PDEs for multiphase advanced materials

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Special section dedicated to the 65th birthday
of Gianni Gilardi

Mathematical modelling of migration and integration

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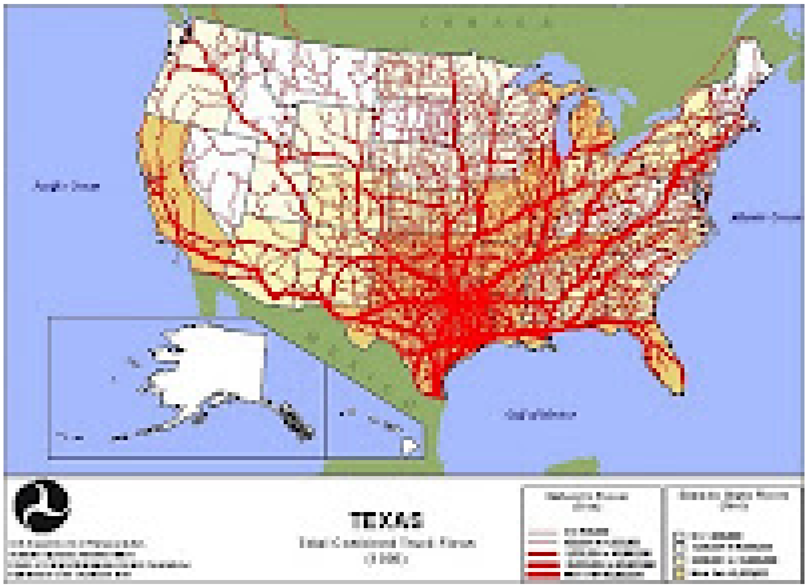
The increasing role of migration and integration in the social and economic development of countries, regions and the whole of the world is becoming more and more apparent, stimulating interest in mathematical modeling of migration and integration.



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The subject of this presentation is to provide a model for studying the integration of migration flows with the resident population.

A key basis for social cohesion in societies is the *cultural and educational* level of the affected populations.

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In our framework, we suppose an analogy between propagation laws of the culture and heat, which are described respectively by a diffusion equation on the knowledge and temperature

The integration of two ethnic groups is studied by a differential system. This issue is represented by a mathematical model consisting of the Cahn-Hilliard equation, which describes the integration or separation law of two ethnic fluxes, according to a control factor given by the cultural levels of two populations, whose evolutions are described by a system of diffusion equations. Moreover, we assume that the homogenization process occurs when the mean of two cultural levels exceeds a critical value.

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A mathematical model of integration

Let us consider, in a bounded domain $\Omega \subset \boldsymbol{R}^2$, two ethnic groups A_1 and A_2 with different cultures (traditions, religions, ecc.). Moreover, we fix a time interval $[0, T]$, in which there is not any new immigration. So, the total mass M_1 and M_2 of two populations will be constants. In the following, we denote by ρ_1 and ρ_2 the local (relative) densities of the two groups, while the specific densities of the populations A_1 and A_2 are the same, denoted by ρ. Finally, the concentration $c \in [-1, 1]$ of the component A_1 is given by

$$c = \frac{2\rho_1 - \rho}{\rho} \qquad (1)$$

of course the concentration of the component A_2 is defined by $(1 - c) = \frac{2\rho_2 - \rho}{\rho}$.

In our framework, we study the evolution of this system of two ethnic groups, as a mixtures two fluids with the same specific density ρ.Thus, we are interested to study the mean velocity $\mathbf{v}$ of the mixture, defined by

$$\mathbf{v} = \frac{\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2}{\rho} \tag{2}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity related with the components A_1 and A_2. Thus, we suppose that the evolution of the mixture may be represented as a motion of a viscous incompressible fluid by the system

$$\rho \dot{\mathbf{v}} = -\nabla p + \rho \nabla \cdot (\nabla c \otimes \nabla c) + \nabla \cdot \upsilon(c) \nabla \mathbf{v} + \rho \mathbf{b} \tag{3}$$

where p is the pressure, $\upsilon(c)$ is the viscosity coefficient depending on c, the vector $\mathbf{b}$ denotes the external body forces. Moreover, because we suppose $\rho(x,t) = const.$, we have $\nabla \cdot \mathbf{v} = 0$. For this problem is natural to suppose slow motions, so $\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t}$ and the term $\nabla \cdot (\nabla c \otimes \nabla c)$ will be negligible.

