Global attractors for Cahn-Hilliard-Navier-Stokes systems with nonlocal interactions

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Once upon a time...

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Gianni and I

What I share with Gianni

- Love for mountain
- Strong (and hot) coffee
- 9 joint papers
- 24 years of friendship (since I moved to Pavia in 1988 ...)
- 2 times in a committee for a researcher position

What I tried to learn from Gianni

- Clarity and rigor in Mathematical Analysis

What I didn’t learn from Gianni

- To be as good as he is in Mathematical Analysis

PLAN

- Cahn-Hilliard-Navier-Stokes systems (model H)
- CHNS systems with nonlocal interactions
- existence of a global weak solution
- dissipative estimate and energy identity
- attractors
- concluding remarks
- future work and open issues

Model H

- isothermal motion of an incompressible homogeneous binary mixture of immiscible fluids (model **H**: Siggia, Halperin & Hohenberg ’76, Halperin & Hohenberg ’77)
- *rigorous* derivation: Gurtin, Polignone & Viñals ’96, Jasnow & Viñals ’96, Morro ’10

$$
\boxed{\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = -\varepsilon\mu\nabla\varphi\\
&\nabla\cdot\mathbf{u} = 0\\
&\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \nabla\cdot(\kappa\nabla\mu)\\
&\mu = -\varepsilon\Delta\varphi + \varepsilon^{-1}F'(\varphi)
\end{aligned}}
$$

- $\mathbf{u}$ (averaged) fluid velocity, density $= 1$
- φ (relative) difference of concentrations of the two species
- viscosity $\nu > 0$, mobility $\kappa > 0$, interface thickness $\varepsilon > 0$
- μ chemical potential , F potential energy density

Regular and singular potentials: basic examples

- regular : the polynomial double-well potential

$$F(s) = (s^2 - 1)^2$$

  for all $s \in \mathbb{R}$

- singular : the logarithmic potential

$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$

  for all $s \in (-1, 1)$, $\theta < \theta_c$

Cahn-Hilliard-Navier-Stokes systems

- CHNS system is a diffuse interface model : the interface is treated as a finite (although thin: $O(\varepsilon)$) region where φ varies from one value (not necessarily of equilibrium) to the other
- taking the limit as $\varepsilon \searrow 0$ one gets a sharp interface model: the **Navier-Stokes-Mullins-Sekerka** system (Abels & Röger '09)
- the free bdry need not be explicitly tracked
- the (diffuse) interface is transported with the material
- **numerical approximation**

  Badalassi, Ceniceros & Banerjee '03, Liu & Shen '03; Kay, Styles & Welford '08; Kim, Kang & Lowengrub '04; Shen & Yang '10, Boyer et al. '11, . . .

CHNS systems: theoretical results

- well-posedness, stability of equilibria: V.N. Starovoitov '97 [$\Omega = \mathbb{R}^2$, smooth F, spatially decaying sols]
- existence and uniqueness, local stability of constant solutions: F. Boyer '99 [degenerate $\kappa = \kappa(\varphi)$, singular or regular F]
- existence and uniqueness: H. Abels '09 [constant κ, singular F]
- unmatched densities: F. Boyer '01 [$\exists$ local strong sols], H. Abels '09 [$\exists$ weak sols]
- compressible case: H. Abels & E. Feireisl '08 [$\exists$ weak sols]

CHNS systems: longtime behavior

convergence to equilibrium of single trajectories

- H. Abels '09 [singular F]
- M.G. & C.G. Gal '09 [2D, regular F, conv. rate estimates]
- L. Zhao, H. Wu & H. Huang '09 [regular F, nonconstant κ, conv. rate estimates]

attractors

- H. Abels '09 [singular F, global attractor à la Foias & Cheskidov]
- M.G. & C.G. Gal '10 [3D, smooth F, nonconstant κ, time-dependent ext. force, trajectory attractor]
- M.G. & C.G. Gal '09 and '11 [2D, regular F, ext. force, smooth global attractor, exp. attractors, dim. bounds]

