Large Deformations

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work in progress with

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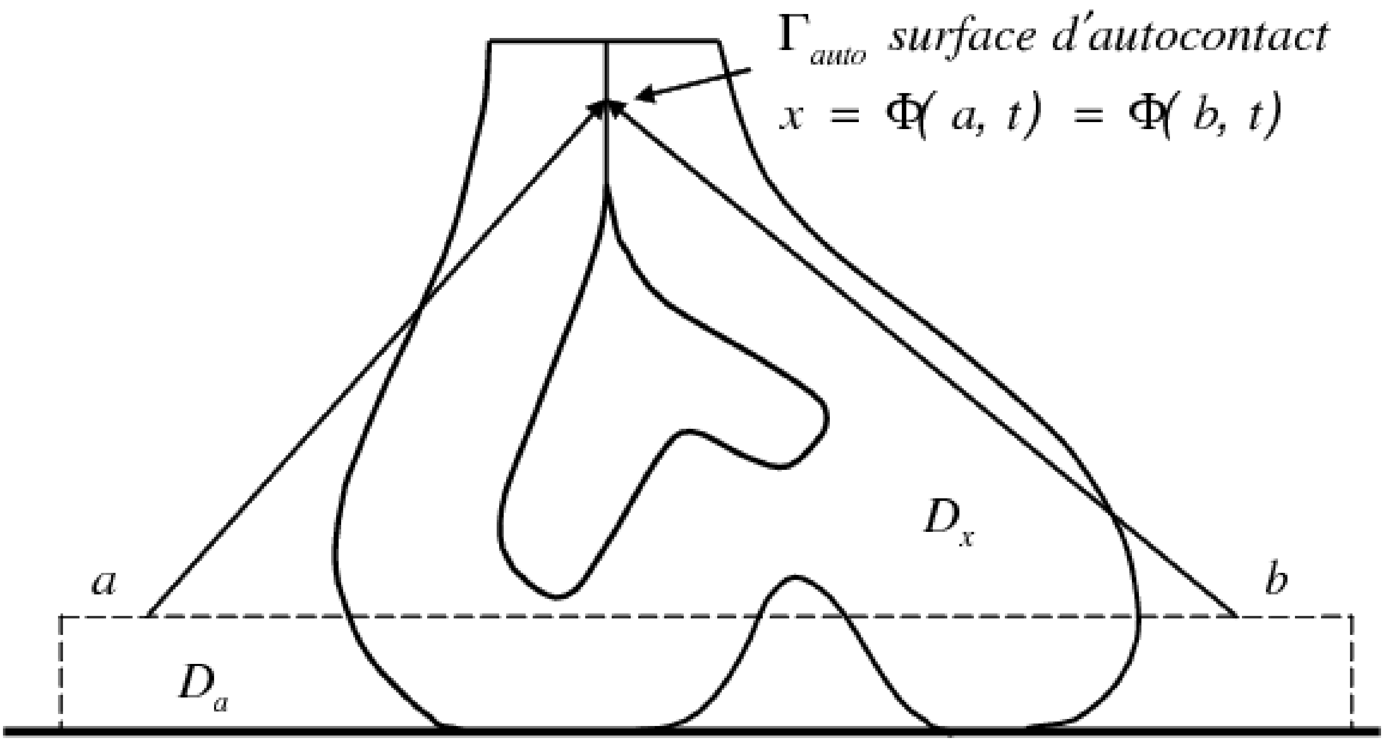
Kinematics



$$x = (x_i),\ a = (a_\alpha),\ x = \Phi(a,t)$$

$$\mathbf{F} = \operatorname{grad}\Phi = (\frac{\partial \Phi_i}{\partial a_\alpha})$$

$$\mathbf{F} = \mathbf{R}\mathbf{W},\ \mathbf{W} = \mathbf{W}^T$$

$$\mathbf{R}\mathbf{R}^T = \mathbf{I},\ \det \mathbf{R} = \mathbf{1}$$

W stretch matrix

R rotation matrix

Kinematically admissible positions Φ

Local non interpenetration

$$\det \mathbf{F} = \det \mathbf{W} \geq 0$$

We assume crushing or flattening

is impossible

$$\det \mathbf{W} \geq \alpha^2, \; tr\mathbf{W} \geq \alpha$$

with $1 > \alpha > 0$

We assume no collision with an obstacle
and no autocollision

Principle of virtual power

Loads are applied by plates, rods, wires, needles,...

Plate equations of motion involve second gratient theories. Velocities of the plate are the trace on the surface of the volume velocities.

It is wise to have a volume third gradient theory.