The behavior of the components of the mixture will be described by the Cahn-Hilliard equation, which allows to study the evolution of the concentration c by

$$\rho \dot{c} = \nabla \cdot M(c) \nabla \mu \tag{4}$$

where the function $M(c)$ is the mobility such that

$$M(c) \geq 0 \quad , \quad M(-1) = M(1) = 0 \tag{5}$$

while μ is called embedded (supplemented) potential. Which, in analogy with chemical potential, describes the slope of the internal energy with respect to variation of composition of two species, defined by

$$\mu(c, \varphi_1, \varphi_2) = \gamma \nabla^2 c - \varphi_0 F'(c) - \frac{\varphi_1 + \varphi_2}{2} G'(c) \tag{6}$$

where the potentials F and G are defined by

$$F(c) = \frac{1}{4}(c^2 - 1)^2 \quad , \quad G(c) = \frac{c^2}{2} \tag{7}$$

while $\varphi_1 > 0$ and $\varphi_2 > 0$ represent the knowledge levels of the two components, while φ_0 is a critical value, which denotes the integration-separation phase transition, controlled by the mean value $u = \frac{\varphi_1 + \varphi_2}{2}$. Hence, we obtain by (4) and (6) the equation on the concentration c

$$\rho \dot{c} = \nabla \cdot M(c) \nabla (\gamma \nabla^2 c - \varphi_0 F'(c) - \frac{\varphi_1 + \varphi_2}{2} G'(c)) \tag{8}$$

The free energy W related with the equation (8) is given for homogeneous states by

$$W(\varphi, u) = \varphi_0 F(c) + u G(c)$$

Then, we study the evolution of the educational level by a diffusion equation. So that, it is possible to observe that the culture show a diffusive behavior similar to the heat diffusion. (Both are extensive variables. On the other side, the knowledge and the temperature are intensive variables.) Thus, from this similarity, we introduce two equations related with the knowledge, which describe the cultural balance laws of two ethnic groups

$$\rho\dot{\varphi}_1 - \frac{1}{2}\dot{G}(c) + \frac{\nu}{2}\nabla^2\mathbf{v} = -\nabla\cdot\mathbf{p}_1 - \alpha(\varphi_1 - \varphi_2) + \rho\gamma_1 \tag{9}$$

$$\rho\dot{\varphi}_2 - \frac{1}{2}\dot{G}(c) + \frac{\nu}{2}\nabla^2\mathbf{v} = -\nabla\cdot\mathbf{p}_2 - \alpha(\varphi_2 - \varphi_1) + \rho\gamma_2 \tag{10}$$

where $\alpha \geq 0$, γ_1 and γ_2 represent the cultural supplies, while $\mathbf{p}_1$ and $\mathbf{p}_2$ denote the cultural fluxes related with the knowledge φ_1 and φ_2 by the constitutive equations

$$\mathbf{p}_1 = -\delta_1\nabla\varphi_1 \tag{11}$$

$$\mathbf{p}_2 = -\delta_2\nabla\varphi_2 \tag{12}$$

with two cultural conductivities δ_1,$\delta_2 > 0$ connected with the components A_1 and A_2.

Therefore, the differential system is given by the equations (3), (8), (9) and (10) with the boundaries conditions

$$
\begin{aligned}
M(c)\nabla\mu\cdot\mathbf{n}|_{\partial\Omega} &= 0\ ,\quad \mathbf{v}(t)|_{\partial\Omega}=0\ ,\quad \nabla c(t)\cdot\mathbf{n}|_{\partial\Omega}=0\ , \\
\nabla\varphi_1(t)\cdot\mathbf{n}|_{\partial\Omega} &= 0\ ,\quad \nabla\varphi_2(t)\cdot\mathbf{n}|_{\partial\Omega}=0
\end{aligned}
\tag{13}
$$

where $\mathbf{n}$ is the unit outward normal.and the initial conditions

$$
\begin{aligned}
\mathbf{v}(x,0) &= \mathbf{v}_0(x)\ ,\ c(x,0)=c_0(x)\quad ,\quad x\in\Omega \\
\varphi_1(x,0) &= \varphi_{10}(x)\ ,\ \varphi_2(x,0)=\varphi_{20}(x)\quad ,\quad x\in\Omega
\end{aligned}
\tag{14}
$$

Dissipation for a phenotype system

Now, there is a second important step. In fact, if we consider the differential system of a binary mixture with two different temperatures, we should impose the laws of thermodynamics.

