s \le t$.

Proof. (i) Take a sequence $\{K_n(t)\} \in (K_S)$ which approximates $\{K(t)\}$ as well as $u_{n0} \in K_n(0) \cap D(\varphi)$ with $u_{n0} \to u_0$ in H. Let u_n be the strong solution of (E) for $\{K_n(t)\}$, f, u_{n0}.

Since $(I + A_\varepsilon)u_m(\tau) \in K_n(\tau)$ for large n, m,

$$(u_n'(\tau), u_n(\tau) - (I + A_\varepsilon)u_m(\tau))_H + \varphi(u_n(\tau))$$
$$\leq (I + C\varepsilon)\varphi(u_m(\tau)) + (f(\tau), u_n(\tau) - (I + A_\varepsilon)u_m(\tau))_H$$
$$(u_m'(\tau), u_m(\tau) - (I + A_\varepsilon)u_n(\tau))_H + \varphi(u_m(\tau))$$
$$\leq (1 + C\varepsilon)\varphi(u_n(\tau)) + (f(\tau), u_m(\tau) - (I + A_\varepsilon)u_n(\tau))_H$$

Add these inequalities and use $A_\varepsilon = A_\varepsilon^*$ to get

$$\frac{1}{2}\frac{d}{d\tau}|u_n(\tau) - u_m(\tau)|_H^2 - \frac{d}{d\tau}(A_\varepsilon u_n(\tau), u_m(\tau))_H$$
$$\leq C'\varepsilon\{\varphi(u_n(\tau)) + \varphi(u_m(\tau)) + |u_n(\tau)|_H^2 + |u_m(\tau)|_H^2 + |f(\tau)|_H^2 + 1\}$$

Integrate this over $[0, t]$ to obtain an inequality of the form

$$\limsup_{n,m\to\infty} |u_n(t) - u_m(t)|_H^2 \leq C''\varepsilon \text{ uniformly in } t \in [0, T]$$

We show that $u_n \to u$ in $C([0, T]; H)$, and u is a weak solution.

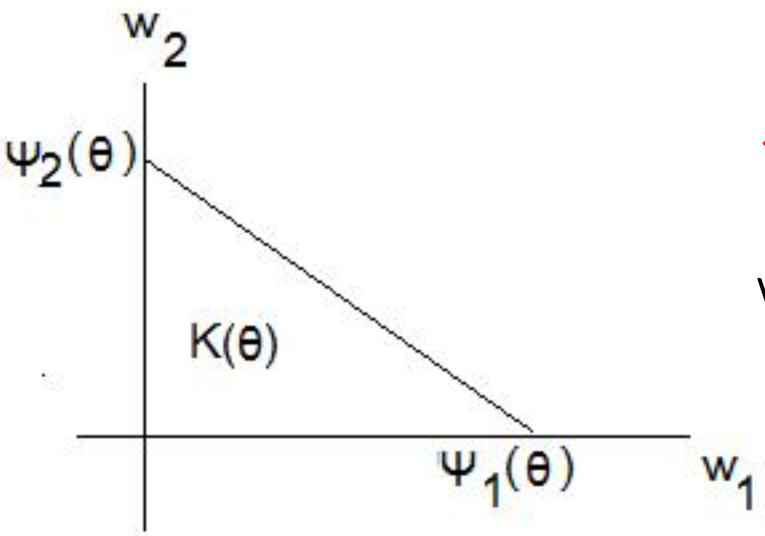
3. Activation of bacteria with environmental constraint

$$H := L^2(\Omega)^2,\ \varphi(w) := \frac{\nu}{2}\sum_{i=1}^{2} |\nabla w_i|^2_{L^2(\Omega)},\ \forall w = (w_1, w_2) \in H^1(\Omega)^2,$$

$\psi_i(\cdot), i = 1, 2$, C^2 on **R**, $0 < c_* \le \psi_i \le c_* < \infty$,

$$K(\theta) := \left\{ w = (w_1, w_2) \in \mathbf{R}^2 \,\middle|\, \begin{array}{l} \dfrac{w_1}{\psi_1(\theta)} + \dfrac{w_2}{\psi_2(\theta)} \le 1, \\ w_i \ge 0,\ i = 1, 2 \end{array} \right\}, \quad \forall \theta \in \mathbf{R},$$

$K(\theta(t)) = \{w \in H \mid w(x) \in K(\theta(x,t))$ a.e. $x \in \Omega\}$, $\forall \theta \in C(\overline{Q})$, $\forall t$.



$$\begin{cases} \theta_t - \kappa\Delta\theta + h(x,\theta,w) = f & \text{in } Q, \\ w_t - \nu\Delta w + \partial I_{K(\theta)}(w) \ni g(w) & \text{in } Q, \end{cases}$$

where

$h(x, \theta, w)$: C^2 on $\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^2$,
$h(\cdot, 0, w) = 0$ on $\partial\Omega$, $\forall w \in \mathbf{R}^2$;
$g(\cdot)$: Lip. cont. in $\mathbf{R}^2$.

Given $\theta_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $w_0 \in H^1(\Omega)^2$ with $w_0 \in K(\theta_0)$ a.e. on Ω, and $f \in L^2(0,T;L^2(\Omega))$, find a solution $\{\theta, w\}$ of problem (1)-(4):

$$\theta_t - \kappa\Delta\theta + h(x,\theta,w) = f(x,t) \quad \text{in } Q, \tag{1}$$

$$\theta = 0 \text{ on } \partial\Omega \times (0,T), \quad \theta(\cdot,0) = \theta_0 \text{ in } \Omega, \tag{2}$$

$$w(t) := (w_1(t), w_2(t)) \in K(\theta(t)),\ \forall t \in [0,T],\ w(0) = w_0, \tag{3}$$

$$\int_0^t \int_\Omega \eta' \cdot (w-\eta)dxd\tau + \sum_{i=1}^{2} \int_0^t \int_\Omega \nu \nabla w_i \cdot \nabla(w_i - \eta_i)dxd\tau \tag{4}$$

$$+\frac{1}{2}|w(t)-\eta(t)|_H^2 \le \int_0^t \int_\Omega g(w)\cdot(w-\eta)dxd\tau + \frac{1}{2}|w_0 - \eta(0)|_H^2,$$

$$\forall t \in [0.T],\ \forall \eta := (\eta_1, \eta_2) \in \mathcal{K}_0(\theta),$$

Theorem 2. Problem (1)-(4) has a solution $\{\theta, w\}$ such that

$\theta \in W^{1,2}(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \subset C(\overline{Q})$,

$w \in C([0,T];H) \cap L^2(0,T;H^1(\Omega)^2)$

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