On a 3D isothermal model for nematic liquid crystals accounting for stretching terms

Cecilia Cavaterra

Dipartimento di Matematica “F. Enriques”
Università degli Studi di Milano
cecilia.cavaterra@unimi.it

ADMAT2012
PDEs for multiphase advanced materials
Cortona, September 17 -21, 2012

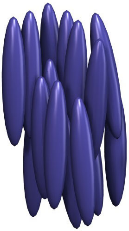
Joint work with Elisabetta Rocca

Liquid crystals

- fourth state of matter besides gas, liquid and solid
- intermediate state between crystalline and isotropic
- different liquid crystal phases, depending on the *amount* of order in the material
- nematic (*thread*), smectic (*soap*), cholesteric, columnar

Nematic liquid crystals

- simplest liquid crystal phase, *close* to the liquid one
- molecules have no positional order but tend to point in the same direction



Literature

- Ericksen (1976) and Leslie (1978) developed the hydrodynamics theory of liquid crystals based on the evolution of the *velocity field* $\boldsymbol{u}$ and the *director field* $\boldsymbol{d}$
- Lin & Liu (1995), (2001) formulated a simplified version in which the stretching and rotation effects are neglected
- Coutand & Shkoller (2001) (local $\exists\,!$ + stretching)
- Shkoller (2002) (global attractor - 2D - no stretching)
- Sun & Liu (2009) (periodic boundary conditions for $\boldsymbol{d}$)
- Climent-Ezquerra & Guillen-Gonzalez & Rodryguez-Bellido (2010)
- Segatti & Wu (2011) (smectic model)
- Grasselli & Wu, Wu & Xu & Liu (2011) (long time behavior)
- Frigeri (2012) (long time behavior)
- Bosia (2012) (exponential attractors - no stretching)

Nematic liquid crystals model

- the evolution of the *velocity field* $\boldsymbol{u}$ is ruled by the 3D incompressible Navier-Stokes system with a stress tensor exhibiting a special coupling between the transport and the induced terms
- the dynamics of the *director field* $\boldsymbol{d}$ is described by a modified Ginzburg-Landau equation with a suitable penalization of the physical constraint $|\boldsymbol{d}| = 1$

The evolution system for $\boldsymbol{u}$ and $\boldsymbol{d}$

$$
\begin{cases}
\operatorname{div} \boldsymbol{u} = 0 \\[2ex]
\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{T} + \boldsymbol{f} \\[2ex]
\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} - \alpha \boldsymbol{d} \cdot \nabla \boldsymbol{u} + (1-\alpha) \boldsymbol{d} \cdot \nabla^T \boldsymbol{u} = \gamma(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))
\end{cases}
$$

in $\Omega \times (0, T)$, $\Omega \subset \mathbb{R}^3$ bdd domain with smooth bdry Γ

$$
\begin{aligned}
\mathbb{T} = \mathbb{S} - \lambda(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) - \alpha\lambda(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} \\
+ (1-\alpha)\lambda \boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))
\end{aligned}
$$

$$
\mathbb{S} = \mu \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u} \right)
$$

$\mathbb{T}$, $\mathbb{S}$ Cauchy and Newtonian viscous stress tensors

$\nabla_{\boldsymbol{d}}$ gradient with respect to the variable $\boldsymbol{d}$

$(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d})_{ij} = \nabla_i \boldsymbol{d} \cdot \nabla_j \boldsymbol{d} \quad i, j = 1, 2, 3$

$(\boldsymbol{u} \otimes \boldsymbol{u})_{ij} = \boldsymbol{u}_i \boldsymbol{u}_j \quad i, j = 1, 2, 3$ (Kronecker product)

Basic features

p hydrodynamic pressure

$\boldsymbol{f}$ external force

$\mu > 0$ viscosity

λ competition between kinetic energy and potential energy

γ microscopic elastic relaxation time (Deborah number)

$\alpha \in [0, 1]$ depends on the shape of the liquid crystal molecules

$\alpha = \frac{1}{2}$ spherical shape

$\alpha = 1$ rod-like shape

$\alpha = 0$ disc-like shape

The function W

The function W penalizes the deviation of the length $|\boldsymbol{d}|$ from 1

Example: *double well potential* $W(\boldsymbol{d}) = (|\boldsymbol{d}|^2 - 1)^2$