As a research field, the thermodynamics of life systems remains obscure within science and esoteric to the researches.

Anyway, in our framework, we need to consider the natural restrictions on the constitutive equations. About this question, we can follow two different ways. The first supposes a complete analogy between heat and culture, temperature and knowledge. So, it is easy to generalize the thermodynamics to these life systems, as the study of the restrictions, which we have to associate with any particular phenotype organism. In the second way, we do not consider any similarity between heat and culture. So, the equations on knowledge will be considered as new diffusive equations, without any connection with temperature. In this framework, the stability conditions need of a Dissipation Principle, which may be defined as the quantitative study of the energy dissipation, that occurs among living structures. Similarly, the Thermodynamics of Life Science studies the restrictions on the set of living processes by a First and Second Law. These two frameworks, although similar, involve different restrictions, which lead to different equations.

Let us begin with the dissipation principle, for which we need to define for the differential system (3), (8), (9) and (10) the internal mechanical power $\mathcal{P}^i_m$, the internal structural power $\mathcal{P}^i_c$, the internal cultural power $\mathcal{P}^i_\varphi$ defined respectively

$$\mathcal{P}^i_m = \upsilon(c)(\nabla \mathbf{v})^2 \tag{15}$$

$$\mathcal{P}^i_c = \rho\gamma\frac{(\nabla c^2)^{\cdot}}{2} + \varphi_0\dot{F}(c) + \frac{1}{2}(\varphi_1+\varphi_2)\dot{G}(c) + M(c)(\nabla\mu)^2 \tag{16}$$

$$\mathcal{P}^i_\varphi = \rho_1\dot{\varphi}_1\varphi_1 + \rho_1\dot{\varphi}_2\varphi_2 - \frac{1}{2}(\varphi_1+\varphi_2)\dot{G}(c) + \alpha(\varphi_1-\varphi_2)^2 + \delta((\nabla\varphi_1)^2) + \delta((\nabla\varphi_2)^2) \quad (17)$$

Thus, we can put out

Dissipation Principle

On any living process, there exists a state function ψ, called free energy such that, we have

$$\rho\dot{\psi} \leq \mathcal{P}^i_m + \mathcal{P}^i_c + \mathcal{P}^i_\varphi \quad (18)$$

Proposition. The differential system (3), (8), (9) and (10) satisfies the Dissipation Principle (19).

Proof. By (15)-(17), the inequality (18) assumes the form

$$\rho\dot{\psi} \leq \upsilon(c)(\nabla\mathbf{v})^2 + \rho\gamma\frac{(\nabla c^2)^{\cdot}}{2} + \varphi_0\dot{F}(c) + M(c)(\nabla\mu)^2 + \rho_1\dot{\varphi}_1\varphi + \rho_2\dot{\varphi}_2\varphi_2 + \alpha(\varphi_1-\varphi_2)^2 + \delta((\nabla\varphi_1)^2) + \delta((\nabla\varphi_2)^2)$$

hence, the free energy is given by

$$\rho\psi = \rho\gamma\frac{(\nabla c^2)}{2} + \varphi_0 F(c) + \frac{1}{2}(\rho_1\varphi_1^2 + \rho_2\varphi_2^2)) \tag{19}$$

Thus, we obtain by (18) the inequality

$$\upsilon(c)(\nabla \mathbf{v})^2 + M(c)(\nabla\mu)^2 + \alpha(\varphi_1 - \varphi_2)^2 + \delta((\nabla\varphi_1)^2) + \delta((\nabla\varphi_2)^2 \geq$$

so, the coefficients $\upsilon(c)$, $M(c), \delta$ and α must be non-negative.

Thermodynamics in life science

Here, the evolution of two ethnic populations can be described by a different view point, where the cultural power is fully assimilated to heat power (*extensive variables*) and the analogy between the knowledge and the temperature is more apparent (*intensive variables*). Moreover, for our mixture, which have two knowledge, the analogy is with a mixture of two fluids with two different temperatures. So, it is evident to use the Laws of Thermodynamics for describing the restrictions and the property of this particular phenotype system.

