Asymptotic analysis of some isothermal models for nematic liquid crystal flows

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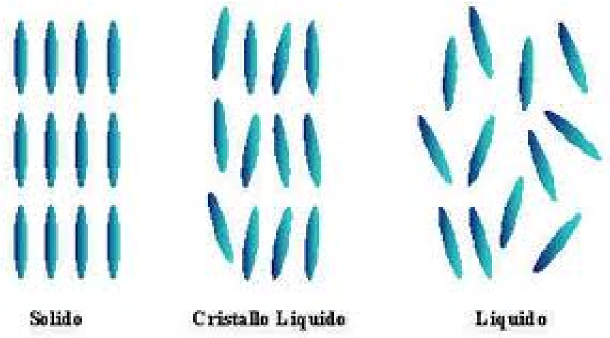
Nematic liquid crystals

- Materials consisting of molecules with elongated shape
- Fluid has anisotropic properties over a limited temperature range: molecules lined up in a specific direction (uniaxial), but no positional order
- the director **d** is the average, over a small volume element, of unit vectors representing the long axis of each molecule



A simplified model for nematic liquid crystals (F.-H. Lin & C. Liu '95, '96)

Semplification of the original Ericksen-Leslie system with termal and e.m. effects neglected

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = -\lambda \mathrm{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) + \mathbf{h}$$
$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \eta(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$$
$$\mathrm{div}\mathbf{u} = 0$$

- $W(\mathbf{d})$ double-well regular potential, e.g. $W(\mathbf{d}) = (|\mathbf{d}|^2 - 1)^2$
- W relaxation of the constraint $|\mathbf{d}| = 1$

Reasonable b.c. are

- no-slip for **u**, Dirichlet for **d** (strong anchoring)
- free-slip for **u**, hom. Neumann for **d**

A corrected model (H. Sun & C. Liu, '09)

In $\Omega \times (0, \infty)$, $\Omega \subset \mathbb{R}^3$

$$\mathbf{u}_t + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{div}(\nu(\nabla \mathbf{u} + \nabla^T \mathbf{u})) - \mathrm{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d})$$
$$- \mathrm{div}(\alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} - (1-\alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) + \mathbf{h}$$
$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \alpha \mathbf{d} \cdot \nabla \mathbf{u} + (1-\alpha)\mathbf{d} \cdot \nabla^T \mathbf{u} = (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$$
$$\mathrm{div}(\mathbf{u}) = 0$$

- $\alpha \in [0, 1]$ related to the shape of liquid crystal molecules
  $\alpha = 1$ rod-like, $\alpha = 1/2$ spherical, $\alpha = 0$ disc-like
- b.c. considered: periodic, or no-slip for **u**+hom. Neumann or non hom. Dirichlet for **d**

$$\partial_{\mathbf{n}} \mathbf{d} = 0 \qquad \text{or} \qquad \mathbf{d} = \mathbf{g} \qquad \text{on } \partial\Omega$$

The Sun-Liu isothermal model

Remarks

- Lin-Liu model neglects kinematic transport: liquid crystal molecules assumed small
- Strechting term $\mathbf{d} \cdot \nabla \mathbf{u}$ included and a new component $\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$ added in the stress tensor to ensure energy balance
- Mathematical difficulty: lack of maximum principle and of $L^\infty-$estimate for $\mathbf{d}$

Mathematical results

- Well-posedness
  - H. Sun & C. Liu 09': ∃ strong sols, periodic b.c., in 2D (3D, ν large)
  - H. Wu, X. Xu & C. Liu '10: uniqueness and continuous dependence on in. data of strong sols, periodic b.c., in 2D
  - C. Cavaterra & E. Rocca '12: ∃ weak sols in 3D, Neumann or non-hom. Dirichlet b.c. for **d** and no-slip b.c. for **u**
  - E. Feireisl, M Frémond, E. Rocca & G. Schimperna '11: ∃ weak sols for non-isothermal system, Neumann b.c. for **d**, free-slip b.c. for **u**
- Asymptotic behavior
  - H. Wu, X. Xu & C. Liu '10: conv. to eq. strong sols, conv. rate, periodic bc, in 2D (3D, ν large)
  - H. Petzeltová, E. Rocca & G. Schimperna '12: conv. to eq. weak sols, conv. rate, Neumann bc for **d** in 2D and 3D
  - M. Grasselli & H. Wu '11: smooth global attractor in 2D for strong sols, with finite fractal dimension, periodic b.c.