In general W may be written as a sum of a convex part and a smooth (possibly) non-convex one

Novelty 1

- we prove the existence of *weak solutions* (in 3D) without any restriction on the size of the coefficients and the data, e.g. we do not need the viscosity coefficient μ in the stress tensor $\mathbb{S}$ big enough (no maximum principle holds for $\boldsymbol{d}$!)
- the main point is an appropriate choice of the space of test functions leading to a rigorous formulation of the system in weak form (for $\boldsymbol{u}$)
- a suitable definition of weak solution is necessary in order to deal with the stretching terms in the tensor $\mathbb{T}$
- in Climent-Ezquerra et al. only formal computations are performed to show the existence of weak solutions but no rigorous definition of the weak formulation, as well as no proof of existence of such solutions are given

Novelties 2 and 3

- our results hold also for non-homogeneous Dirichlet or homogeneous Neumann bdry conditions on the director field $\boldsymbol{d}$ (more meaningful from the application viewpoint)
- all the previous contributions in the literature were obtained assuming periodic bdry conditions on the director field $\boldsymbol{d}$
- the techniques employed for the proof of existence of solutions is based on the combination of a Faedo-Galerkin approximation and a regularization procedure, necessary to treat the stretching terms
- a non standard but physically meaningful regularization is obtained by adding in the stress tensor the r-Laplacian operator $|\nabla \boldsymbol{u}|^{r-2}\nabla \boldsymbol{u}$, as in J.-L.Lions models (1965), (1969) or in the Ladyzhenskaya models (1969), where $\nabla \boldsymbol{u}$ is replaced by $\dfrac{\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}}{2}$

The initial and boundary value problem

PROBLEM (P) $\quad (\gamma = \lambda = 1)$

$$
\begin{cases}
\operatorname{div} \boldsymbol{u} = 0, \\[2ex]
\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div}(\mu(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})) - \operatorname{div}(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) \\
-\operatorname{div}(\alpha(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} - (1-\alpha)\boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))) + \boldsymbol{f} \\[2ex]
\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} - \alpha \boldsymbol{d} \cdot \nabla \boldsymbol{u} + (1-\alpha)\boldsymbol{d} \cdot \nabla^T \boldsymbol{u} = (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \\[2ex]
\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0, \ \boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \qquad \text{in } \Omega \\[2ex]
\boldsymbol{u} = \mathbf{0}, \qquad \text{on } (0,T) \times \Gamma \\[2ex]
(1) \quad \partial_{\mathbf{n}} \boldsymbol{d} = \mathbf{0} \quad \text{or} \quad (2) \quad \boldsymbol{d}_{|\Gamma} = \mathbf{h} \quad \text{on } (0,T) \times \Gamma
\end{cases}
$$

Weak formulation of problem (P)

$$
\left\{
\begin{array}{l}
\displaystyle\int_\Omega \boldsymbol{u}(t,\cdot)\cdot\nabla\varphi = 0, \quad \forall\,\varphi\in W^{1,3}_0(\Omega), \text{ a.a. } t\in(0,T) \\[2ex]
\displaystyle\langle\partial_t\boldsymbol{u},\mathbf{v}\rangle - \int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\mathbf{v} + \int_\Omega \mu\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right) : \nabla\mathbf{v} = \\
\displaystyle\int_\Omega (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) : \nabla\mathbf{v} + \alpha\int_\Omega (\Delta\boldsymbol{d}-\nabla_{\boldsymbol{d}}W(\boldsymbol{d}))\otimes\boldsymbol{d} : \nabla\mathbf{v} \\
\displaystyle -(1-\alpha)\int_\Omega \boldsymbol{d}\otimes(\Delta\boldsymbol{d}-\nabla_{\boldsymbol{d}}W(\boldsymbol{d})) : \nabla\mathbf{v} + \int_\Omega \boldsymbol{f}\cdot\mathbf{v}, \\[2ex]
\forall\,\mathbf{v}\in W^{1,3}_0(\Omega;\mathbb{R}^3) : \operatorname{div}\mathbf{v} = 0, \text{ a.a. } t\in(0,T) \\[2ex]
\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = \Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}}W(\boldsymbol{d}), \\
\text{a.e. in } (0,T)\times\Omega \\[2ex]
\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0, \quad \boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0, \quad \text{a.e. in } \Omega \\[2ex]
(1) \quad \partial_{\mathbf{n}}\boldsymbol{d} = \mathbf{0}, \quad \text{or} \quad (2) \quad \boldsymbol{d}_{|\Gamma} = \mathbf{h}, \quad \text{a.e. on } (0,T)\times\Gamma
\end{array}
\right.
$$