First Law. On any life process, there exits a state function e, called internal energy, such that

$$\rho \dot{e} = \mathcal{P}^i_m + \mathcal{P}^i_c + \tilde{\mathcal{P}}^i_\varphi \tag{20}$$

where the power $\tilde{\mathcal{P}}^i_\varphi$ is the new representation of the internal cultural power related with this new life system, defined by

$$\tilde{\mathcal{P}}^i_\varphi = \tilde{\mathcal{P}}^i_{\varphi_1} + \tilde{\mathcal{P}}^i_{\varphi_2}$$

where $\tilde{\mathcal{P}}^i_{\varphi_1}$ and $\tilde{\mathcal{P}}^i_{\varphi_2}$ have to satisfy the cultural balance laws

$$\tilde{\mathcal{P}}^i_{\varphi_1} = \nabla \cdot \mathbf{p}_1 + \rho s_1 \tag{21}$$

$$\tilde{\mathcal{P}}^i_{\varphi_2} = \nabla \cdot \mathbf{p}_2 + \rho s_2 \tag{22}$$

For this model, $\tilde{\mathcal{P}}^i_\varphi$ will be defined by the First Law by the equation

$$\tilde{\mathcal{P}}^i_\varphi = \rho\dot{e} - \mathcal{P}^i_m - \mathcal{P}^i_c = \tag{23}$$

$$\rho\dot{e} - \upsilon(c)(\nabla\mathbf{v})^2 - \rho\gamma\frac{(\nabla c^2)^{\cdot}}{2} - \varphi_0\dot{F}(c) + \frac{1}{2}(\varphi_1+\varphi_2)\dot{G}(c) + M(c)(\nabla\mu)$$

Thus, we suppose

$$\rho e(c, \nabla c, \varphi_1, \varphi_2) = \rho_1\tilde{e}_1(\varphi_1) + \rho_2\tilde{e}_2(\varphi_2) + \varphi_0 F(c) + \gamma\frac{(\nabla c)^2}{2} \tag{24}$$

So, we can define

$$\tilde{\mathcal{P}}^i_{\varphi_1} = \rho_1 \tilde{e}_{1\varphi_1}(\varphi_1)\dot{\varphi}_1 - \frac{\upsilon(c)}{2}(\nabla \mathbf{v})^2 - \frac{M(c)}{2}(\nabla \mu)^2 - \frac{\rho}{2}\varphi_1 \dot{G}(c) \tag{25}$$

$$\tilde{\mathcal{P}}^i_{\varphi_2} = \rho_2\ \tilde{e}_{2\varphi_2}(\varphi_2)\dot{\varphi}_2 - \frac{\upsilon(c)}{2}(\nabla \mathbf{v})^2 - \frac{M(c)}{2}(\nabla \mu)^2 - \frac{\rho}{2}\varphi_2 \dot{G}(c) \tag{26}$$

Finally, by (22), (23) and (26), (27) the equations on φ_1 and φ_2 assume the new forms

$$\rho_1\ \tilde{e}_{1\varphi_1}\dot{\varphi}_1 - \frac{\upsilon(c)}{2}(\nabla \mathbf{v})^2 - \frac{M(c)}{2}(\nabla \mu)^2 - \frac{\varphi_1}{2}\dot{G}(c) = k_1 \nabla^2 \varphi_1 + \rho_1 s_1 \tag{27}$$

$$\rho_2\ \tilde{e}_{2\varphi_2}\dot{\varphi}_2 - \frac{\upsilon(c)}{2}(\nabla \mathbf{v})^2 - \frac{M(c)}{2}(\nabla \mu)^2 - \frac{\varphi_2}{2}\dot{G}(c) = k_2 \nabla^2 \varphi_2 + \rho_2 s_2 \tag{28}$$

Thus, the system (3), (8) is integrated with the equations (28) and (29).

Second law. On any life process, there exits a state function η, called entropy, such that

$$\rho\dot{\eta} \geq \frac{\tilde{\mathcal{P}}^i_{\varphi_1}}{\varphi_1} + \frac{\tilde{\mathcal{P}}^i_{\varphi_2}}{\varphi_2} + \mathbf{p}_1 \cdot \nabla\varphi_1 + \mathbf{p}_2 \cdot \nabla\varphi_2 \tag{29}$$

Proposition. The differential system (3), (8), (28) and (29) satisfies the First and Second Law.

