Blowup & Stationary Solutions in Aggregation Equations

J. A. Carrillo

ICREA - Universitat Autònoma de Barcelona

ADMAT2012, Cortona, 2012

Outline

Outline

1. Macroscopic Models: measure solutions.
   - Origin & Main Questions
   - Gradient Flows
   - Finite versus Infinite time Blow-up
2. Measure Solutions
   - Not too singular potentials
   - Singular repulsive potentials in 1D
3. Attractive-Repulsive Potentials
   - Particle Simulations
   - Stability/Instability of Delta Rings
   - Dimensionality of the support
4. Conclusions

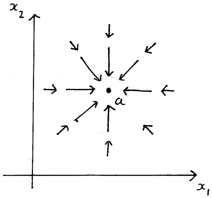
Aggregation for particles - Continuum Model

One particle attracted by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$



$\rho(t, x)$ = density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x - y) \, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$ with Morse potential $U(x) = 1 - e^{-|x|}$:

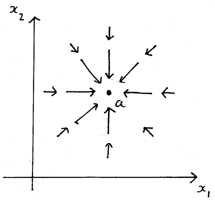
$$\begin{cases} \rho_t + \text{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

Aggregation for particles - Continuum Model

One particle attracted by a fixed location $x = a$

$$\dot{X} = -\nabla U(X-a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another



$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$

$\rho(t,x) =$ density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x-y)\, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$ with Morse potential $U(x) = 1 - e^{-|x|}$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

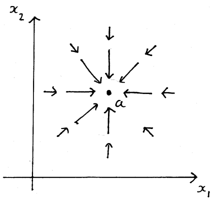
Aggregation for particles - Continuum Model

One particle attracted by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \nabla U(X_i - X_j)$$



$\rho(t, x) =$ density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x - y)\, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$ with Morse potential $U(x) = 1 - e^{-|x|}$:

$$\begin{cases} \rho_t + \text{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

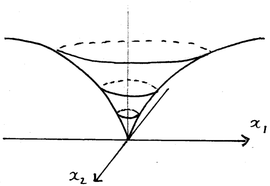
Aggregation Equation

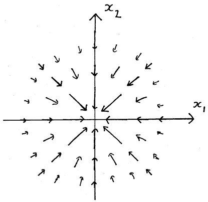
$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction attractive potentials do we get finite time blowup?
For which interaction attractive/repulsive potentials do we get convergence towards some nontrivial steady states?

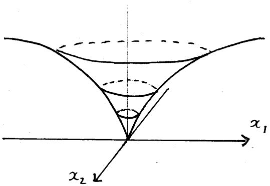
Aggregation Equation

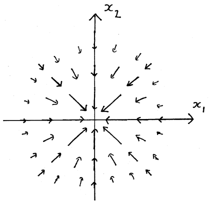
$$\begin{cases} \rho_t + \mathrm{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction attractive potentials do we get finite time blowup?
For which interaction attractive/repulsive potentials do we get convergence towards some nontrivial steady states?

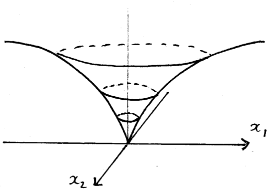
Aggregation Equation

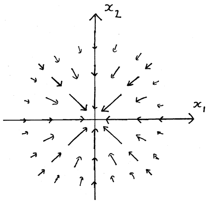
$$\begin{cases} \rho_t + \text{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction attractive potentials do we get finite time blowup?

For which interaction attractive/repulsive potentials do we get convergence towards some nontrivial steady states?

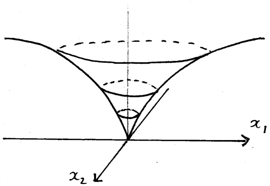
Aggregation Equation

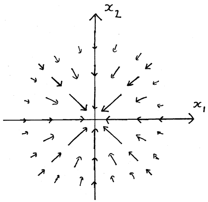
$$\begin{cases} \rho_t + \mathrm{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction attractive potentials do we get finite time blowup?
For which interaction attractive/repulsive potentials do we get convergence towards some nontrivial steady states?

Outline

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

with respect to the Wasserstein distance W_2..

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \text{div}\left(\rho(t,x)\nabla\left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right]\right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x)\left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right|^2 dx .$$

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

with respect to the Wasserstein distance W_2..