Equations of motion

Linear momentum

$$\frac{\partial^2 \Phi}{\partial t^2} = \operatorname{div} \mathbf{\Pi} + \operatorname{div} \Delta \mathbf{Z} + \vec{f}$$

Angular momentum

$$0 = \operatorname{div} \Lambda + \mathbf{M}$$

plus boundary and initial conditions

We investigate 2-D motions

$\mathcal{M}$ set of 2×2 matrices

scalar product $\mathbf{\Pi} : \mathbf{D} = \Pi_{i\alpha} D_{i\alpha}$

$\mathcal{S} \subset \mathcal{M}$ symmetric matrices

$\mathcal{A} \subset \mathcal{M}$ antisymmetric matrices

$\mathcal{A} \perp \mathcal{M}$

Constitutive laws

Stretch matrix $\mathbf{W}$ satisfies

$$\mathbf{W} \in \mathcal{S}, \; tr\mathbf{W} \geq \alpha, \; \det \mathbf{W} \geq \alpha^2$$

Internal constraints are taken into account by free energy

$$\frac{1}{2}\left|\mathbf{M} - \mathbf{I}\right|^2 + \hat{\Psi}(\mathbf{M}) + I_{\mathcal{S}}(\mathbf{M}) + ..$$

$$\hat{\Psi}(\mathbf{M}) = I_1(tr\mathbf{M}) + I_2(\det \mathbf{M})$$

$I_{\mathcal{S}}$ is the indicator function of $\mathcal{S} \subset \mathcal{M}$

I_1 and I_2 are approximation from the interior of the indicator functions of segments $[2\alpha, \infty]$ and $[\alpha^2, \infty]$

Constitutive laws

$$\mathbf{\Pi} = \mathbf{R}\left\{(\mathbf{W}-\mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\Psi}}{\partial\mathbf{M}}(\mathbf{W}) + \mathbf{A}\right\}$$

$$\mathbf{A} \in \partial I_{\mathcal{S}}(\mathbf{W}) = \mathcal{A}$$

$$\mathbf{Z} = \operatorname{grad}\Delta\Phi$$

$$\Lambda = (\operatorname{grad}\mathbf{R})\,\mathbf{R}^T + \operatorname{grad}\mathbf{\Omega}$$

$$\mathbf{M} = \mathbf{\Pi}\mathbf{F}^T - \mathbf{F}\mathbf{\Pi}^T$$

with

$$\dot{\mathbf{W}} = \frac{\partial\mathbf{W}}{\partial t},\quad \mathbf{\Omega} = \frac{\partial\mathbf{R}}{\partial t}\mathbf{R}^T$$

Equations

$$\mathbf{F} = \operatorname{grad}\Phi,\ \ \mathbf{W} = \sqrt{\mathbf{F}^T\mathbf{F}},\ \ \mathbf{R} = \mathbf{F}\mathbf{W}^{-1},$$

$$\frac{\partial^2\Phi}{\partial t^2} = \operatorname{div}\mathbf{\Pi} + \operatorname{div}(\Delta\mathbf{Z}) + \vec{f},\ \ in\ \Omega_a,$$

$$\dot{\Phi} = 0,\ \ \operatorname{grad}\dot{\Phi} = 0,\ \ \frac{\partial}{\partial N}\left(\operatorname{grad}\dot{\Phi}\right) = 0,\ \ on\ \Gamma_0,$$

$$\mathbf{\Pi}\vec{N} + \Delta\mathbf{Z}\vec{N} = 0,\ \ \operatorname{grad}\dot{\Phi} = 0,\ \ \mathbf{Z} = 0, on\ \Gamma_1,$$

$$\mathbf{\Pi} = \mathbf{R}\left\{(\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\mathbf{\Psi}}}{\partial\mathbf{M}}(\mathbf{W}) + \mathbf{A}\right\},$$

$$\mathbf{A} \in \partial I_{\mathcal{S}}(\mathbf{W}) = \mathcal{A},$$

$$\mathbf{Z} = \operatorname{grad}\Delta\Phi,$$

$$\operatorname{div}((\operatorname{grad}\mathbf{R})\,\mathbf{R}^T) + \Delta\mathbf{\Omega}$$

$$+\mathbf{R}\left\{\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} + \dot{\mathbf{W}}\mathbf{W} - \mathbf{W}\dot{\mathbf{W}})\right\}\mathbf{R}^{\mathbf{T}} = \mathbf{0},$$

$$\Phi(a,0) = a,\ \ \frac{d\Phi}{dt}(a,0) = 0.$$

Two unknowns Φ and $\mathbf{A}$

Two equations

In 2-D

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \ \boldsymbol{\Omega} = \dot{\theta}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{A} = z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The unknowns are Φ and z

