Asymptotics of the fractional perimeter functionals

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Definition

Given $s \in (0,1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the s-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in Ω is defined as

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) := {} & L(E\cap\Omega, (\mathcal{C}E)\cap\Omega) \\ & + L(E\cap\Omega, (\mathcal{C}E)\cap(\mathcal{C}\Omega)) + L(E\cap(\mathcal{C}\Omega), (\mathcal{C}E)\cap\Omega), \end{aligned} \tag{1}$$

where $\mathcal{C}E = \mathbb{R}^n \setminus E$ denotes the complement of E, and $L(A,B)$ denotes the following nonlocal interaction term

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy \qquad \forall A,B \subseteq \mathbb{R}^n. \tag{2}$$

This notion of s-perimeter and the corresponding minimization problem were introduced in (Caffarelli-Roquejoffre-Savin, CPAM 2010).

Motivations

The limits as $s \searrow 0$ and $s \nearrow 1$ are somehow the critical cases for the s-perimeter, since the functional in (1) diverges as it is.

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In (Caffarelli-Valdinoci, CVPDE 2011) and (Ambrosio-De Philippis-Martinazzi, MM 2011) it was shown that

$$(1-s)\mathrm{Per}_s \to \mathrm{Per}, \qquad \text{as } s \nearrow 1$$

(up to normalizing multiplicative constants).

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$$\Downarrow$$

Surfaces of minimal s-perimeter inherit the regularity properties of the classical minimal surfaces for s sufficiently close to 1; see (Caffarelli-Valdinoci, Preprint).

Preliminaries

We are interested in the quantity

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whenever the limit exists. Of course, if it exists then

$$\mu(E) = \mu(\mathcal{C}E),$$

since

$$\mathrm{Per}_s(E; \Omega) = \mathrm{Per}_s(\mathcal{C}E; \Omega).$$

A special case

First we take $E \subset \Omega$. We have

$$sPer_s(E;\Omega) = sL(E,\mathcal{C}E) = \frac{s}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+s}} = \frac{s}{2}\,[\chi_E]_{H^s(\mathbb{R}^n)}\,.$$

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A result in (Maz'ya-Shaposhnikova, JFA 2002) implies that

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$$\lim_{s \searrow 0} s\mathrm{Per}_s(E;\Omega) = \omega_{n-1}\,|\chi_E|^2_{L^2(\mathbb{R}^n)}\,,$$

that is,

$$\mu(E) = \omega_{n-1}|E| =: \mathcal{M}(E),$$

the *normalized Lebesgue measure*.

Some properties of μ

We define $\mathcal{E}$ to be the family of sets $E \subseteq \mathbb{R}^n$ for which the limit defining $\mu(E)$ in (3) exists.

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Hence, a natural question: is μ a measure? No.

Some properties of μ

Proposition

μ is not necessarily additive on separated sets in $\mathcal{E}$, i.e. there exist $E, F \in \mathcal{E}$ such that $\operatorname{dist}(E, F) \geq c > 0$, but $\mu(E \cup F) < \mu(E) + \mu(F)$.

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For this, we observe that if $x \in B_1$ and $y \in \mathcal{C}B_2$ then $|x - y| \leq |x| + |y| \leq 2|y|$, therefore

$$sL(B_1, \mathcal{C}B_2) \geq c_1 s \int_{B_1} dx \int_{\mathcal{C}B_2} dy \frac{1}{|y|^{n+s}} \geq c_2 s \int_2^{+\infty} \frac{d\rho}{\rho^{1+s}} \geq c_3,$$

for some positive constants c_1, c_2 and c_3.

Some properties of μ

Now we take $E := \mathcal{C}B_2$, $F := \Omega := B_1$. Then

$$\begin{aligned}
\mathrm{Per}_s(E;\Omega) &= L(B_1, \mathcal{C}B_2),\\
\mathrm{Per}_s(F;\Omega) &= L(B_1, \mathcal{C}B_1) = L(B_1, \mathcal{C}B_2) + L(B_1, B_2 \setminus B_1)\\
\mathrm{Per}_s(E \cup F;\Omega) &= L(B_1, B_2 \setminus B_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
s\,\mathrm{Per}_s(E;\Omega) + s\,\mathrm{Per}_s(F;\Omega) &= 2sL(B_1, \mathcal{C}B_2) + sL(B_1, B_2 \setminus B_1)\\
&\geq 2c_3 + s\,L(B_1, B_2 \setminus B_1)\\
&= 2c_3 + s\,\mathrm{Per}_s(E \cup F;\Omega).
\end{aligned}$$

By sending $s \searrow 0$, we conclude that $\mu(E) + \mu(F) \geq 2c_3 + \mu(E \cup F)$, so μ is not additive.