Further comments

- known results for NS can be extended to CHNS
- CHNS longterm dynamics is more complex (as expected)
- similar considerations hold for the Ladyzhenskaya variant where

$$\mathbf{T}_q(D\mathbf{u}) = \nu D\mathbf{u} + \delta |D\mathbf{u}|^{q-2} D\mathbf{u}$$

  with $\nu, \delta \geq 0$ and $q > 2$ is large enough (G. & Pražák '11)

Remark

Standard CH eq can be derived through a phenomenological argument, **however** *a nonlocal CH eq can be rigorously justified as a macroscopic limit of microscopic of suitable phase segregation models (Giacomin & Lebowitz '97, '98)*

Free energies: local vs. nonlocal

μ is the first variation of the (local) free energy

$$E(\varphi) = \int_\Omega \left(\frac{\xi}{2} |\nabla \varphi(x)|^2 + \eta F(\varphi(x)) \right) dx$$

but the hydrodynamic limit "gives" the nonlocal free energy

$$\mathcal{E}(\varphi) = \frac{1}{4} \int_\Omega \int_\Omega K(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \eta \int_\Omega F(\varphi(x)) dx$$

where $K : \mathbb{R}^N \to \mathbb{R}$ s.t. $K(x) = K(-x)$

examples

$$K(x) = e^{-\sigma |x|^2}, \quad K(x) = \sigma |x|^{-1} \quad \sigma > 0$$

Nonlocal chemical potential

The chemical potential given by the nonlocal free energy is

$$\boxed{\mu = a\varphi - K * \varphi + \eta F'(\varphi)}$$

where

$$(K * \varphi)(x) := \int_\Omega K(x-y)\varphi(y)dy, \quad a(x) := \int_\Omega K(x-y)dy$$

Remark

The term

$$\int_\Omega \frac{\xi}{2}|\nabla\varphi(x)|^2 dx$$

can be viewed as the first approximation of

$$\int_\Omega \int_\Omega K(x-y)(\varphi(x)-\varphi(y))^2 dxdy$$

Nonlocal interactions: some math literature

- nonlocal Cahn-Hilliard eqs : Giacomin & Lebowitz '97 and '98; Chen & Fife '00; Gajewski '02; Gajewski & Zacharias '03; Han '04; Bates & Han '05; Colli, Krejčí; Rocca & Sprekels '07; Londen & Petzeltová '11; Gal & G. '12
- Binary fluids with long range segregating interactions : Bastea et al. '00
- Navier-Stokes-Korteweg systems (liquid-vapour phase transitions): Rohde '05, Haspot '10
- nonlocal Allen-Cahn eqs and phase-field systems : Bates et al.; Sprekels et al.; Feireisl, Issard-Roch & Petzeltová '04; G. & Schimperna '11

Nonlocal CHNS systems

$\Omega \subset \mathbb{R}^N$ bdd ($N = 2, 3$)

$$\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{g}(t) \\
&\nabla \cdot \mathbf{u} = 0 \\
&\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta \mu \\
&\mu = -K * \varphi + a\varphi + F'(\varphi) \\
&\\
&\text{in } \Omega \times (0, +\infty)
\end{aligned}$$

subject to

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial \mu}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega \times (0, +\infty)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega$$

Basic assumptions

interaction kernel

- $K \in W^{1,1}(\mathbb{R}^N)$ s.t. $a(x) = \int_\Omega K(x-y)dy \geq 0$

potential

- $F = F_1 + F_2$, $F_1 \in C^4(-1,1)$, $F_2 \in C^2([-1,1])$
- $\lim_{s\to\pm 1} F_1'(s) = \pm\infty$
- $F_1^{(2)}(s) \geq 0$ and $F_1^{(4)}(s) \geq c_1 > 0$ near $s = \pm 1$
- $F_1^{(3)}(s) \geq 0 (\leq 0)$ near $s = 1$ $(s = -1)$
- $F_1^{(4)}$ non-decreasing (increasing) near $s = 1$ $(s = -1)$
- $\exists\, \alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > -\min_{[-1,1]} F_2^{(2)}$ s.t.