Equations in 2-D

$$\mathbf{F} = \operatorname{grad}\Phi,\ \ \mathbf{W} = \sqrt{\mathbf{F}^T\mathbf{F}},\ \ \mathbf{R} = \mathbf{F}\mathbf{W}^{-1},$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

$$\frac{\partial^2\Phi}{\partial t^2} = \operatorname{div}\mathbf{\Pi} + \operatorname{div}(\Delta\mathbf{Z}) + \vec{f},\ \ in\ \Omega_a,$$

$$\dot{\Phi} = 0,\ \ \operatorname{grad}\dot{\Phi} = 0,\ \ \frac{\partial}{\partial N}\left(\operatorname{grad}\dot{\Phi}\right) = 0,\ \ on\ \Gamma_0,$$

$$\mathbf{\Pi}\vec{N} + \Delta\mathbf{Z}\vec{N} = 0,\ \ \operatorname{grad}\dot{\Phi} = 0,\ \ \mathbf{Z} = 0, on\ \Gamma_1,$$

$$\mathbf{\Pi} = \mathbf{R}\left\{(\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\mathbf{\Psi}}}{\partial\mathbf{M}}(\mathbf{W}) + \mathbf{z}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$$

$$\mathbf{Z} = \operatorname{grad}\Delta\Phi,$$

$$\Delta\dot{\theta} + \Delta\theta + z(tr\mathbf{W})$$

$$+(\dot{w}_{11} - \dot{w}_{22})w_{12} + \dot{w}_{12}(w_{22} - w_{11}) = 0,$$

$$\Phi(a,0) = a,\ \ \frac{d\Phi}{dt}(a,0) = 0.$$

The last pde does nor give θ, it gives z!

Virtual velocities

$$\mathcal{V}(T) = \Big\{ \vec{\varphi} \in L^2(0,T;H^3(\Omega_a)),\ \frac{d\vec{\varphi}}{dt} \in L^2(0,T;L^2(\Omega_a)),$$

$$\vec{\varphi} = 0,\ \mathrm{grad}\,\vec{\varphi} = 0, \frac{\partial}{\partial N}(\mathrm{grad}\,\vec{\varphi}) = 0,\ on\ \Gamma_0,\ \mathrm{grad}\,\vec{\varphi} = 0,\ on\ \Gamma_1 \Big\}$$

$$\mathcal{V}_r(T) = \{ \varphi \in L^2(0,T;H^1(\Omega_a)),\ \varphi = 0,\ on\ \partial\Omega_a \}$$

Variational formulations

$$(\Phi - a) \in \mathcal{V}(T),\ \forall \vec{\varphi} \in \mathcal{V}(T),$$

$$\int_0^t \int_{\Omega_a} \frac{\partial^2 \Phi}{\partial t^2} \cdot \vec{\varphi} da d\tau$$

$$+ \int_0^t \int_{\Omega_a} \mathbf{R} \left\{ (\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}) + \mathbf{A} \right\} : \mathrm{grad}\,\vec{\varphi}$$

$$+ \mathrm{grad}\,\Delta\Phi : \mathrm{grad}\,\Delta\vec{\varphi} da d\tau$$

$$= \int_0^t \int_{\Omega_a} \vec{f} \cdot \vec{\varphi} da d\tau,$$

$$\dot{\theta} \in \mathcal{V}_r(T),\ \forall \varphi \in \mathcal{V}_r(T),\ \theta(a,0) = 0,$$

$$\int_0^t \int_{\Omega_a} \Big\{ \mathrm{grad}\,\dot{\theta} + \mathrm{grad}\,\theta) \cdot \mathrm{grad}\,\varphi \Big\} da d\tau$$

$$= \int_0^t \int_{\Omega_a} z(w_{11} + w_{22}) \varphi da d\tau$$

$$+ \int_0^t \int_{\Omega_a} (\dot{w}_{11} - \dot{w}_{22}) w_{12} + \dot{w}_{12}(w_{22} - w_{11}) \varphi da d\tau$$

Let γ such that $1 > \gamma > \alpha > 0$.

If $tr\mathbf{W} \geq 2\gamma$, $\det \mathbf{W} \geq \gamma^2$

and $|\mathbf{W}| \leq c$ then functions

$$\mathbf{F} \to \mathbf{W},\ \mathbf{F} \to \mathbf{W}^{-1},\ \mathbf{F} \to \mathbf{R},$$

$$\mathbf{F} \to \frac{\partial \hat{\Psi}}{\partial \mathbf{M}}(\mathbf{W}),$$

are C^∞

with globally bounded derivatives

Approximation

$$\Phi_n(a,t) = a + \sum_{i=1}^{n} x_i(t)\vec{u}_i(a)$$

Displacements $\vec{u}_i(a)$ are dense in $H^3(\Omega_a)$ with the convenient boundary conditions

One unknown $x = (x_i(t))$

We have

$$\mathbf{F}_n(a,t) = f(a, x(t))$$

$$\mathbf{W}_n(a,t) = f(a, x(t))$$

$$\mathbf{R}_n(a,t) = f(a, x(t))$$

$$\dot{\mathbf{W}}_n(a,t) = f(a, x(t))\dot{x}(t)$$

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{M}}\left(\mathbf{W}_n(a,t)\right) = f(a, x(t))$$