Some properties of μ

Proposition

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For this we take E such that $\mu(E) > 0$ (for instance, one can take E a small ball inside Ω), and $F := \mathbb{R}^n$: with this choice, $E \subset F$ and $\mathrm{Per}_s(F; \Omega) = 0$, so $\mu(E) > 0 = \mu(F)$.

Some properties of μ

On the other hand, in some circumstances the additivity property holds true:

Proposition

μ is additive on bounded, separated sets in $\mathcal{E}$, i.e. if $E, F \in \mathcal{E}$, E and F are bounded, disjoint and $\operatorname{dist}(E, F) \geq c > 0$, then $E \cup F \in \mathcal{E}$ and $\mu(E \cup F) = \mu(E) + \mu(F)$.

The main result

There is a natural condition under which $\mu(E)$ does exist, based on the behavior of the set E towards infinity, which is encoded in the quantity

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (\mathcal{C}B_1)} \frac{1}{|y|^{n+s}}\, dy. \tag{4}$$

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We set

$$\tilde{\alpha}(E) := \frac{\alpha(E)}{\omega_{n-1}}. \tag{5}$$

The main results

We have the following:

Theorem

Suppose that $\mathrm{Per}_{s_0}(E;\Omega) < \infty$ *for some* $s_0 \in (0,1)$*, and that the limit in (4) exists. Then* $E \in \mathcal{E}$ *and*

$$\mu(E) = \big(1 - \tilde{\alpha}(E)\big)\,\mathcal{M}(E \cap \Omega) + \tilde{\alpha}(E)\,\mathcal{M}(\Omega \setminus E). \tag{6}$$

The main results

In particular,

Corollary

Let E be a bounded set, and $\mathrm{Per}_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0,1)$. Then $E \in \mathcal{E}$ and

$$\mu(E) = \mathcal{M}(E \cap \Omega).$$

In particular, if $E \subseteq \Omega$ and $\mathrm{Per}_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0,1)$, then $\mu(E) = \mathcal{M}(E)$.

The main results

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Theorem

Suppose that $\mathrm{Per}_{s_0}(E;\Omega) < \infty$*, for some* $s_0 \in (0,1)$*. Then:*

(i) *If* $|\Omega \setminus E| = |E \cap \Omega|$*, then* $E \in \mathcal{E}$ *and* $\mu(E) = \mathcal{M}(E \cap \Omega)$*.*

(ii) *If* $|\Omega \setminus E| \neq |E \cap \Omega|$ *and* $E \in \mathcal{E}$*, then the limit in (4) exists and*

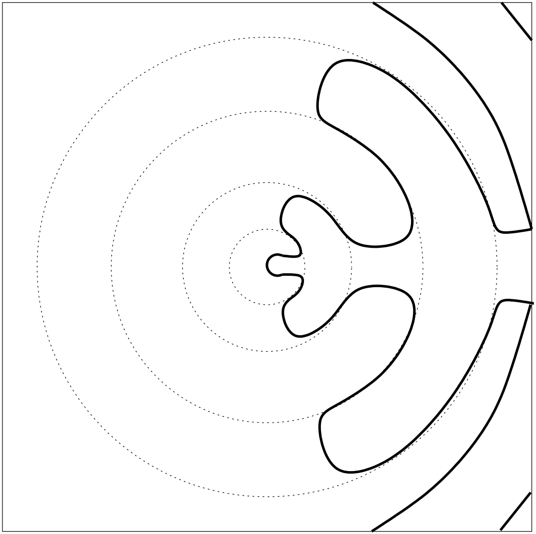
$$\alpha(E) = \frac{\mu(E) - \mathcal{M}(E \cap \Omega)}{|\Omega \setminus E| - |E \cap \Omega|}.$$

Example 1

Example

There exists a set E with C^∞-boundary for which the limits defining $\mu(E)$ in (3) and $\alpha(E)$ in (4) do not exist.