Proof. By the definition of $\tilde{\mathcal{P}}^i_{\varphi_1}$ and $\tilde{\mathcal{P}}^i_{\varphi_2}$, we have from the inequality (30)

$$\begin{aligned} \rho\dot{\eta} \ \geq \ & \rho_1 \ \frac{\tilde{e}_{1\varphi_1}(\varphi_1)}{\varphi_1}\dot{\varphi}_1 - \frac{\upsilon(c)}{2\varphi_1}(\nabla\mathbf{v})^2 - && (30) \\ & \frac{M(c)}{2\varphi_1}(\nabla\mu)^2 - \frac{\rho}{2}\dot{G}(c) + \mathbf{p}_1 \cdot \nabla\varphi_1 + \\ & \rho_2\frac{\tilde{e}_{2\varphi_2}(\varphi_2)}{\varphi_2}\dot{\varphi}_2 - \frac{\upsilon(c)}{2\varphi_2}(\nabla\mathbf{v})^2 - && (31) \\ & \frac{M(c)}{2\varphi_2}(\nabla\mu)^2 - \frac{\rho}{2}\dot{G}(c) + \mathbf{p}_2 \cdot \nabla\varphi_2 \end{aligned}$$

from which the entropy is defined by

$$\rho\eta(c, \varphi_1, \varphi_2) = \rho G(c) + \int \ \frac{\rho_1\tilde{e}_{1\varphi_1}(\varphi_1)}{\varphi_1}d\varphi_1 + \int \ \frac{\rho_2\tilde{e}_{1\varphi_1}(\varphi_1)}{\varphi_1}d\varphi_1$$

while from (31), we have

$$\left(\frac{1}{\varphi_1}+\frac{1}{\varphi_2}\right)\frac{\upsilon(c)}{2}(\nabla\mathbf{v})^2+ \tag{32}$$

$$\left(\frac{1}{\varphi_1}+\frac{1}{\varphi_2}\right)\frac{M(c)}{2}(\nabla\mu)^2-\mathbf{p}_1\cdot\nabla\varphi_1-\mathbf{p}_2\cdot\nabla\varphi_2\geq 0$$

Hence, for the arbitrariness of the processes $\nabla\mathbf{v}$,$\nabla\mu$, $\nabla\varphi_1$, $\nabla\varphi_2$, we obtain the restrictions

$$\upsilon(c)\geq 0\ ,\quad M(c)\geq 0\ ,\ \mathbf{p}_1\cdot\nabla\varphi_1\leq 0\ ,\ \mathbf{p}_2\cdot\nabla\varphi_2\leq 0 \tag{33}$$

Maximum principle

If we like that the Cahn–Hilliard equation describes a natural physical problem, we have to prove a maximum theorem, namely we have to show that the evolution equations imply that the concentration c is always defined into the interval $[-1,1]$.

To this aim, remembering that the chemical potential is given by

$$\mu = -\frac{\gamma}{\rho}\nabla\cdot(\rho\nabla c) + \varphi_0 F_c(c) + \left[\frac{\varphi_1+\varphi_2}{2}\right] G_c(c), \tag{34}$$

we recall the definition of F and G by letting

$$F(c) = \frac{1}{4}(c^2-1)^2 \ , \qquad c\in\mathbb{R} \tag{35}$$

$$G(c) = \frac{1}{2}\begin{cases} c^2 & -1\le c\le 1 \\ 1 & c<-1 \ \ or \ \ c>1 \end{cases} \tag{36}$$

Hence $F\ge 0$ and F vanishes only at $c=-1,1$. Moreover, by (36)–(37) we have

$$F_c(c) = 4c(c^2-1) \ , \qquad c\in\mathbb{R} \tag{37}$$

$$G_c(c) = \begin{cases} c & -1<c<1 \\ 0 & c<-1 \ \ or \ \ c>1 \end{cases} \tag{38}$$

We denote by W the $c-$dependent part of the free energy, that is

$$W(c) = \varphi_0 F(c) + uG(c), \qquad u = \frac{\varphi_1+\varphi_2}{2}.$$

The function W has a unique minimum when $u \geq 4\varphi_0$, while for $u < 4\varphi_0$ it has two minima in $c_\pm$, with $|c_\pm| < 1$. It is known that the unique minimum in the potential corresponds to the situation without a miscibility gap, while in the regime with two minima there is a miscibility gap.