$\exists$ weak sols in 3D

Consider, e.g., hom. Neumann b.c. for $\mathbf{d}$ (no-slip for $\mathbf{u}$)

(A1) $W = W_1 + W_2 \qquad W \in C^2(\mathbb{R}^3), \quad W_1$ convex, $W_2 \in C^{1,1}$

(A2) $\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div}) \qquad \mathbb{R}^+ := [0, \infty)$

Theorem (C. Cavaterra & E. Rocca '12)

Assume (A1), (A2) and that

$$\mathbf{u}_0 \in \boldsymbol{H}_{div}, \qquad \mathbf{d}_0 \in H^1(\Omega)^3, \qquad W(\mathbf{d}_0) \in L^1(\Omega)$$

Then, $\exists$ *a weak sol* $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ *corresponding to* $\mathbf{u}_0, \mathbf{d}_0$ *s.t.*

$\mathbf{u} \in L^\infty_{loc}(\mathbb{R}^+; \boldsymbol{H}_{div}) \cap L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}_{div})$

$\mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3)$

$\mathbf{d} \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)^3) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3), W(\mathbf{d}) \in L^\infty_{loc}(\mathbb{R}^+; L^1(\Omega))$

$\mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3)$

Theorem (C. Cavaterra & E. Rocca '12)

and satisfying the energy inequality

$$\mathcal{E}(\mathbf{w}(t)) + \int_s^t \Big(\| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla \mathbf{u}\|^2 \Big) d\tau$$
$$\leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau$$

for all $t \geq s$, for a.e. $s \in (0, \infty)$, including $s = 0$. We have set

$$\mathcal{E}(\mathbf{w}(t)) := \frac{1}{2}\|\mathbf{u}(t)\|^2 + \frac{1}{2}\|\nabla \mathbf{d}(t)\|^2 + \int_\Omega W(\mathbf{d}(t)), \qquad \mathbf{w} = [\mathbf{u}, \mathbf{d}]$$

Remark

The space of test functions for weak sols is $W^{1,3}_{0,div}(\Omega)$ (uniqueness or strong-weak uniqueness in 2D not known)

Trajectory attractor approach (V.V. Chepyzhov & M.I. Vishik)

Abstract evolution equation in a Banach space E

$$w_t = A_\sigma w, \qquad \sigma \in \Sigma, \quad \Sigma \text{ space of symbols}$$

- *Space of sols*: sols $w : \mathbb{R}^+ \to E$ sought in a space $\mathcal{W}^+_{loc}$ endowed with local convergence topology Θ^+_{loc} (weak or strong)
- *Trajectory space*: for each $\sigma \in \Sigma$, $\mathcal{K}^+_\sigma$ is the set of *some sols* from $\mathcal{W}^+_{loc}$ and $\mathcal{K}^+_\Sigma := \cup_{\sigma\in\Sigma}\mathcal{K}^+_\sigma$
- The translation semigroup $\{T(t)\}$ acts on $\mathcal{K}^+_\Sigma$ (if the family $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}$ is translation coordinated)
- Introduce a subspace $\mathcal{W}^+_b$ of $\mathcal{W}^+_{loc}$: usually Banach, but also metric space with metric $\rho_{\mathcal{W}^+_b}$. We assume that $\mathcal{K}^+_\sigma \subset \mathcal{W}^+_b$, for all $\sigma \in \Sigma$. The subspace $\mathcal{W}^+_b$ is used to define bounded subsets of the trajectory space $\mathcal{K}^+_\Sigma$