$$F_1^{(2)}(s) \geq \alpha \qquad \forall s \in (-1,1), \quad a(x) \geq \beta \qquad \text{a.e. } x \in \Omega$$

the logarithmic potential fulfills the assumptions above

Notion of weak solution 1

- $H = L^2(\Omega)$, $V = H^1(\Omega)$, $Q = \Omega \times (0, T)$, $T > 0$
- $(\mathbf{u}_0, \varphi_0) \in H_{div} \times H$ s.t. $F(\varphi_0) \in L^1(\Omega)$
- $\mathbf{g} \in L^2(0, T; V'_{div})$

$(\mathbf{u}, \varphi)$ is a weak sol if

$$\mathbf{u} \in L^\infty(0, T; H_{div}) \cap L^2(0, T; V_{div})$$

$$\mathbf{u}_t \in L^{4/3}(0, T; V'_{div}), \quad N = 3, \quad \mathbf{u}_t \in L^2(0, T; V'_{div}), \quad N = 2$$

$$\varphi \in L^\infty(0, T; L^p) \cap L^2(0, T; V) \cap L^\infty(Q), \quad p \in [1, \infty)$$

$$|\varphi| < 1 \text{ a.e. in } Q$$

$$\varphi_t \in L^{4/3}(0, T; V'), \quad N = 3, \quad \varphi_t \in L^2(0, T; V'), \quad N = 2$$

$$\mu \in L^2(0, T; V)$$

Notion of weak solution 2

and $\forall \psi \in V$, $\forall v \in V_{div}$ we have

$$\langle \varphi_t, \psi \rangle + (\nabla \rho, \nabla \psi) = ((\mathbf{u}, \nabla \psi), \varphi) + ((\nabla K * \varphi), \nabla \psi)$$

$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla \mathbf{u}, \nabla \mathbf{u}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -((\mathbf{v} \cdot \nabla \mu), \varphi) + \langle \mathbf{g}, \mathbf{v} \rangle$$

for a.a. $t \in (0, T)$ with

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0, \quad \bar{\varphi}(t) = \bar{\varphi}_0, \quad \forall t \in [0, T]$$

where

$$\rho(x, \varphi) := a(x)\varphi + F'(\varphi)$$

Existence of a global weak solution

Theorem (Frigeri & G. '12)

$\forall\, T > 0\ \exists$ a weak solution $(\mathbf{u}, \varphi)$ on $(0, T)$ which satisfies the energy inequality for all $t \geq s$ and a.a. $s \geq 0$ (including $s = 0$)

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) := \frac{1}{2}\|\mathbf{u}(t)\|^2 + \mathcal{E}(\varphi(t))$$

$$+ \int_s^t (\nu\|\nabla\mathbf{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2)d\tau$$

$$\leq \mathcal{E}(\mathbf{u}(s), \varphi(s)) + \int_s^t \langle\mathbf{g}(\tau), \mathbf{u}(\tau)\rangle d\tau$$

Remark

The proof is based on a previous global existence result on regular potentials (Colli, Frigeri & G. '12)

$N = 2$: energy identity

Corollary

The weak solution $(\mathbf{u}, \varphi)$ *satisfies the energy identity*

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}, \varphi) + \nu\|\nabla \mathbf{u}\|^2 + \|\nabla \mu\|^2 = \langle \mathbf{g}, \mathbf{u}\rangle$$

Remark

Thanks to the energy identity and to the strong continuity

$$\mathbf{u} \in C([0, +\infty); H_{div}), \quad \varphi \in C([0, +\infty); H)$$

we can use the generalized semiflow approach devised by J.M. Ball to establish the existence of a global attractor in the autonomous case

Generalized semiflows (J.M. Ball '97)