Equation of motion

$$\Delta\dot{\theta} + \Delta\theta + z(tr\mathbf{W})$$
$$+(\dot{w}_{11} - \dot{w}_{22})w_{12} + \dot{w}_{12}(w_{22} - w_{11}) = 0$$

with boundary condition $\theta = 0$ on $\partial\Omega_a$

gives

$$z_n(a,t) = f(a, x(t))\dot{x}(t) + f(a, x(t))$$

because $tr\mathbf{W}_n \geq 2\alpha > 0$

Functions f are C^∞
with derivatives globally bounded

Variational formulation for Φ gives

$$M\ddot{x} + G_1(x)\dot{x} + G_2(x) = f$$
$$x(0) = 0,\ \dot{x}(0) = 0$$

Functions G_k are uniformly Lipschitz in open set of $\mathbb{R}^n$

$$|\Phi_n(x) - a| < c,\ tr\mathbf{W}_n(x) > 2\gamma$$
$$\det \mathbf{W}_n(x) > \gamma^2$$

which contains $x = 0$

There is a solution up to time T_n which depends on c, n and γ

A priori estimates

Principle of virtual power
with the actual velocities
and property

$$\frac{\partial\hat{\Psi}}{\partial\mathbf{M}}(\mathbf{W})\,\mathbf{W} = \mathbf{W}\frac{\partial\hat{\Psi}}{\partial\mathbf{M}}(\mathbf{W})$$

gives

$$\frac{1}{2}\int_{\Omega_a}\dot{\Phi}_n^2(t)da + \int_{\Omega_a}\left\{\frac{1}{2}\left\|\mathbf{W}_n(t)-\mathbf{I}\right\|^2 + \hat{\Psi}(\mathbf{W}_n(t))\right\}da$$

$$+\int_0^t\int_{\Omega_a}\dot{\mathbf{W}}_n:\dot{\mathbf{W}}_n da d\tau + \int_{\Omega_a}\frac{1}{2}\left\|\mathrm{grad}\,\Delta\Phi_n(t)\right\|^2 da$$

$$+\int_{\Omega_a}\frac{1}{2}\left\|(\mathrm{grad}\,\theta_n(t))\right\|^2 da + \int_0^t\int_{\Omega_a}\frac{1}{2}\mathrm{grad}\,\dot{\theta}_n\cdot\mathrm{grad}\,\dot{\theta}_n da d\tau$$

$$= \int_0^t\int_{\Omega_a}\vec{f}\cdot\dot{\Phi}_n da d\tau. + \int_{\Omega_a}\hat{\Psi}(\mathbf{I})da.$$

We assume

$$\vec{f} \in L^\infty(0,\infty; L^2(\Omega_a))$$

- $\dot{\Phi}_n$ is bounded in $L^2(0,T_n; H^1(\Omega_a))$;;
- Φ_n is bounded in $L^\infty(0,T_n; H^3(\Omega_a))$;
- $\mathbf{W}_n$ bounded in $L^\infty(0,T_n; H^2(\Omega_a))$;
- $\dot{\mathbf{W}}_n$ is bounded in $L^2(0,T_n; L^2(\Omega_a))$;
- $\hat{\Psi}(\mathbf{W}_n)$ is bounded in $L^\infty(0,T_n; L^1(\Omega_a))$;
- θ_n is bounded in $L^\infty(0,T_n; H^2(\Omega_a))$;
- $\dot{\theta}_n$ is bounded in $L^2(0,T_n; H^1(\Omega_a))$.
- z_n is bounded in the dual space of $\mathcal{V}_r(T_n)$

The bounds do not depend on n

Due to the a priori estimates

for $0 < t \leq T = \inf\left\{\frac{1-\gamma^2}{c_1}, \frac{2-2\gamma}{c_2}\right\}$

$$\det \mathbf{F}_n(a,t) = \det \mathbf{W}_n(a,t) \geq 1 - c_1 t \geq \gamma^2$$

$$tr\mathbf{W}_n \geq 2 - c_2 t \geq 2\gamma$$

and

$$|\Phi_n(a,t) - a| \leq c_3 \leq c$$

Thus $T_n > T > 0$

Because the c_k do not depend on n

T does not depend on c and n

The impenetrability reaction

Due to the a priori estimates

$$\forall t \in [0,T], \quad \left| \frac{\partial \hat{\Psi}}{\partial \mathbf{M}} \left(\mathbf{W}_n(a,t) \right) \right| \leq c$$

With the Lebesgue theorem, we have

$$\forall \vec{\varphi} \in \mathcal{V}(T),$$

$$\lim_{n \to \infty} \int_0^T \int_{\Omega_a} \mathbf{R}_n \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}_n) : \mathrm{grad}\, \vec{\varphi} da d\tau$$

$$= \int_0^T \int_{\Omega_a} \mathbf{R} \frac{\partial \hat{\Psi}}{\partial \mathbf{W}}(\mathbf{W}) : \mathrm{grad}\, \vec{\varphi} da d\tau$$