Assumptions on the data

- $\Omega \subset \mathbb{R}^3$ bounded domain of class $C^{1,1}$
- $W \in C^2(\mathbb{R}^3), \quad W \geq 0$
- $W = W_1 + W_2$ such that
  W_1 is convex and $W_2 \in C^1(\mathbb{R}^3)$, $\nabla W_2 \in C^{0,1}(\mathbb{R}^3; \mathbb{R}^3)$
- $\mathbf{f} \in L^2(0, T; W^{-1,2}(\Omega; \mathbb{R}^3))$
- $\boldsymbol{u}_0 \in L^2(\Omega; \mathbb{R}^3), \quad \mathrm{div}\, \boldsymbol{u}_0 = 0$ in $L^2(\Omega)$
- $\boldsymbol{d}_0 \in W^{1,2}(\Omega; \mathbb{R}^3), \quad W(\boldsymbol{d}_0) \in L^1(\Omega)$

For the case (2), in addition:

- $\mathbf{h} \in H^1(0, T; H^{-1/2}(\Gamma; \mathbb{R}^3)) \cap L^\infty(0, T; H^{3/2}(\Gamma; \mathbb{R}^3))$
- $\mathbf{h}(0) = \boldsymbol{d}_{0|\Gamma}$

Main result - case (1)

Theorem

Problem (P) admits a global in time weak solution ($\boldsymbol{u}$, $\boldsymbol{d}$) s.t.

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3))$$

$$\partial_t \boldsymbol{u} \in L^2(0, T; W^{-1,3/2}(\Omega; \mathbb{R}^3))$$

$$W(\boldsymbol{d}) \in L^\infty(0, T; L^1(\Omega)), \quad \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2((0, T) \times \Omega; \mathbb{R}^3)$$

$$\boldsymbol{d} \in L^\infty(0, T; W^{1,2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,2}(\Omega; \mathbb{R}^3))$$

$$\partial_t \boldsymbol{d} \in L^2(0, T; L^{3/2}(\Omega; \mathbb{R}^3))$$

and satisfying, for a.a. $t \in (0, T)$, the energy inequality

$$\frac{d}{dt}\int_\Omega \left(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\right)(t) + 2\left\|(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))(t)\right\|^2_{L^2(\Omega)}$$

$$+\mu\|\nabla \boldsymbol{u}(t)\|^2_{L^2(\Omega)} \le C\|\boldsymbol{f}(t)\|^2_{W^{-1,2}(\Omega)}$$

Main result - case (2)

Theorem

Problem (P) admits a global in time weak solution ($\boldsymbol{u}$, $\boldsymbol{d}$) with the same regularity as in case (1) and satisfying, for a.a. $t \in (0, T)$, the energy inequality

$$\frac{d}{dt}\int_\Omega \left(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\right)(t) + 2\left\|(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))(t)\right\|^2_{L^2(\Omega)}$$

$$+\mu\|\nabla \boldsymbol{u}(t)\|^2_{L^2(\Omega)} \leq C\|\boldsymbol{f}(t)\|^2_{W^{-1,2}(\Omega)}$$

$$+\|\mathbf{h}(t)\|^2_{H^{3/2}(\Gamma)} + \|\mathbf{h}_t(t)\|^2_{H^{-1/2}(\Gamma)} + \|\nabla_{\boldsymbol{d}} W(\mathbf{h})(t)\|^2_{L^2(\Gamma)}$$