Trajectory attractor approach

- $\mathcal{A}_\Sigma \subset \mathcal{W}^+_{loc}$ *uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor* if
  1) $\mathcal{A}_\Sigma$ is compact in Θ^+_{loc}
  2) $\mathcal{A}_\Sigma$ is a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}$ in the topology Θ^+_{loc}
  3) $\mathcal{A}_\Sigma$ is the minimal compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{\mathcal{K}^+_\sigma\}_{\sigma\in\Sigma}$ in Θ^+_{loc}
- If $\mathcal{A}_\Sigma$ exists, it is unique
- If $T(t)$ is continuous in Θ^+_{loc}, then $T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma$, $\forall t \in \mathbb{R}^+$
- For existence of $\mathcal{A}_\Sigma$: prove $\exists\ P \subset \mathcal{W}^+_{loc}$ compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting in Θ^+_{loc}

Trajectory attractor approach

To this aim we need

- A dissipative estimate of the form

$$\rho_{\mathcal{W}_b^+}(T(t)w, w_0) \leq \Lambda_0\Big(\rho_{\mathcal{W}_b^+}(w, w_0)\Big)e^{-kt} + \Lambda_1, \quad \forall t \geq t_0$$

  for every $w \in \mathcal{K}_\Sigma^+$
  $\Lambda_0 : \mathbb{R}^+ \to \mathbb{R}^+$ locally bdd; k, Λ_0, Λ_1 independent of w

- That the ball

$$B_{\mathcal{W}_b^+}(w_0, 2\Lambda_1) := \{w \in \mathcal{W}_b^+ : \rho_{\mathcal{W}_b^+}(w, w_0) \leq 2\Lambda_1\}$$

  is compact in Θ_{loc}^+

If, in addition, $\{\mathcal{K}_\sigma^+\}_{\sigma\in\Sigma}$ is $(\Theta_{loc}^+, \Sigma)-$closed, Σ compact, then $\mathcal{A}_\Sigma \subset \mathcal{K}_\Sigma^+$ and

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$$

+further properties of the trajectory attractor

GENERAL SMOOTH POTENTIALS

Banach-metric setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and introduce the space

$$\mathcal{W}^+_{loc} := \Big\{ \mathbf{w} \in L^\infty_{loc}(\mathbb{R}^+; \mathbf{H}_{div} \times H^1(\Omega)^3) \cap L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3) \Big\}$$

endowed with the topology Θ^+_{loc} of local weak convergence.
In $\mathcal{W}^+_{loc}$ we consider the following metric subspace

$$\mathcal{W}^+_b := \Big\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times H^1(\Omega)^3) \cap L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{tb}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{tb}(\mathbb{R}^+; L^{3/2}(\Omega)^3), \\ W(\mathbf{d}) \in L^\infty(\mathbb{R}^+; L^1(\Omega)) \Big\}$$

used to define bounded subsets of the space of trajectories $\mathcal{K}^+_\Sigma$.

Trajectory attractor in 3D-Hom. Neumann b.c. for **d**

Definition

For every $\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$ the trajectory space $\mathcal{K}^+_{\mathbf{h}}$ with external force $\mathbf{h}$ is the set of all weak sols $\mathbf{w} = [\mathbf{u}, \mathbf{d}]$ satisfying the energy inequality for all $t \geq s$ and for a.a. $s \in (0, \infty)$

Set $\mathcal{K}^+_\Sigma := \cup_{\mathbf{h}\in\Sigma}\mathcal{K}^+_{\mathbf{h}}$. We have $\mathcal{K}^+_\Sigma \subset \mathcal{W}^+_b$. We take

$$\Sigma = \mathcal{H}_+(\mathbf{h}_0) := \left[\{T(t)\mathbf{h}_0, t \geq 0\}\right]_{L^2_{loc,w}(\mathbb{R}^+;\mathbf{V}'_{div})}$$

$\mathbf{h}_0$ translation bounded in $L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e.

$$\|\mathbf{h}_0\|^2_{L^2_{tb}(\mathbb{R}^+;\mathbf{V}'_{div})} := \sup_{t\geq 0}\int_t^{t+1} \|\mathbf{h}_0(\tau)\|^2_{\mathbf{V}'_{div}} d\tau < \infty$$

$\Leftrightarrow$ $\mathbf{h}_0$ translation compact (tr.-c.) in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e., Σ compact in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$