The acceleration

With the a priori estimates
and the approximated equation for Φ

$$\lim_{n\to\infty}\int_0^T\int_{\Omega_a}\frac{d^2\Phi_n}{dt^2}\cdot\vec{\varphi}\,da\,d\tau=<\frac{d^2\Phi}{dt^2},\vec{\varphi}>$$

The limit equations

$$(\Phi - a) \in \mathcal{V}(T),\ \forall \vec{\varphi} \in \mathcal{V}(T),$$

$$< \frac{\partial^2 \Phi}{\partial t^2} \cdot \vec{\varphi} >$$

$$+ \int_0^T \int_{\Omega_a} \mathbf{R} \left\{ (\mathbf{W} - \mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial \hat{\Psi}}{\partial \mathbf{W}} (\mathbf{W}) \right\} : \operatorname{grad} \vec{\varphi}$$

$$+ \operatorname{grad} \Delta \Phi : \operatorname{grad} \Delta \vec{\varphi} da d\tau + << \mathbf{A} : \mathbf{R}^T \operatorname{grad} \vec{\varphi} >>$$

$$= \int_0^T \int_{\Omega_a} \vec{f} \cdot \vec{\varphi} da d\tau,$$

and

$$\dot{\theta} \in \mathcal{V}_r(T),\ \forall \varphi \in \mathcal{V}_r(T),\ \theta(a, 0) = 0,$$

$$\int_0^T \int_{\Omega_a} \left\{ \operatorname{grad} \dot{\theta} + \operatorname{grad} \theta) \cdot \operatorname{grad} \varphi \right\} da d\tau$$

$$= < z, (w_{11} + w_{22}) \varphi >$$

$$+ \int_0^T \int_{\Omega_a} (\dot{w}_{11} - \dot{w}_{22}) w_{12} + \dot{w}_{12} (w_{22} - w_{11}) \varphi da d\tau$$

What occurs after time T?

Mechanics provides the solution
A collision due to crushing
occurs inside the solid.
Collisions theory has to be applied.