- In the weak formulation, we test on Ω the equation for $\boldsymbol{u}$ by $\boldsymbol{u}$ and the equation for $\boldsymbol{d}$ by $-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})$
- Summing up, an application of the divergence theorem gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \Big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d}) \Big) + \mu \int_\Omega |\nabla \boldsymbol{u}|^2$$

$$+ \int_\Omega |-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d})|^2 =_{H^{-1}(\Omega)} \langle \mathbf{f}, \boldsymbol{u} \rangle_{W_0^{1,2}(\Omega)}$$

- Applying Schwarz and Poincaré inequalities we deduce the energy estimate

Sketch of the proof - 2

- Integrating over $(0, T)$ the previous equality we get

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)) \cap L^{10/3}((0, T) \times \Omega; \mathbb{R}^3)$$

$$\boldsymbol{d} \in L^\infty(0, T; W^{1,2}(\Omega; \mathbb{R}^3))$$

$$-\Delta \boldsymbol{d} + \nabla_{\boldsymbol{d}} W(\boldsymbol{d}) \in L^2(0, T; L^2(\Omega; \mathbb{R}^3))$$

$$\partial_t \boldsymbol{d} \in L^2(0, T; L^{3/2}(\Omega; \mathbb{R}^3))$$

- Using suitable interpolation inequalities one has

$$\nabla \boldsymbol{d} \in L^{10/3}(0, T; L^{10/3}(\Omega; \mathbb{R}^{3\times 3}))$$

- The previous estimate is crucial for the proof of existence of solutions since we can deduce

$$(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) - \alpha(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} + (1-\alpha)\boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))$$

$$\in L^{5/3}((0,T) \times \Omega; \mathbb{R}^{3\times 3})$$

$$(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) - \alpha(\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \otimes \boldsymbol{d} + (1-\alpha)\boldsymbol{d} \otimes (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))$$

$$\in L^{2}(0,T; L^{3/2}(\Omega; \mathbb{R}^{3\times 3}))$$

Sketch of the proof - case (2)

- In the case (2) we get the equality

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \Big(|\boldsymbol{u}|^2 + |\nabla \boldsymbol{d}|^2 + 2W(\boldsymbol{d})\Big) + \mu\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} + \|\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})\|^2_{L^2(\Omega)}$$

$$= {}_{H^{-1}(\Omega)}\langle \mathbf{f}, \boldsymbol{u}\rangle_{W_0^{1,2}(\Omega)} + {}_{H^{-1/2}(\Gamma)}\langle \mathbf{h}_t, \partial_{\mathbf{n}} \boldsymbol{d}\rangle_{H^{1/2}(\Gamma)}$$

- Using standard trace theorems and regularity results for elliptic equations, we estimate from above the right hand side and and from below the left hand side by

$$ {}_{H^{-1/2}(\Gamma)}\langle \mathbf{h}_t, \partial_{\mathbf{n}} \boldsymbol{d}\rangle_{H^{1/2}(\Gamma)}$$

$$\leq C\left(\|\mathbf{h}_t\|^2_{H^{-1/2}(\Gamma)} + \|\mathbf{h}\|^2_{H^{3/2}(\Gamma)}\right) + \frac{1}{4}\|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)}$$

$$\|\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})\|^2_{L^2(\Omega)}$$

$$\geq \frac{3}{4}\|\Delta \boldsymbol{d}\|^2_{L^2(\Omega)} - C\|\nabla \boldsymbol{d}\|^2_{L^2(\Omega)} - C\|\mathbf{h}\|^2_{H^{3/2}(\Gamma)} - C\|\nabla_{\boldsymbol{d}} W(\mathbf{h})\|^2_{L^2(\Gamma)}$$

Sketch of the proof - The approximation scheme

- We introduce a double approximation scheme: a standard Faedo-Galerkin method coupled with an approximation of the convective term and a regularization of the momentum equation by adding an r-Laplacian operator acting on $\boldsymbol{u}$
- For the approximation of the convective term we follow the classical approach by Leray
- All the a priori bounds and estimates obtained for (P) hold also for the solution $u_{N,M}$ of the approximation scheme
- Passing to the limits $\lim_{N\to\infty} u_{N,M} = u_M, \ \lim_{M\to\infty} u_M = u$, we can prove that u is a solution to problem (P)

