# A variational approach to a Cahn-Hilliard model in a domain with non-permeable walls

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joint work with
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# Our model

- The Cahn-Hilliard model : phase-field $u$ (pure-phases $u = \pm 1$)

  $\Omega \subset \mathbb{R}^3$ bounded with smooth $\partial\Omega = \Gamma$, $\quad \lambda \geq 0$

$$\begin{cases} \partial_t u - \Delta\mu = 0, & \text{in} \quad \Omega \\ \mu = -\Delta u + f(u) - \lambda u, & \text{in} \quad \Omega \end{cases}$$

Physically relevant instance

$$f(s) - \lambda s = -\theta_c s + \frac{\theta}{2} \ln \frac{1+s}{1-s}, \quad s \in (-1,1), \quad \theta_c > \theta > 0$$

- $f$ singular at $\pm 1$

$$f \in C^2(-1,1) \qquad \lim_{s \to \pm 1} f(s) = \pm\infty \qquad \lim_{s \to \pm 1} f'(s) = +\infty$$

- $f$ monotone increasing in $(-1,1)$ $\qquad f'(s) \geq 0$

Further assumptions : $\quad f(0) = 0 \quad$ and $\quad f''(s) \begin{cases} \geq 0, & s \geq 0 \\ \leq 0, & s \leq 0 \end{cases}$

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- The two-phase system is confined in a non-permeable vessel
  $\rightarrow$ dynamic boundary conditions

  ◀ Some mass on the boundary (add surface free energy)

  ◀ Comply with conservation of total mass.

  If $U(t) = (u(t), u(t)|_\Gamma)$ starts at $U(0) = (u(0), u(0)|_\Gamma)$

  $$\int_\Omega u(t)dx + \int_\Gamma u|_\Gamma(t)d\Sigma = \int_\Omega u(0)dx + \int_\Gamma u(0)|_\Gamma d\Sigma$$

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(★) $(u_0, u_0|_\Gamma) \in (-1,1) \times (-1,1) \Rightarrow (u(t), u(t)|_\Gamma) \in (-1,1) \times (-1,1)$

Without (★) possible $\nexists$ of classical solutions [Miranville-Zelik 2010]

Strong singularities of $u$ close to the boundary may be produced

The jumps in the normal derivatives close to the boundary prevent the existence of solution in the sense of distribution

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◂ Most general assumptions on $f$ and $g$

◂ Weak formulation of the problem (Duality techniques)

◂ Existence of global attractor only if $\exists p_0 \in (0,2): \quad |f'(s)| \leq c(1 + |f(s)|^{p_0})$

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# Literature

- Cahn-Hilliard

Elliott-Zheng, Nicolaenko-Scheurer, Nicolaenko-Scheurer-Temam, Novick-Cohen, Brochet-Hilhorst-Novick Cohen, Brochet-Hilhorst-Chen Alt-Pawlow, Kenmochi-Niezgodka-Pawlow, Rybka-Hoffmann, Colli-Gilardi-Grasselli-Schimperna...

- Singular Cahn-Hilliard

Elliott-Luckhaus, Elliott-Garcke, Debussche-Dettori, Abels-Wielke, Li-Zhong, Miranville-Zelik,...

- Cahn-Hilliard with dynamic boundary conditions

Racke-Zheng, Chill-Fašangová-Prüss, Prüss-Racke-Zheng, Miranville-Zelik...

- Singular Cahn-Hilliard with dynamic boundary conditions

Gilardi-Miranville-Schimperna, Miranville-Zelik, Ruiz Goldstein-Miranville-Schimperna

Review on Singular Cahn Hilliard with different boundary conditions : Cherfils-Miranville-Zelik

# Our results

1. Variational solutions

   Approximate singular $(P)$ by regular $(P_N)$ ($\leftarrow$ replace $f$ with $f_N$)

   $\exists! U_N$ solution to $(P_N)$, Lipschitz continuous dependence on the initial data at any fixed time, a priori estimates, smoothing, dissipativity

   uniformly in $N$

   $\exists U_{N_k} \to U$ but $U$ is NOT classical solution what solution is $U$ ?