Before collision,
negative stretch up to

$$-(1-\gamma) = -99,999..\%$$

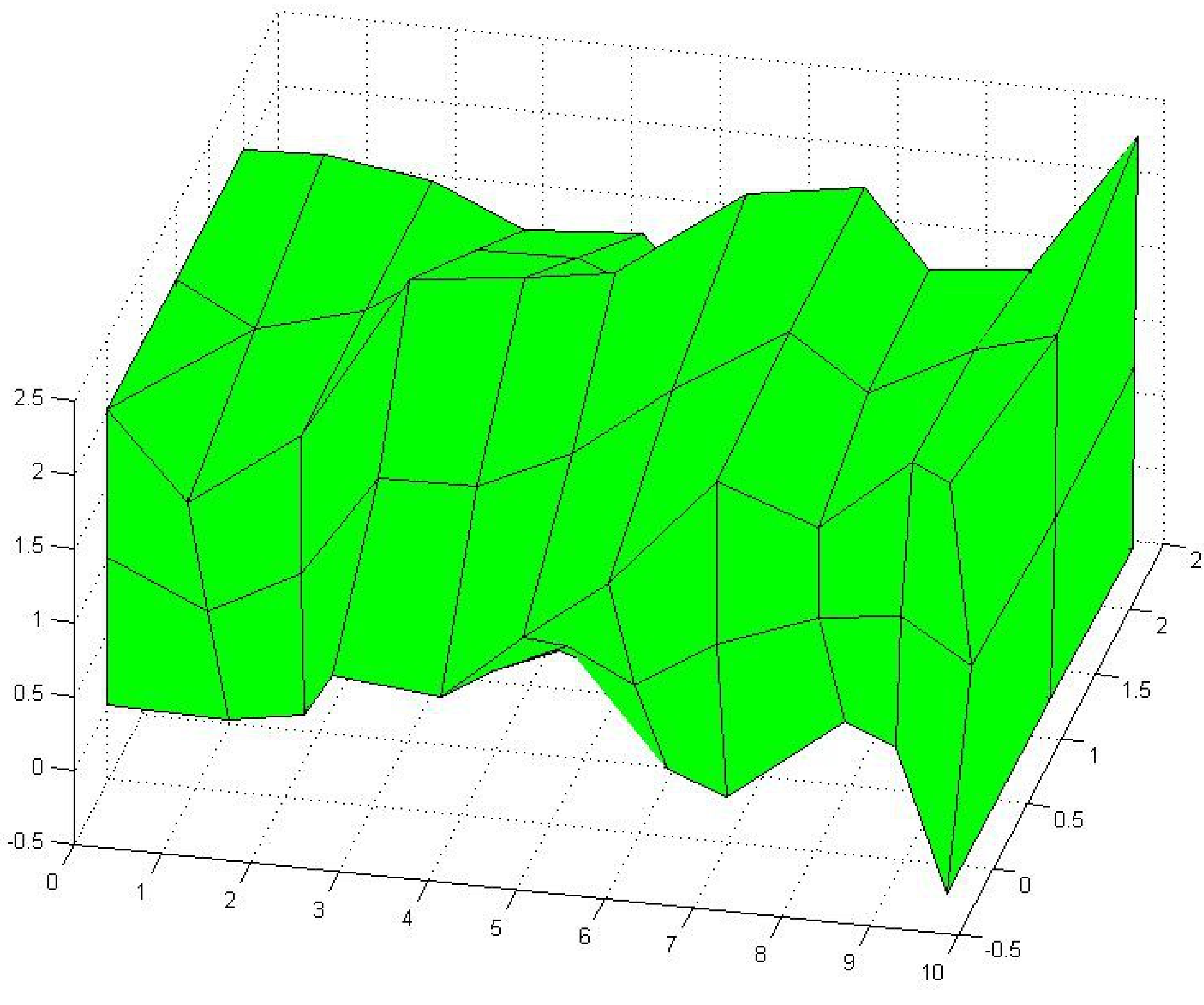
is possible

No limitation for positive stretch

Conclusion

The new quantity **A** is the reaction to the bilateral internal constraint: **W** is symmetric.

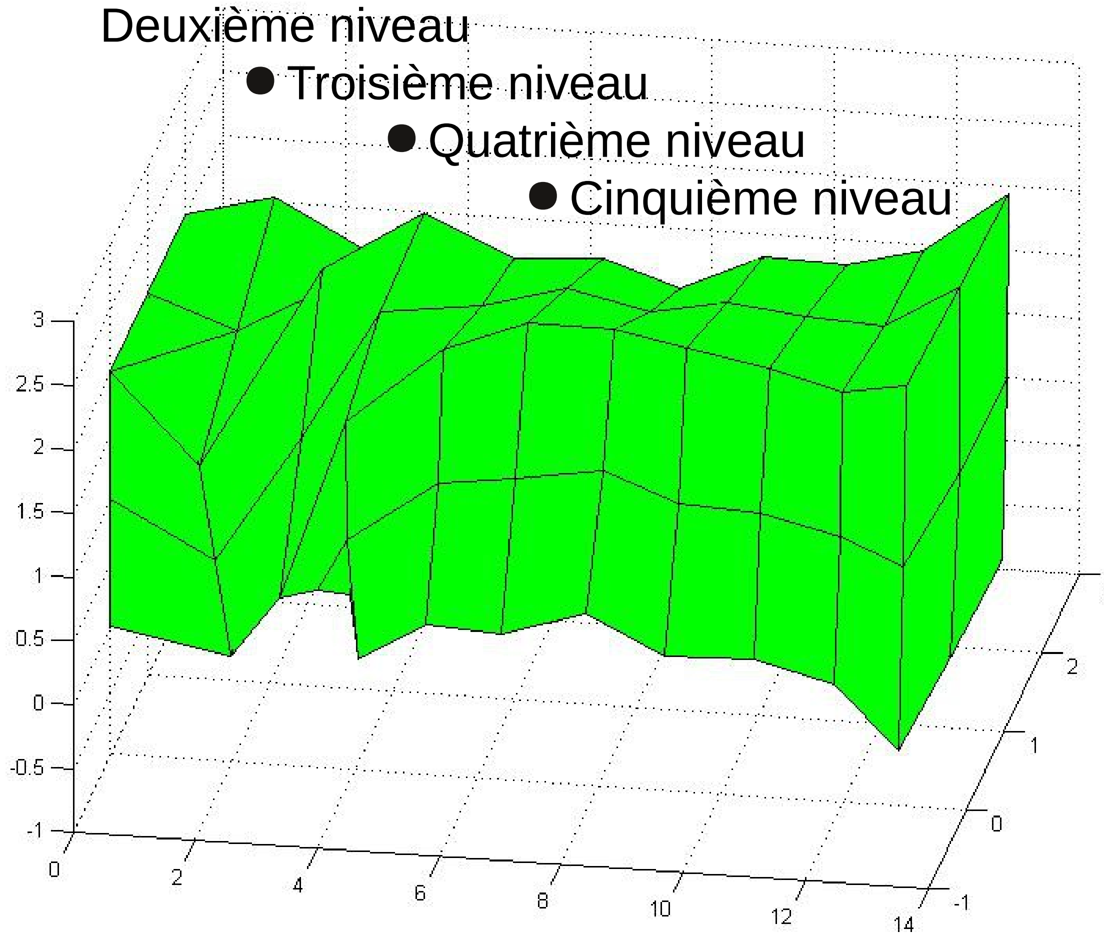
The third order gradient theory is rational because of the exterior actions



Torque

Cliquez pour modifier les styles du texte du masque

- Deuxième niveau
  - Troisième niveau
    - Quatrième niveau
      - Cinquième niveau