Open issues

- 2D: Uniqueness of weak solutions
- 2D: $\exists$ strong solutions with Neumann / Dirichlet boundary conditions for $\boldsymbol{d}$
- 2D: istantaneous regularization of the weak solutions with periodic boundary conditions for $\boldsymbol{d}$
- 3D: regularization in finite time of the weak solutions

Remark

Our technique has been used in

- Petzeltová & Rocca & Schimperna (2011), where the authors prove, via Łojasiewicz-Simon techniques, the convergence of the trajectories to the stationary states of (P) with suitable boundary conditions

and in

- Feireisl & Frémond & Rocca & Schimperna (2011), where it is proved the existence of weak solutions for the case of a non-isothermal system with Neumann (for $\boldsymbol{d}$) and complete slip (for $\boldsymbol{u}$) boundary conditions

Work in progress

- With Elisabetta Rocca and Hao Wu we considered a more general Ericksen–Leslie system modeling nematic liquid crystal flows
- We proved existence of global-in-time weak solutions under physically meaningful boundary conditions on the velocity field $\boldsymbol{u}$, the director field $\boldsymbol{d}$ and on the Leslie coefficients

The Ericksen–Leslie system (E-L)

$$\begin{cases} \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \operatorname{div} \mathbb{T}_1 \\ \partial_t \boldsymbol{d} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{d} - \alpha \boldsymbol{d} \cdot \nabla \boldsymbol{u} + (1-\alpha)\boldsymbol{d} \cdot \nabla^T \boldsymbol{u} = \gamma\, (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d})) \end{cases}$$

where

$$\gamma = \frac{1}{2\lambda_1} \qquad \alpha = \frac{1}{2}\left(1 + \frac{\lambda_2}{\lambda_1}\right) \quad \text{with} \quad \lambda_1 > 0$$

$$\mathbb{T}_1 = -(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) + \sigma \qquad \mathbb{S} = \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)$$

$$\sigma = \mu_1(\boldsymbol{d}^T \mathbb{S} \boldsymbol{d})\boldsymbol{d} \otimes \boldsymbol{d} + \mu_2 \mathcal{N} \otimes \boldsymbol{d} + \mu_3 \boldsymbol{d} \otimes \mathcal{N}$$
$$+ \mu_4 \mathbb{S} + \mu_5 (\mathbb{S}\boldsymbol{d}) \otimes \boldsymbol{d} + \mu_6 \boldsymbol{d} \otimes (\mathbb{S}\boldsymbol{d})$$

(μ_i : Leslie coefficients $\qquad \mu_4 > 0$: viscosity coefficient)

$$\mathcal{N} = \lambda_2 \gamma\, \mathbb{S}\boldsymbol{d} + \gamma\, (\Delta \boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))$$

Initial and boundary conditions

- Initial conditions

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0 \qquad \boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0 \ \text{ in } \Omega$$

- Homogeneous Dirichlet b.c. for the velocity field

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma$$

- Nonhomogeneous Dirichlet b.c. for the director field

$$\boldsymbol{d}|_\Gamma = \boldsymbol{d}_0|_\Gamma \quad \text{on } (0,T)\times\Gamma$$

(or homogeneous Neumann b.c. $\partial_{\mathbf{n}}\boldsymbol{d} = \mathbf{0} \quad \text{on } (0,T)\times\Gamma$)

Literature

- First result on existence of weak solutions to (E–L) + i.c. + b.c is due to Lin & Liu [Arch. Ration. Mech. Anal., 2008] under the assumption $\lambda_2 = 0$
- Physically, $\lambda_2 = 0$ indicates that the stretching due to the flow field is neglected
- From the mathematical point of view, if $\lambda_2 = 0$ then the maximum principle for $|\boldsymbol{d}|$ holds

Our contribution

- Our result shows that, without any restriction on the size of the fluid viscosity μ_4 and the initial data it is possible to obtain the existence of suitably defined weak solutions
- We used a technique analogous to the one applied in the previous model

Conditions on Leslie cofficients and parameters λ_i

(1) $\lambda_1 > 0 \qquad -\lambda_1 \leq \lambda_2 \leq \lambda_1 \; (\Rightarrow \alpha \in [0,1]\,)$