Example 1



Example 1

We start with some preliminary comutations. Let $a_k := 10^{k^2}$, for any $k \in \mathbb{N}$. For any $j \in \{0, 1, 2, 3\}$, let

$$I_j := \Big\{ x \in \mathbb{R} \text{ s.t. } \exists k \in \mathbb{N} \text{ s.t. } x \in \big[a_{4k+j}, a_{4k+j+1} \big) \Big\}.$$

Notice that $[1, +\infty)$ may be written as the disjoint union of the I_j's.

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Notice that $[1, +\infty)$ may be written as the disjoint union of the I_j's.
Let $\varphi \in C^\infty\big([0, +\infty), [0, 1]\big)$ be such that $\varphi = 0$ in $[0, 1] \cup I_0$, $\varphi = 1$ in I_2, and then φ smoothly interpolates between 0 and 1 in $I_1 \cup I_3$.

Example 1

We claim that there exist two sequences $\nu_{0,k} \to +\infty$ and $\nu_{1,k} \to +\infty$ such that

$$\lim_{k\to+\infty} \int_0^{+\infty} \varphi(\nu_{0,k}x)e^{-x}\,dx = 0 \quad \text{and} \quad \lim_{k\to+\infty} \int_0^{+\infty} \varphi(\nu_{1,k}x)e^{-x}\,dx = 1. \tag{7}$$

Example 1

We take $\nu_{0,k} := a_{4k+1}/k$ and $\nu_{1,k} := a_{4k+3}/k$. We observe that, by construction, $\varphi = 0$ in $\left[a_{4k}, a_{4k+1}\right)$ and $\varphi = 1$ in $\left[a_{4k+2}, a_{4k+3}\right)$, so $\varphi(\nu_{0,k}x) = 0$ for any $x \in [kb_{0,k},\, k)$ and $\varphi(\nu_{1,k}x) = 1$ in $[kb_{1,k},\, k)$, where

$$b_{0,k} := \frac{a_{4k}}{a_{4k+1}} = 10^{-(8k+1)} \text{ and } b_{1,k} := \frac{a_{4k+2}}{a_{4k+3}} = 10^{-(8k+5)}.$$

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We deduce that

$$\int_0^{+\infty} \varphi(\nu_{0,k}x)e^{-x}\,dx \le \int_0^{kb_{0,k}} e^{-x}\,dx + \int_k^{+\infty} e^{-x}\,dx = 1 - e^{-kb_{0,k}} + e^{-k}$$

$$\text{and } \int_0^{+\infty} \varphi(\nu_{1,k}x)e^{-x}\,dx \ge \int_{kb_{1,k}}^{k} e^{-x}\,dx = e^{-kb_{1,k}} - e^{-k}.$$

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This implies (7) by noticing that

$$\lim_{k\to+\infty} kb_{0,k} = 0 = \lim_{k\to+\infty} kb_{1,k}.$$

Example 1

Now we construct our example. We take $\Omega := B_{1/2}$ and $E := \left\{x = (\rho\cos\gamma, \rho\sin\gamma)\,, \rho > 1, \gamma \in [0, \theta(\rho)]\,\right\} \subset \mathbb{R}^2$, where $\theta(\rho) := \varphi(\log\rho)$. Then,

$$
\begin{aligned}
s \int_{E\cap(\mathcal{C}B_1)} \frac{1}{|y|^{n+s}} dy &= s\int_1^{+\infty}\int_0^{\theta(\rho)} \frac{\rho^{n-1}}{\rho^{n+s}}\, d\theta\, d\rho \\
&= s\int_1^{+\infty} \theta(\rho)\,\frac{1}{\rho^{1+s}}\, d\rho = s\int_0^{+\infty} \varphi(r)\, e^{-rs}\, dr \\
&= \int_0^{+\infty} \varphi\left(\frac{x}{s}\right) e^{-x}\, dx,
\end{aligned}
$$

by the changes of variable $\log\rho = r$ and $rs = x$.

Example 1

If we set $\nu = 1/s$, the limit in (4) becomes the following:

$$\alpha(\boldsymbol{E}) = \lim_{\nu\to\infty} \int_0^{+\infty} \varphi\,(\nu x)\, e^{-x}\, dx,$$

and, by (7), we get that such a limit does not exist.

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and, by (7), we get that such a limit does not exist.