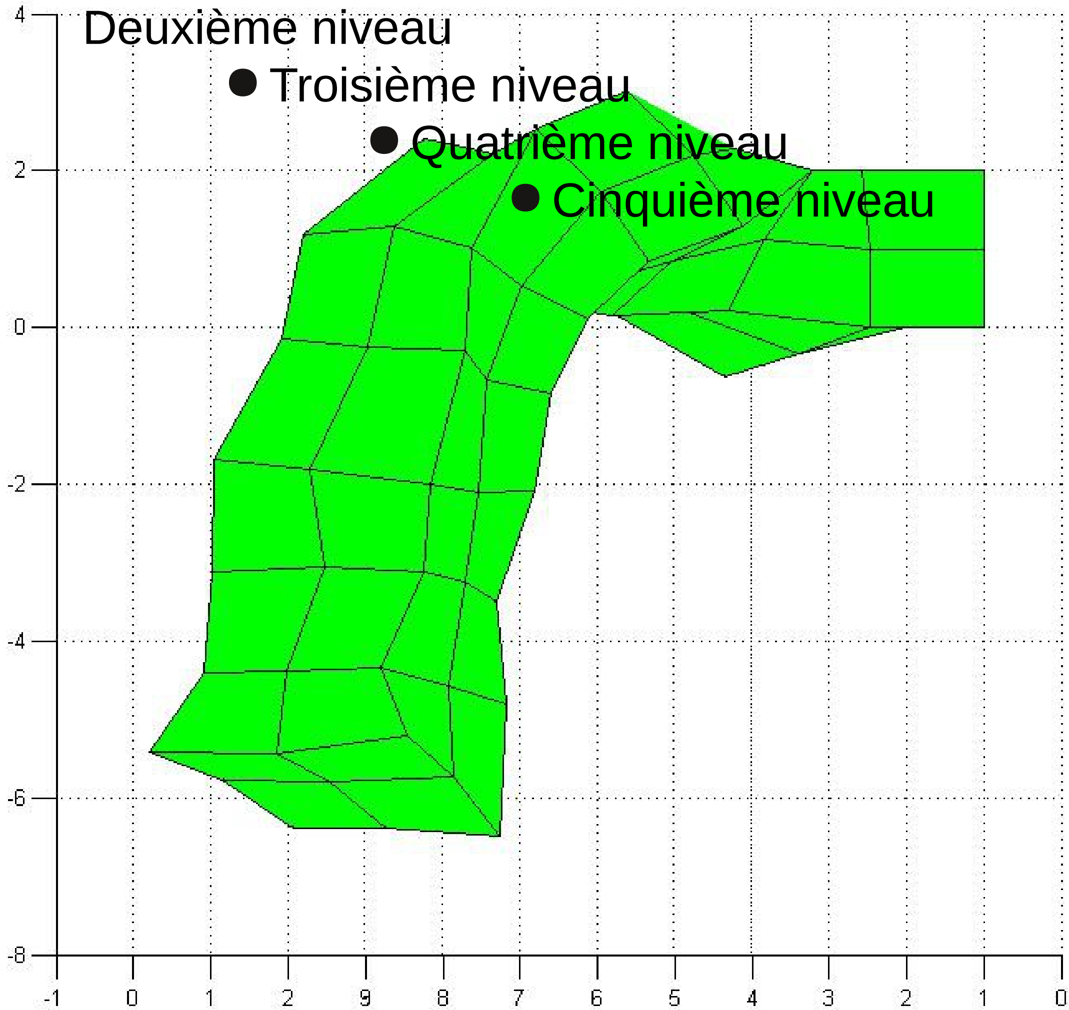


Tension and torque

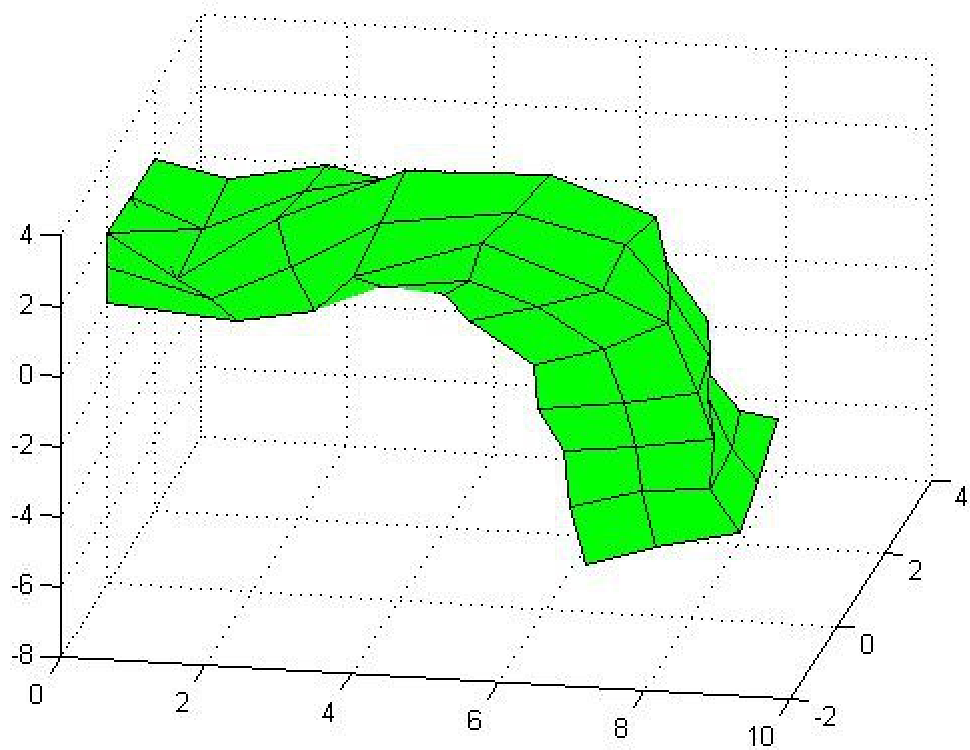
Cliquez pour modifier les styles du texte du masque