   The **monotonicity** of $f_N \uparrow$ and $f \uparrow$ allows to associate $(P_N)$ with $(V_N)$ and $(P)$ with $(V)$

   Since $U_N$ solves $(V_N) \Rightarrow U$ is (the variational) solution to $(V)$

2. Relation between variational and classical solutions

3. Asymptotic analysis for variational solutions

   For any fixed total mass $I \in (-1, 1)$ $\exists\, \mathcal{A}_I$ regular global attractor and $\exists\, \mathcal{E}_I$ exponential attractor $\Rightarrow$ Bound on the fractal dimension of $\mathcal{A}_I$

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# Abstract problem

- Let $U = (u, u|_\Gamma)$ and $\mathbf{M} = (\mu, \mu|_\Gamma)$

$$m(U) = \frac{1}{|\Omega| + |\Gamma|}\left(\int_\Omega u dx + \int_\Gamma u|_\Gamma d\Sigma\right) \text{ and } \langle U\rangle = (m(U), m(U))$$

$$\begin{cases} \partial_t U + \mathbf{A}\mathbf{M} = 0 \\ \mathbf{M} = \mathbf{A}U + \mathbf{f}(U) \\ U(0) = U_0 \end{cases} \qquad \mathbf{f}(U) = \begin{cases} f(u) - \lambda u & \text{in } \Omega \\ g(\psi) & \text{on } \Gamma \end{cases}$$

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# Exploit monotonicity

$$\begin{cases} \mathbf{A}^{-1}\partial_t U + \mathbf{M} - \langle \mathbf{M} \rangle = 0 & \times U - V \\ \mathbf{M} = \mathbf{A}U + \mathbf{f}(U) & \times U - V \qquad V: \quad m(V) = m(U) \\ U(0) = U_0 \end{cases}$$

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$$\exists \sigma > 0 \; \exists L > 0: \; \langle \mathbf{A}U, U\rangle + \langle \mathbf{f}(U), U\rangle + L\|\mathbf{A}^{-1/2}U\|^2$$
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$$+ \underbrace{(f(u), u)_\Omega + (g(\psi) + \sigma\psi, \psi)_\Gamma}_{\exists \sigma > 0: \text{ monotone increasing}} \quad \text{for } U: \; m(U) = 0$$

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# The variational inequality

Since

$$B(U,U-V) \geq B(V,U-V)\,\forall U,V \quad \text{such that} \quad m(U)=m(V)$$

and

$$(f(u),u-v)_\Omega + (g(\psi)+\sigma\psi,\psi-w)_\Gamma \geq$$
$$(f(v),u-v)_\Omega + (g(w)+\sigma w,\psi-w)_\Gamma \qquad U=(u,\psi)\ V=(v,w)$$

$\Rightarrow$

$$(V)\begin{cases}\langle \mathbf{A}^{-1}\partial_t U, U-V\rangle + B(V,U-V) + \langle f(v),u-v\rangle_\Omega \\ \quad +(g(w)+\sigma w,\psi-w)_\Gamma \leq L\langle U-\langle U\rangle, \mathbf{A}^{-1}(U-V)\rangle \\ \text{for a.a. } t>0\ \ \forall V=(v,v|_\Gamma)\in\mathcal{H}^1 \quad \text{such that} \\ m(V)=m(U_0) \qquad \text{and} \qquad f(v)\ \in L^1(\Omega)\end{cases}$$

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# Our notion of a variational solution

$\forall U_0 \quad U(t) = (u(t), \psi(t))$ is a variational solution if $U(0) = U_0$ and

- $u(t)_{|\Gamma} = \psi(t)$ and $m(U(t)) = m(U_0)$ for a.a. $t > 0$
- $-1 < u(x,t) < 1,$ for almost all $(x,t) \in \Omega \times [0, \infty)$
- $U \in C([0, +\infty), \mathcal{H}^{1^*}) \cap L^2([0,T], \mathcal{H}^1), \ \forall T > 0,$
- $f(u) \in L^1(\Omega \times [0,T]),$ for any $T > 0$
- $\partial_t U \in L^2([\tau, T], \mathcal{H}^{1^*}) : \quad \langle \partial_t U, 1 \rangle_{\mathcal{H}^{1^*}, \mathcal{H}^1} = 0 \quad \forall \tau \in (0, T], \quad \forall T > 0,$
- $U(t)$satisfies $(V)$ :