Finally, the mobility can be chosen as a positive function depending on c. The dependence of mobility on the concentration is not new in literature: it appeared for the first time in the original derivation of the Cahn-Hilliard equation and later other authors considered different expressions for $M(c)$. Here the mobility $M(c)$ is taken in the form

$$M(c) = M_0(c^2 - 1)^2 \ , \qquad M_0 > 0,$$

which implies that both M and ∇M vanish at $c = -1, 1$. Furthermore, the mass density is such that

$$\rho(c) = \rho_{20} \ , \quad c < -1 \ , \qquad \rho(c) = \rho_{10} \ , \quad c > -1 \ .$$

In such a way ρ is extended to $\mathbb{R}$.

Now we consider the initial value problem

$$\rho(c)\dot{c} = \nabla \cdot [M(c)\nabla\mu(c)] \qquad c(\mathbf{x}, 0) = c_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega \quad (39)$$

Theorem. Let $c_0(\mathbf{x}) \in [-1,1]$ for any $\mathbf{x} \in \Omega$, then the solution $c(\mathbf{x},t)$ of equation (40) takes value in $[-1,1]$ a.e $\mathbf{x} \in \Omega$ and for each $t \in \mathbb{R}^+$.

Proof. We introduce

$$c_- = \begin{cases} -1, & c \geq -1 \\ c, & c < -1 \ . \end{cases}$$

Accordingly,

$$c_-(\mathbf{x},0) = -1, \qquad F(c_-(\mathbf{x},0)) = 0 \quad \forall \mathbf{x} \in \Omega;$$

moreover

$$\frac{\partial c}{\partial t} G_c(c) = 0 \ , \qquad c \notin (-1,1)$$

hence

$$\frac{\partial c}{\partial t}\mu\left(c_-\right) = \theta_0 \frac{\partial c}{\partial t} F_c\left(c_-\right) - \frac{\partial c}{\partial t}\frac{\gamma}{\rho}\nabla \cdot \left[\rho \nabla c_-\right] \ , \qquad c \notin (-1,1) \ . \tag{40}$$

Multiply the differential equation in (40) by $\mu(c_-)$. Then, after an integration on Ω and by the divergence theorem we obtain

$$\int_\Omega \rho \frac{\partial c}{\partial t}\mu\left(c_-\right) dv = -\int_\Omega M(c)\nabla\mu\left(c\right) \cdot \nabla\mu\left(c_-\right) dv \tag{41}$$

Looking at the left–hand side of (42) and applying (41) we obtain

$$\begin{aligned} & \int_\Omega \rho \, \frac{\partial c}{\partial t} \mu\left(c_-\right) dv \\ = \; & \theta_0 \int_\Omega \rho \frac{\partial c}{\partial t} F_c\left(c_-\right) dv - \gamma \int_\Omega \frac{\partial c}{\partial t} \nabla \cdot \left[\rho \nabla \, c_-\right] dv \end{aligned}$$

Since $\nabla c_- \cdot \mathbf{n} = 0$ at $\partial\Omega$, the divergence theorem gives

$$\int_\Omega \frac{\partial c}{\partial t} \nabla \cdot \left[\rho \, \nabla \, c_-\right] dv = - \int_\Omega \rho \nabla \left[\frac{\partial c}{\partial t}\right] \nabla \, c_- dv$$

Because

$$\nabla \left[\frac{\partial c}{\partial t} - (c_-)^{\cdot}\right] \cdot \nabla \, c_- = 0$$

so, we have

$$\int_\Omega \frac{\partial c}{\partial t} \nabla \cdot \left[\rho \, \nabla \, c_-\right] dv = -\frac{1}{2} \frac{d}{dt} \int_\Omega \rho \, \left|\nabla \, c_-\right|^2 dv \qquad (42)$$

Moreover, since $F_c(-1) = 0$ then

$$\left[\frac{\partial c}{\partial t} - \frac{\partial (c_-)}{\partial t}\right] F_c(c_-) = 0;$$

so we have

$$\int_{\Omega} \rho \frac{\partial c}{\partial t} F_c (c_-)\, dv = \frac{d}{dt} \int_{\Omega} \rho\, F(c_-)\, dv \tag{43}$$