Trajectory attractor in 3D-Hom. Neumann b.c. for $\mathbf{d}$

(A3) W satisfies (A1) and $\exists\, c_0 \geq 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$ and $\delta > 0$ s.t. for all $\mathbf{d} \in \mathbb{R}^3$

$$W_1(\mathbf{d}) \leq c_0(1 + |\nabla_{\mathbf{d}} W_1(\mathbf{d})|^2) \qquad W_1(\mathbf{d}) \geq c_1|\mathbf{d}|^{2+\delta} - c_2$$

(A4) $W(\mathbf{d}) \leq b(1 + |\mathbf{d}|^6) \qquad \forall \mathbf{d} \in \mathbb{R}^3 \qquad b > 0$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}'_{div})$. Then, the semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ possesses the uniform (w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)}$. This set is strictly invariant, bounded in $\mathcal{W}^+_b$ and compact in Θ^+_{loc}.
In addition, if (A4) holds and $\mathbf{h}_0$ is tr.-c. in $L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$ or $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{H}_{div})$, then $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ is closed in Θ^+_{loc}, $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ and

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

POLYNOMIAL POTENTIALS

Banach-Banach setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and, for $p \geq 2$, introduce the space

$$\mathcal{W}^+_{p,loc} := \Big\{ \mathbf{w} \in L^\infty_{loc}(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \\ \mathbf{w} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3) \Big\}$$

endowed with its inductive limit topology $\Theta^+_{p,loc}$.

Bdd subsets of $\mathcal{K}^+_\Sigma$ defined w.r.t the Banach subspace of $\mathcal{W}^+_{p,loc}$

$$\mathcal{W}^+_{p,b} := \Big\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \\ \mathbf{w} \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \\ \mathbf{u}_t \in L^2_{tb}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L^2_{tb}(\mathbb{R}^+; L^{3/2}(\Omega)^3) \Big\}$$

Trajectory attractor in 3D-Hom. Neumann b.c. for **d**

(A5) $\exists\, C_1, C_2 > 0$ and $p \in (2, +\infty)$ s.t.

$$C_1(|\mathbf{d}|^p - 1) \le W(\mathbf{d}) \le C_2(1 + |\mathbf{d}|^p), \qquad \forall \mathbf{d} \in \mathbb{R}^3$$

$\mathcal{K}^+_{p,\mathbf{h}}$ and $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)} := \bigcup_{\mathbf{h}\in\mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}^+_{p,\mathbf{h}}$ are the trajectory spaces

Theorem (S.F. & E. Rocca '12)

Assume (A3), (A5) and $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{V}'_{div})$. Then, $\{T(t)\}$ acting on $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ possesses the uniform (w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) trajectory attractor $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)}$. This set is strictly invariant, bounded in $\mathcal{W}^+_{p,b}$, compact in $\Theta^+_{p,loc}$.
In addition, if $\mathbf{h}_0$ is tr.-c. in $L^2_{loc}(\mathbb{R}^+; \boldsymbol{V}'_{div})$ or $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \boldsymbol{H}_{div})$, then $\mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ is closed in $\Theta^+_{p,loc}$, $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{p,\mathcal{H}_+(\mathbf{h}_0)}$ and

$$\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{p,\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

If $\mathbf{g} \in H^1_{loc}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3) \cap L^2_{loc}(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ (same assumptions for $\mathbf{u}_0$, $\mathbf{d}_0$ and $\mathbf{h}$), then $\exists$ a weak sol $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ satisfying

$$\mathcal{E}(\mathbf{w}(t)) + \int_s^t \left(\| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla \mathbf{u}\|^2 \right) d\tau$$
$$\leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{g}_t, \partial_{\boldsymbol{n}} \mathbf{d} \rangle_{H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3} d\tau + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau$$

for all $t \geq s$, for a.e. $s \in (0, \infty)$, including $s = 0$

Symbol space for the Dirichlet datum **g**

$$\mathcal{H}_+(\mathbf{g}_0) := [\{T(t)\mathbf{g}_0, t \geq 0\}]_{\Xi^+_{loc,w}}$$

$$\Xi^+_{loc,w} := \{\mathbf{g} \in C(\mathbb{R}^+; H^{3/2}(\Gamma)^3) : \mathbf{g}_t \in L^2_{loc,w}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)\}$$