$$\langle \mathbf{A}^{-1} \partial_t U, U - V \rangle + B(V, U - V) + \langle f(v), u - v \rangle_\Omega$$
$$+ \langle g(v|_\Gamma) + \sigma v|_\Gamma, \psi - v|_\Gamma \rangle_\Gamma \leq L \langle U - \langle U \rangle, \mathbf{A}^{-1}(U - V) \rangle$$

for a.a. $t > 0, \ \forall V = (v, v|_\Gamma) \in \mathcal{H}^1$ such that

$m(V) = m(U_0)$ and $f(v) \in L^1(\Omega)$

# Relation between variational and classical solutions

- $\exists U_{N_k} \to U$ but $U(t) = (u(t), \psi(t))$ is NOT necessarily a classical solution, since $u$ may reach $\pm 1$ on regions of $\Gamma \times \mathbb{R}^+$ with positive measure. The normal derivative may have discontinuities
- $u \in L^\infty((\tau, T]; W^{2,1}(\Omega))$ for any $0 < \tau < T$
  $\Rightarrow \exists [\partial_n u]_{int} := \partial_n u_{|\Gamma} \in L^\infty([\tau, T], L^1(\Gamma))$
- $U_{N_k} \to U$, the a priori estimates and the dynamic boundary condition

$$\Rightarrow \qquad \exists [\partial_n u]_{ext} := \lim_{N_k \to +\infty} \partial_n u_{N_k|\Gamma} \in L^\infty([\tau, T], L^2(\Gamma))$$

$$\begin{cases} \partial_t u - \Delta \mu = 0 \quad \text{in } L^2_{loc}(\Omega \times (\tau, T)) \\ \mu = -\Delta u + f(u) - \lambda u \quad \text{in } L^2_{loc}(\Omega \times (\tau, T)) \\ u(t)|_\Gamma = \psi(t), \quad t > 0 \\ \partial_t \psi - \Delta_\Gamma \mu|_\Gamma = -\partial_{\boldsymbol{n}} \mu, \quad \text{in } L^2_{loc}(\Gamma \times (\tau, T)) \\ \mu|_\Gamma = -\Delta_\Gamma \psi + g(\psi) + [\partial_n u]_{ext} \quad \text{on } \Gamma, \quad T > \tau > 0 \end{cases}$$

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$$\begin{cases} \partial_t u - \Delta\mu = 0 \quad \text{in } L^2_{loc}(\Omega \times (\tau, T)) \\ \mu = -\Delta u + f(u) - \lambda u \quad \text{in } L^2_{loc}(\Omega \times (\tau, T)) \\ u(t)|_\Gamma = \psi(t), \quad t > 0 \\ \partial_t \psi - \Delta_\Gamma \mu|_\Gamma = -\partial_{\boldsymbol{n}} \mu, \quad \text{in } L^2_{loc}(\Gamma \times (\tau, T)) \\ \mu|_\Gamma = -\Delta_\Gamma \psi + g(\psi) + [\partial_n u]_{ext} \quad \text{on } \Gamma, \quad T > \tau > 0 \end{cases}$$

- Unfortunately $[\partial_n u]_{int}$ is not necessarily equal to $[\partial_n u]_{ext}$.

# Sufficient condition for $U$ to be classical

$$\text{If} \quad (\bigstar) \quad |u(x,t)| < 1 \quad \text{a.e. in} \quad \Gamma \times \mathbb{R}^+ \quad \Rightarrow \quad [\partial_n u]_{int} = [\partial_n u]_{ext}$$

$$\Rightarrow \quad U \quad \text{classical solution}$$

$$\text{If} \quad (\blacksquare) \quad \lim_{s \to \pm 1} F(s) = +\infty \quad (F' = f) \quad \Rightarrow \quad (\bigstar) \quad \text{holds true}$$

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# The semigroup

Let $U = (u,\, \psi)$

$$\Phi = \{U \in L^\infty(\Omega) \times L^\infty(\Gamma) :\ \|u\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Gamma)} \leq 1,\, m(U) \in (-1,1)\}$$

$$\Phi_I = \{U \in \Phi : \quad m(U) = I\}, \qquad I \in (-1,1)$$

$$\mathcal{H}^1 = H^1(\Omega) \times H^1(\Gamma) \subset \mathcal{L}^2 = L^2(\Omega) \times L^2(\Gamma) \subset (\mathcal{H}^1)^*$$

$S(t) : (\Phi_I, \mathcal{H}^{1^*}) \to (\Phi_I, \mathcal{H}^{1^*})$ closed semigroup

$U_0 \mapsto U(t)$ solution to the variational problem $(V)$

$(S(t), \Phi_I)$ admits a compact absorbing set

$\Rightarrow \quad \exists \mathcal{A}_I \quad (\Phi_I, \mathcal{H}^{1^*})-$global attractor

$\mathcal{A}_I$ bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma)$ has finite fractal dimension

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# Exponential attractors

- [Eden, Foias, Nicolaenko, Temam 1994]

We are dealing with variational solutions

How can we prove the existence of an exponential attractor ?