Since $|\Omega \setminus \boldsymbol{E}| = |\boldsymbol{B}_{1/2}| > 0 = |\boldsymbol{E} \cap \Omega|$, the limit defining $\mu(\boldsymbol{E})$ in (3) does not exist either.

Example 2

Example

There exists a set E with C^∞-boundary for which the limit defining $\mu(E)$ in (3) exists and the limit $\alpha(E)$ in (4) does not exist.

Example 2

It is sufficient to modify Example 1 inside $\Omega = B_{1/2}$ in such a way that $|\Omega \setminus E| = |E \cap \Omega|$. Then (4) is not affected by this modification and so the limit in (4) does not exist in this case too. On the other hand, the limit in (3) exists, thanks to Theorem 3(i).

Example 3

Example

There exists a set E for which $\mathrm{Per}_s(E;\Omega) = +\infty$ for any $s \in (0,1)$.

Example 3

We take a decreasing sequence β_k such that $\beta_k > 0$ for any $k \geq 1$,

$$M := \sum_{k=1}^{+\infty} \beta_k < +\infty$$

but

$$\sum_{k=1}^{+\infty} \beta_{2k}^{1-s} = +\infty \qquad \forall s \in (0,1). \tag{8}$$

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For instance, one can take $\beta_1 := \dfrac{1}{\log^2 2}$ and $\beta_k := \dfrac{1}{k \log^2 k}$ for any $k \geq 2$.

Example 3

Now, we define

$$\Omega := (0, M) \subset \mathbb{R}, \qquad \sigma_m := \sum_{k=1}^{m} \beta_k,$$

$$I_m := (\sigma_m, \sigma_{m+1}), \qquad E := \bigcup_{j=1}^{+\infty} I_{2j}.$$

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Notice that $E \subset \Omega$ and

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) &= L(E, \mathcal{C}E) \geq \sum_{j=1}^{+\infty} L(I_{2j}, I_{2j+1}) \\ &= \sum_{j=1}^{+\infty} \int_{\sigma_{2j}}^{\sigma_{2j+1}} \int_{\sigma_{2j+1}}^{\sigma_{2j+2}} \frac{1}{|x-y|^{1+s}}\, dx\, dy. \end{aligned} \tag{9}$$

Example 3

An integral computation shows that if $a < b < c$ then

$$\int_a^b \int_b^c \frac{1}{|x-y|^{1+s}}\,dx\,dy = \frac{1}{s(1-s)}\Big[(c-b)^{1-s} + (b-a)^{1-s} - (c-a)^{1-s}\Big].$$

Example 3

By plugging this into (9), we obtain

$$\begin{aligned} & s(1-s)\mathrm{Per}_s(E;\Omega) \\ & \quad \geq \sum_{j=1}^{+\infty} \Big[(\sigma_{2j+2}-\sigma_{2j+1})^{1-s} + (\sigma_{2j+1}-\sigma_{2j})^{1-s} - (\sigma_{2j+2}-\sigma_{2j})^{1-s} \Big] \\ & \quad = \sum_{j=1}^{+\infty} \beta_{2j+2}^{1-s} + \beta_{2j+1}^{1-s} - (\beta_{2j+2}+\beta_{2j+1})^{1-s}. \end{aligned} \tag{10}$$

Example 3

Now we observe that the map $[0,1) \ni t \mapsto (1+t)^{1-s}$ is concave, therefore

$$(1+t)^{1-s} \leq 1+(1-s)t \leq 1+(1-s)t^{1-s}$$

for any $t \in [0,1)$, that is

$$1+t^{1-s}-(1+t)^{1-s} \geq st^{1-s}.$$

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By taking $t := \beta_{2j+2}/\beta_{2j+1}$ and then multiplying by β_{2j+1}^{1-s}, we obtain

$$\beta_{2j+1}^{1-s}+\beta_{2j+2}^{1-s}-(\beta_{2j+1}+\beta_{2j+2})^{1-s} \geq s\beta_{2j+2}^{1-s}.$$

Example 3

By plugging this into (10) and using (8), we conclude that

$$\mathrm{Per}_s(E;\Omega) \geq \frac{1}{1-s}\sum_{j=1}^{+\infty}\beta_{2j+2}^{1-s} = +\infty \qquad \forall\, s\in(0,1),$$

as desired.

Thank you very much for your attention!