Consider, e.g., general smooth potentials. Trajectory spaces

$$\mathcal{K}^+_{\mathbf{g},\mathbf{h}} \quad \text{and} \quad \mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} := \bigcup_{\mathbf{g}\in\mathcal{H}_+(\mathbf{g}_0),\mathbf{h}\in\mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}^+_{\mathbf{g},\mathbf{h}}$$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and that $\mathbf{g}_0$ *is tr.-c. in* $C(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ *with* $\partial_t\mathbf{g}_0 \in L^2_{tb}(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)$, *and* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div})$. *Then,* $\{T(t)\}$ *acting on* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *possesses the uniform (w.r.t.* $[\mathbf{g},\mathbf{h}] \in \mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)$*) trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$. *This set is strictly invariant, bounded in* $\mathcal{W}^+_b$ *and compact in* Θ^+_{loc}.

In addition, if (A4) holds, if $\mathbf{g}_0$ *is tr.-c. in* Ξ^+_{loc} *and if* $\mathbf{h}_0$ *is tr.-c. in* $L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div})$ *or* $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{H}_{div})$, *then* $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *is closed in* Θ^+_{loc}, $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)}$ *and*

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{g}_0)\times\mathcal{H}_+(\mathbf{h}_0))}$$

Robustness of the trajectory attractor

Open issue: eventual regularization and energy identity for weak sols in 2D $\Rightarrow$ existence of trajectory attractor in 2D for the *strong* topology of $\mathcal{W}^{+}_{loc}$ not known
Consider then the problem P_{ϵ}

$$
\begin{aligned}
&\mathbf{u}_t + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p \\
&= 2\nu \mathrm{div}(D\mathbf{u})) + \epsilon \mathrm{div}(|Du|^{q-2} Du) - \mathrm{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) \\
&- \mathrm{div}(\alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} - (1-\alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) + \mathbf{h} \\
&\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \alpha \mathbf{d} \cdot \nabla \mathbf{u} + (1-\alpha)\mathbf{d} \cdot \nabla^T \mathbf{u} = (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \\
&\mathrm{div}(\mathbf{u}) = 0
\end{aligned}
$$

where $q > 3$

Preliminary results

- *Strong trajectory attractor for P_ϵ*

Every weak sol to P_ϵ satisfies the energy identity

$$\frac{d}{dt}\mathcal{E}(\mathbf{w}(t)) + \|-\Delta\mathbf{d} + \nabla_{\mathbf{d}}W(\mathbf{d})\|^2 + \nu\|\nabla\mathbf{u}\|^2 + \epsilon\|D\mathbf{u}\|_{L^q}^q = \langle\mathbf{h},\mathbf{u}\rangle$$

Then, under assumption (A1) and

$$|\partial^2_{d_i d_j}W(\mathbf{d})| \le c(1+|\mathbf{d}|^{4-\sigma}) \qquad \forall\mathbf{d}\in\mathbb{R}^3 \quad \forall i,j \qquad \sigma>0$$

Problem P_ϵ admits a (unique) strong trajectory attractor $\mathcal{A}^\epsilon_{\mathcal{H}_+(\mathbf{h}_0)}$ (uniform w.r.t. $\mathbf{h}\in\mathcal{H}_+(\mathbf{h}_0)$) in the space of weak sols $\mathcal{Z}^+_{q,loc}$ with its strong topology $\Theta^+_{q,loc,s}$

Robustness of the trajectory attractor

- *Convergence of the family of strong trajectory attractors*

  Introducing a suitable space $\widetilde{\mathcal{Z}}^{+}_{q,loc}$ of sols of both P_ϵ and P_0, endowed with its weak topology $\widetilde{\Theta}^{+}_{q,loc}$, so that $\mathcal{A}^{\epsilon}_{\mathcal{H}_+(\mathbf{h}_0)}, \mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \subset \widetilde{\mathcal{Z}}^{+}_{q,loc}$, we have

$$\mathcal{A}^{\epsilon}_{\mathcal{H}_+(\mathbf{h}_0)} \to \mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \qquad \text{in} \quad \widetilde{\Theta}^{+}_{q,loc}$$