(2) $\mu_5 + \mu_6 \geq 0$

(3) $\mu_1 \geq 0 \quad \mu_4 > 0$

(4) $\lambda_1 = \mu_3 - \mu_2 \quad \lambda_2 = \mu_5 - \mu_6$

(5) $\mu_2 + \mu_3 = \mu_6 - \mu_5$

- (1) (2) (3) are necessary for the dissipation of the system
- (4) is necessary to satisfy the equation of motion identically and to guarantee the existence of a Lyapunov functional
- (5) derives from Onsager relation and expresses equality between flows and forces (*Parodi's relation*)

Two cases

- **Case 1** (with Parodi's relation)

  (1)–(5)

  (5bis) $$\frac{(\lambda_2)^2}{\lambda_1} \leq \mu_5 + \mu_6$$

- **Case 2** (without Parodi's relation)

  (1)–(4)

  (4bis) $$|\lambda_2 - \mu_2 - \mu_3| < 2\sqrt{\lambda_1}\sqrt{\mu_5 + \mu_6}$$

Weak solutions

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$
$$\partial_t \boldsymbol{u} \in L^2(0,T;W^{-1,\frac{6}{5}}(\Omega;\mathbb{R}^3))$$
$$\boldsymbol{d} \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3))$$
$$\partial_t \boldsymbol{d} \in L^2(0,T;L^{\frac{3}{2}}(\Omega;\mathbb{R}^3))$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0 \quad \boldsymbol{d}(0,\cdot) = \boldsymbol{d}_0 \quad \text{a.e. in } \Omega$$

$$\int_\Omega \boldsymbol{u}(t,\cdot)\cdot\nabla\varphi = 0, \quad \text{for a.e. } t \in (0,T), \quad \text{for any } \varphi \in W_0^{1,6}(\Omega)$$

$$\langle \partial_t \boldsymbol{u}, \mathbf{v}\rangle - \int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\mathbf{v}dx = \int_\Omega (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) : \nabla\mathbf{v}dx - \int_\Omega \sigma : \nabla\mathbf{v}dx$$

for a.e. $t \in (0,T)$, for any $\mathbf{v} \in W_0^{1,6}(\Omega;\mathbb{R}^3)$ s.t. $\operatorname{div}\mathbf{v} = 0$

$$\boldsymbol{d}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} - \alpha\boldsymbol{d}\cdot\nabla\boldsymbol{u} + (1-\alpha)\boldsymbol{d}\cdot\nabla^T\boldsymbol{u} = \gamma(\Delta\boldsymbol{d} - \nabla_{\boldsymbol{d}} W(\boldsymbol{d}))$$

a.e. in $(0,T)\times\Omega$

$$\boldsymbol{d} = \boldsymbol{d}_0|_\Gamma \qquad \text{a.e. on } (0,T)\times\Gamma$$

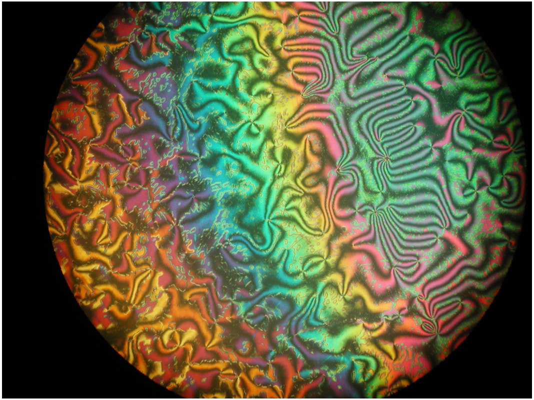
Main results

Theorem

Assume

$$\boldsymbol{u}_0 \in L^2(\Omega;\mathbb{R}^3), \quad \operatorname{div} \boldsymbol{u}_0 = 0 \ in \, L^2(\Omega)$$
$$\boldsymbol{d}_0 \in W^{1,2}(\Omega;\mathbb{R}^3), \quad W(\boldsymbol{d}_0) \in L^1(\Omega)$$
$$\boldsymbol{d}_0|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$$

*For both **Case 1** and **Case 2**, problem (E–L) + i.c. + b.c. possesses a global-in-time weak solution* $(\boldsymbol{u}, \boldsymbol{d})$



THANKS FOR YOUR ATTENTION!

HAPPY 65th BIRTHDAY GIANNI!!!