Finally, we have from (43) and (44)

$$\frac{d}{dt} \int_{\Omega} \rho\ (c) \left[\theta_0 F(c_-) + \frac{\gamma}{2} |\nabla\, c_-|^2 \right] dv =$$

$$- \int_{\Omega} M(c_-) \nabla \mu\ (c_-) \cdot \nabla \mu\, (c_-)\, dv \le 0$$

After a time integration on $[0, T]$ and by the initial conditions we obtain

$$\int_{\Omega} \rho\ \left[\theta_0 F(c_-) + \frac{\gamma}{2} |\nabla\, c_-|^2 \right] |_{t=T}\ dv \le 0 \qquad T \in \mathbb{R}^+\ .$$

This implies $F(c_-(\mathbf{x}, T)) = 0$, $\nabla c_-(\mathbf{x}, T) = 0$. Since, for any $T > 0$ we have

$$c_-(\mathbf{x}, T) = -1 \qquad c(\mathbf{x}, T) \ge -1\ .$$

By a very similar proof we can show that $c(\mathbf{x}, T) \le 1$.

The differential system for the separation-integration model

In the framework of the Section 3, the differential system is given in the 2-dimensional domain $\Omega \times (0, T)$ by

$$\rho \dot{\mathbf{v}} = -\nabla p + \rho \nabla \cdot (\nabla c \otimes \nabla c) + \nabla \cdot \upsilon(c) \nabla \mathbf{v} + \rho \mathbf{b}, \;\; \nabla \cdot \mathbf{v} = 0 \quad (44)$$

$$\rho \dot{c} = \nabla \cdot M(c) \nabla (\gamma \nabla^2 c - \varphi_0 F'(c) - \frac{\varphi_1 + \varphi_2}{2} G'(c)) \quad (45)$$

$$\rho_1 \, \tilde{e}_{1\varphi_1} \dot{\varphi}_1 - \frac{\upsilon(c)}{2} (\nabla \mathbf{v})^2 - \frac{M(c)}{2} (\nabla \mu)^2 - \frac{\varphi_1}{2} \dot{G}(c) = k_1 \nabla^2 \varphi_1 + \rho_1 s_1 \quad (46)$$

$$\rho_2 \, \tilde{e}_{2\varphi_2} \dot{\varphi}_2 - \frac{\upsilon(c)}{2} (\nabla \mathbf{v})^2 - \frac{M(c)}{2} (\nabla \mu)^2 - \frac{\varphi_2}{2} \dot{G}(c) = k_2 \nabla^2 \varphi_2 + \rho_2 s_2 \quad (47)$$

with the boundary conditions (13). While the initial conditions are given in (14).

There are meaningful problems for which is correct to neglect the mean velocity $\mathbf{v}$ of the mixture. Then, in a such a case, we obtain the new system

$$\rho\frac{\partial c}{\partial t} = \nabla \cdot M(c)\nabla(\gamma\nabla^2 c - \varphi_0 F'(c) - \frac{\varphi_1 + \varphi_2}{2}G'(c)) \qquad (48)$$

$$\rho_1\ \tilde{e}_{1\varphi_1}(\varphi_1)\frac{\partial\varphi_1}{\partial t} - \frac{M(c)}{2}(\nabla\mu)^2 - \frac{\varphi_1}{2}\dot{G}(c) = k_1\nabla^2\varphi_1 + \rho_1 s_1 \qquad (49)$$

$$\rho_2\ \tilde{e}_{2\varphi_2}(\varphi_2)\frac{\partial\varphi_2}{\partial t} - \frac{M(c)}{2}(\nabla\mu)^2 - \frac{\varphi_2}{2}\dot{G}(c) = k_2\nabla^2\varphi_2 + \rho_2 s_2 \qquad (50)$$

For this new system, the boundary conditions are given on $\partial\Omega$

$$\begin{aligned} M(c)\nabla\mu\cdot\mathbf{n} &= 0 \quad \nabla c(x,t)\cdot\mathbf{n}(x)|_{\partial\Omega} = 0\ , \\ \nabla\varphi_1(x,t)\cdot\mathbf{n}(x)|_{\partial\Omega} &= 0\ , \ \ \nabla\varphi_2(x,t)\cdot\mathbf{n}(x)|_{\partial\Omega} = 0 \end{aligned} \qquad (51)$$

Finally, we assign the initial conditions

$$\begin{aligned} \mathbf{v}(x,0) &= \mathbf{v}_0(x)\ ,\ c(x,0) = c_0(x) \quad , \quad x \in \Omega \\ \varphi_1(x,0) &= \varphi_{10}(x)\ ,\ \varphi_2(x,0) = \varphi_{20}(x) \quad , \quad x \in \Omega \end{aligned} \tag{52}$$

In order to describe the integration-separation process, it is crucial for this system to study the numerical simulation of Cahn-Hilliard equation (49). For this purpose, we will consider only the Cahn-Hilliard equation, where $(\varphi_1+\varphi_2)$ is a parameter, which represents the mean knowledge level of two populations. In the Fig. 1, 2, 3, 4, we have represented the evolution of the integration-separation phase fields, with different values of the control $u = \frac{\varphi_1+\varphi_2}{2}$, such that $u_1 < u_2 < u_3 < u_4 < \varphi_0$.