Main Idea [Efendiev-Zelik 2008], [Miranville-Zelik 2010] :

- close to $\pm 1$ $f'$ goes to $+\infty$

$\Rightarrow f'(1-s)$ and $f'(-1+s)$ as large as we want if $s > 0$ is small enough

- far from $\pm 1$ standard parabolic smoothing property

To exploit this idea, we need a local procedure.

- $\exists \mathbb{B}_0$ compact, absorbing and positively invariant, where, in particular, uniform bounds for the solutions hold true and $u|_\Gamma = \psi$.

- $\{B_{\mathcal{H}^{1*}}(U_0, \rho)\}_{z_0 \in \mathbb{B}_0}$ are an open covering of $\mathbb{B}_0$, for any $0 < \rho \leq \rho_0 \ll 1$

$\Rightarrow$ we work on a finite number of balls

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# Main Idea

- $\mathbb{B}_0$ is such that $\|u\|_{C^\alpha(\Omega\times[0,T])} \leq R, \forall T \geq 0$
- This and Lipschitz continuous dependence, interpolation $\Rightarrow$ $\forall U_0 \in \mathbb{B}_0, \forall \delta \in (0,1), \exists T(\delta) > 0, \exists \rho_0(\delta) > 0$ such that $\forall \rho \in (0,\rho_0), \forall U(0) \in B_{\mathcal{H}^{1*}}(U_0,\rho), \forall t \in [0,T]\ S(t)U(0) = (u(t),\psi(t))$ satisfies

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# Construction of an exponential attractor

## Theorem (Málek-Prážak 2002, Efendiev-Zelik 2008)

*Let $X$, $\mathbb{H}_1, \mathbb{H}$ be Banach spaces with $\mathbb{H}_1 \Subset \mathbb{H}$, $\mathbb{B}_0 \Subset X$ such that $\mathbb{S}\mathbb{B}_0 \subset \mathbb{B}_0$ and $\forall U_0 \in \mathbb{B}_0$ $\forall \rho \in (0, \rho_0)$ $\exists \mathbb{K}_{U_0} : B_X(U_0, \rho) \to \mathbb{H}_1$ such that*

- $(\bullet)$ $\|\mathbb{K}_{U_0}(U_1) - \mathbb{K}_{U_0}(U_2)\|_{\mathbb{H}_1} \le c\|U_1 - U_2\|_X, \quad U_1, U_2 \in B_X(U_0, \rho)$
- $(\bullet)$ $\exists \gamma \in (0, 1)$ *such that, for any* $U_1, U_2 \in B_X(U_0, \rho)$

$$\|\mathbb{S}U_1 - \mathbb{S}U_2\|_X \le \gamma\|U_1 - U_2\|_X + c\|\mathbb{K}_{U_0}(U_1) - \mathbb{K}_{U_0}(U_2)\|_{\mathbb{H}}$$

*$\Rightarrow \exists \mathcal{E}^d$ discrete exponential attractor with basin $\mathbb{B}_0$ endowed with the $X$-topology.*

$X = \mathcal{H}^{1^*}$, $\mathbb{S} = S(T(\delta))$ where $\delta$ small enough, $\rho \in (0, \rho_0(\delta))$ and $T = T(\delta)$

$$\mathbb{K}_{U_0} : B_{\mathcal{H}^{1*}}(U_0, \rho) \to \mathcal{L}^2 \quad \text{is} \quad \mathbb{K}_{U_0}U = (\theta u(\cdot), 0)$$

$$\mathbb{H}_1 = L^2([0, T], \mathcal{H}^1) \cap H^1([0, T], \mathcal{H}^{3^*}) \Subset \mathbb{H} = L^2([0, T], \mathcal{L}^2)$$