Deuxième niveau

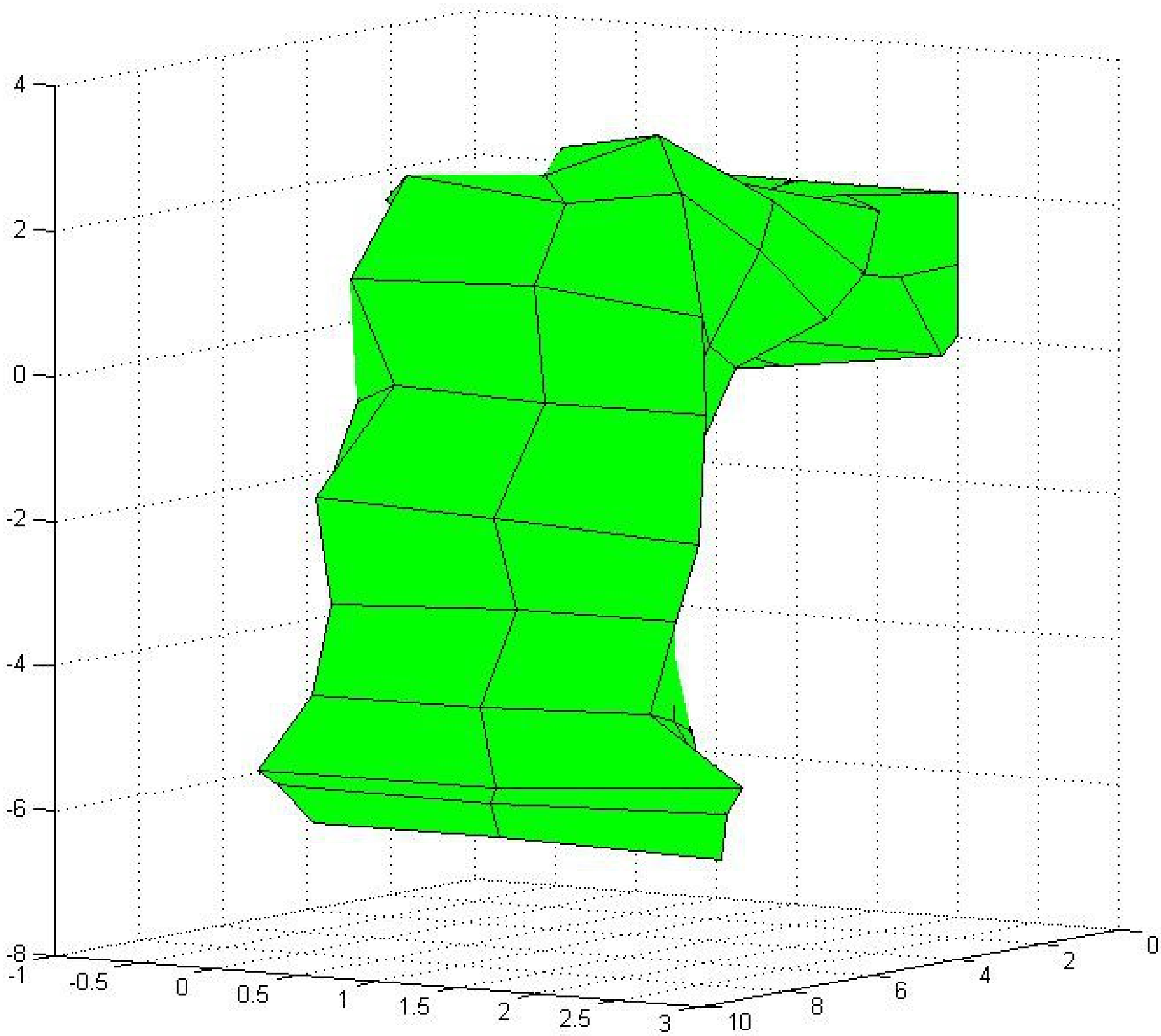
- Troisième niveau
  - Quatrième niveau
    - Cinquième niveau



Shear and torque



Shear and torque



Shear and torque