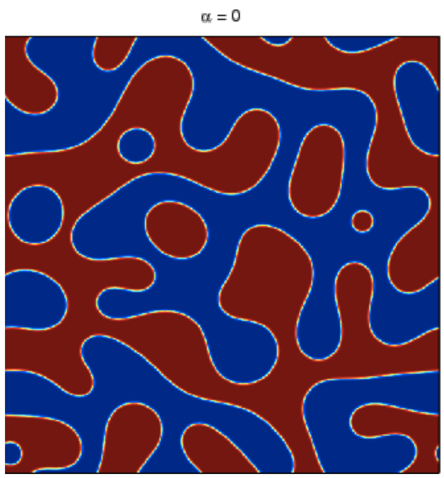

Fig.1

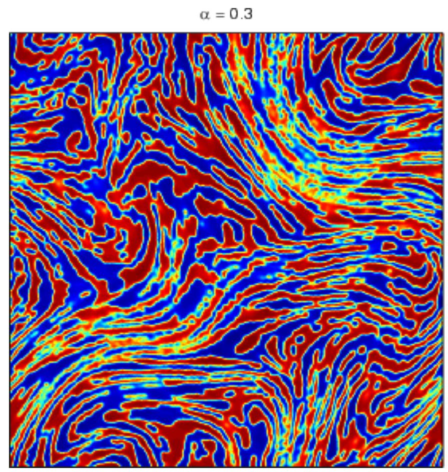

Fig.2

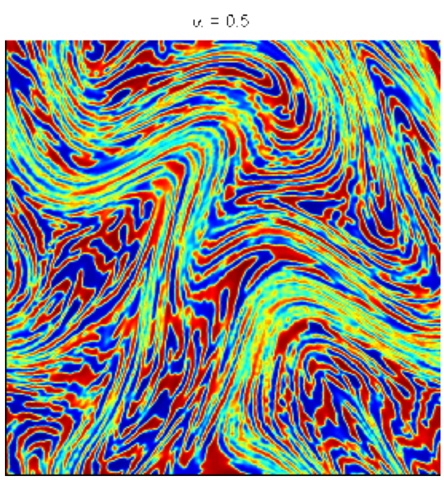

Fig.3

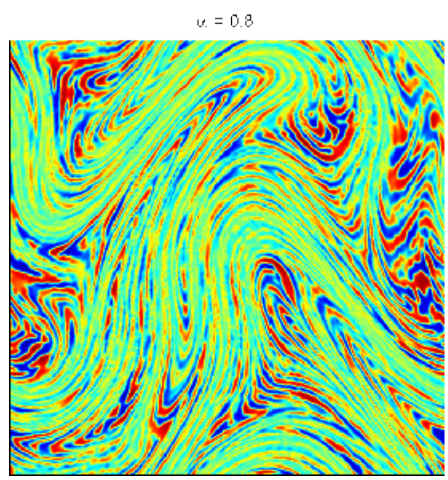

Fig.4

Finally, when $u \geq \varphi_0$ we obtain the complete homogenization represented in the Fig. 5.

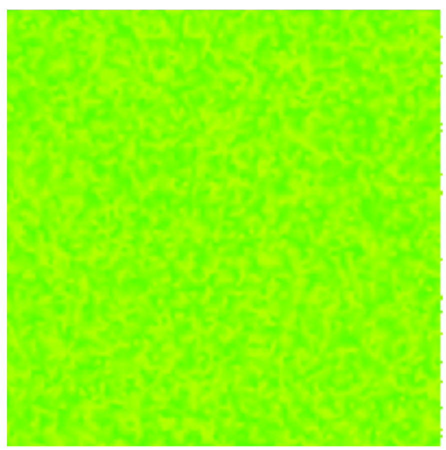

Fig.5