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**Theorem (Málek-Prážak 2002, Efendiev-Zelik 2008)**

*Let* $X$, $\mathbb{H}_1, \mathbb{H}$ *be Banach spaces with* $\mathbb{H}_1 \Subset \mathbb{H}$, $\mathbb{B}_0 \Subset X$ *such that* $\mathbb{S}\mathbb{B}_0 \subset \mathbb{B}_0$ *and* $\forall U_0 \in \mathbb{B}_0 \; \forall \rho \in (0, \rho_0) \; \exists \mathbb{K}_{U_0} : B_X(U_0, \rho) \to \mathbb{H}_1$ *such that*

$$(\bullet) \quad \|\mathbb{K}_{U_0}(U_1) - \mathbb{K}_{U_0}(U_2)\|_{\mathbb{H}_1} \leq c\|U_1 - U_2\|_X, \quad U_1, U_2 \in B_X(U_0, \rho)$$

$$(\bullet) \quad \exists \gamma \in (0,1) \quad \text{such that, for any} \quad U_1, U_2 \in B_X(U_0, \rho)$$

$$\|\mathbb{S}U_1 - \mathbb{S}U_2\|_X \leq \gamma \|U_1 - U_2\|_X + c\|\mathbb{K}_{U_0}(U_1) - \mathbb{K}_{U_0}(U_2)\|_{\mathbb{H}}$$

$\Rightarrow \exists \mathcal{E}^d$ *discrete exponential attractor with basin* $\mathbb{B}_0$ *endowed with the* $X$*-topology.*

$X = \mathcal{H}^{1^*}$, $\mathbb{S} = S(T(\delta))$ where $\delta$ small enough, $\rho \in (0, \rho_0(\delta))$ and $T = T(\delta)$

$$\mathbb{K}_{U_0} : B_{\mathcal{H}^{1*}}(U_0, \rho) \to \mathcal{L}^2 \quad \text{is} \quad \mathbb{K}_{U_0}U = (\theta u(\cdot), 0)$$

$$\mathbb{H}_1 = L^2([0,T], \mathcal{H}^1) \cap H^1([0,T], \mathcal{H}^{3^*}) \Subset \mathbb{H} = L^2([0,T], \mathcal{L}^2)$$

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## Theorem

*For any* $U_0 \in \mathbb{B}_0$ $\exists \delta \in (0,1),\ T(\delta) > 0,\ \rho_0(\delta) > 0 :\ \forall \rho \in (0, \rho_0)$

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$$\|S(T)U_1 - S(T)U_2\|^2_{\mathcal{H}^{1*}} \leq e^{-\beta T}\|U_1 - U_2\|^2_{\mathcal{H}^{1*}} + c\|\mathbb{K}_{U_0}(U_1) - \mathbb{K}_{U_0}(U_2)\|^2_{\mathbb{H}}$$

*for some* $\beta > 0$ *and for any* $U_1, U_2 \in B_{\mathcal{H}^{1*}}(U_0, \rho)$, *where*

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As usual, from $\mathcal{E}_I^d$ we obtain the continuous exponential attractor $\mathcal{E}_I$

$$\mathcal{E}_I \subset C^\alpha(\Omega) \times C^\alpha(\Gamma)$$

**Transitivity of exponential attraction [Fabrie, Galusinski, Miranville, Zelik 2004]**

1. $\mathbb{B}_0$ exponentially attracts the bounded sets in $\Phi_I$
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# An ill-posed CH equation

Assume $\quad f \in C^1(-1,1) \quad \lim_{r\to\pm 1} f(r) = \pm\infty \quad \lim_{r\to\pm 1} f'(r) = \infty$

$$f(0) = 0 \quad f' \geq 0 \qquad g \in C^2[-1,1] \qquad \lambda \geq 0$$

If $f$ is odd with $F(u) = \int_0^u f(s)ds$ such that $F(1) < \infty$ and $g = -K$ with large enough $K$

$$\Rightarrow \quad \nexists \text{classical solution to } \begin{cases} -y''(x) + f(y) = 0, & x \in (-1,1) \ (\lambda = 0) \\ y'(\pm 1) = K \end{cases}$$

- If $K$ is not too large $\exists y_K$ odd, regular solution separated from $\pm 1$
- If $K$ is large enough $\Rightarrow y_K \equiv y_+$ singular solution to

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