Definition

Let $(\mathcal{X}, d)$ be metric space, a family of maps $z : [0, +\infty) \to \mathcal{X}$ is a generalized semiflow $\mathcal{G}$ if

- $\forall z_0 \in \mathcal{X}$, $\exists\, z \in \mathcal{G}$ s.t. $z(0) = z_0$
- translates of elements of $\mathcal{G}$ still belong to $\mathcal{G}$
- concatenation property holds
- upper semicontinuity w.r.t. initial data

We set

$$T(t)\Theta = \{z(t) \,:\, z \in \mathcal{G},\ z(0) \in \Theta\}, \quad \forall\, \Theta \subset \mathcal{X}$$

Definition

$\mathcal{A} \subset \mathcal{X}$ is the global attractor for $\mathcal{G}$ if it is compact, fully invariant and attracts $T(t)B$ for any bdd set $B \subset \mathcal{X}$

$N = 2$: the generalized semiflow

- $\mathbf{g} \in V'_{div}$
- phase space ($m \in [0, 1)$ given)

$$\mathcal{X}_m = H_{div} \times \mathcal{Y}_m$$

where $\mathcal{Y}_m = \{\varphi \in H \ : \ F(\varphi) \in L^1(\Omega), \ |\bar{\varphi}| \leq m\}$

- metric ($z = (\mathbf{u}, \varphi)$)

$$d(z_1, z_2) = \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\varphi_1 - \varphi_2\| + \left| \int_\Omega (F(\varphi_1) - F(\varphi_2)) \right|^{1/2}$$

$\mathcal{G} = \{$ all weak sols corresponding to all $(\mathbf{u}_0, \varphi_0) \in \mathcal{X}_m\}$

$N = 2$: existence of the global attractor

Theorem (Frigeri & G. '12)

$\mathcal{G}$ is a generalized semiflow on $(\mathcal{X}_m, d)$ which has the global attractor $\mathcal{A}_m$

Remark

*The convective nonlocal Cahn-Hilliard equation (i.e. **u** is given and smooth enough) is s.t.*

- *the energy identity still holds if $N = 3$*
- *the (weak) solution is unique*

thus we have a flow $S(t)$ on $\mathcal{Y}_m$ which possesses the connected global attractor A_m

$N = 3$: trajectory attractors

The energy inequality and a suitable generalized Gronwall's lemma are the basic tools to prove the existence of the trajectory attractor (cf. Foias & Temam '87, Sell '96, Chepyzhov & Vishik '97)

- regular potentials: Frigeri & G., '11
- singular potentials: Frigeri & G., '12

Remark

$\mathbf{g} = \mathbf{g}(t)$ *and the trajectory attractor is strong if* $N = 2$

Remark

ALL the results still hold if the viscosity depends smoothly on φ

Concluding remarks

- the regularity $L^\infty(L^p)$ of φ is lower than the one in the local model ($\varphi \in L^\infty(H^1)$)
- the Korteweg force $\mu\nabla\phi$ is as nasty as the convective one
- $\exists$ (and !) of a strong sol in 2D is nontrivial : it requires $K \in W^{2,1}$ and regular potentials [Frigeri, G. & Krejčí, in preparation]
- the above result also entails that $\mathcal{A}_m$ is bdd in $V_{div} \times H^2$

Work ahead

- $N = 2$ and $\mathbf{g} \equiv \mathbf{0}$: convergence of a weak sol to a single equilibrium
- log potential and degenerate mobility ($\kappa(\varphi) = 1 - \varphi^2$) [Frigeri, G. & Rocca, in progress]
- 2D: finite fractal dimension of $\mathcal{A}_m$ and $\exists$ exp. attr.
- Cahn-Hilliard-Hele-Shaw systems (Wang et al. '10, '11) accounting for nonlocal interactions

Open issues

- 2D: uniqueness of weak sols
- 2D: *complete* regularity theory
- nonsmooth interaction kernels (e.g. fractional Laplacian)
- unmatched densities
- sharp interface limits
- numerical simulations and comparison with standard models

THANK YOU GIANNI!