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x) \nabla \left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x) \right] \right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x) \right|^2 dx .$$

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

with respect to the Wasserstein distance W_2..

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x) \nabla \left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x) \right] \right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x) \right|^2 dx .$$

JKO scheme (Jordan-Kinderlehrer-Otto)

A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step Δt.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2^{\varrho}(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} W_2^2(\rho, \rho_k) + \mathcal{F}(\rho) \right\}$$

- As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

The convergence for smooth C^1 potentials U with at most quadratic growth at infinity given in "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré.

JKO scheme (Jordan-Kinderlehrer-Otto)

A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step Δt.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2^{o}(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} W_2^2(\rho, \rho_k) + \mathcal{F}(\rho) \right\}$$

- As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

The convergence for smooth C^1 potentials U with at most quadratic growth at infinity given in "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré.

JKO scheme (Jordan-Kinderlehrer-Otto)

A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step Δt.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2^\varrho(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} W_2^2(\rho, \rho_k) + \mathcal{F}(\rho) \right\}$$

- As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

The convergence for smooth C^1 potentials U with at most quadratic growth at infinity given in "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré.

JKO scheme (Jordan-Kinderlehrer-Otto)

A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step Δt.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2^{\rho}(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} W_2^2(\rho, \rho_k) + \mathcal{F}(\rho) \right\}$$

- As $\Delta t \to 0$ it converges to the solution of a weak form of

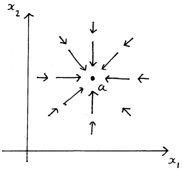
$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

The convergence for smooth C^1 potentials U with at most quadratic growth at infinity given in "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré.

Outline

Osgood condition

$$\dot{X} = -\nabla U(X - a)$$



Question: how long does it take for a particle to reach the bottom of a fixed potential?

$$\begin{cases} \dot{r} = -k'(r) \\ r(0) = L \end{cases}$$

$$U(x) = k(|x|)$$

Answer: $$T = \int_0^L \frac{dr}{k'(r)}$$

because to move by a distance dr, it takes the particle a time $\frac{dr}{k'(r)}$

Finite/Infinite time Blow-up

Sharp condition on the interaction potential in order to get blowup

- If $\boxed{\displaystyle\int_0^L \frac{dr}{k'(r)} = +\infty,}$ then we have global existence in

$$C([0,\infty), L^1 \cap L^p) \cap C^1([0,\infty), W^{-1,p}) \qquad \text{for } p > \tfrac{d}{d-1}.$$

$L^1 \cap L^\infty$(Bertozzi, C., Laurent; Nonlinearity 2009)
$L^1 \cap L^p$(Bertozzi, Laurent, Rosado; CPAM 2011)

- If $\boxed{\displaystyle\int_0^L \frac{dr}{k'(r)} < +\infty,}$ then $\rho(t) \to \delta_{X_0}$ in finite time.

(C., DiFrancesco, Figalli, Laurent, Slepcev; Duke Math. J. 2011)

Outline

Gradient Flow Solutions

Let U be a potential with at most quadratic behavior at infinity such that its only possible singularity is at zero. Moreover, assume that U is λ-convex:

$$U(x) - \frac{\lambda}{2}|x|^2 \text{ is convex.}$$

The typical example in our applications in swarming: the attractive Morse potential

$$U(x) = 1 - e^{-|x|} \text{ is -1-convex.}$$

Let us denote $\partial^0 U(x) = \nabla U(x)$ for all $x \neq 0$ and $\partial^0 U(0) = 0$.

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 U * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot \partial^0 U(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

Gradient Flow Solutions

Let U be a potential with at most quadratic behavior at infinity such that its only possible singularity is at zero. Moreover, assume that U is λ-convex:

$$U(x) - \frac{\lambda}{2}|x|^2 \text{ is convex.}$$

The typical example in our applications in swarming: the attractive Morse potential

$$U(x) = 1 - e^{-|x|} \text{ is -1-convex.}$$

Let us denote $\partial^0 U(x) = \nabla U(x)$ for all $x \neq 0$ and $\partial^0 U(0) = 0$.

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 U * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot \partial^0 U(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

Gradient Flow Solutions

Let U be a potential with at most quadratic behavior at infinity such that its only possible singularity is at zero. Moreover, assume that U is λ-convex:

$$U(x) - \frac{\lambda}{2}|x|^2 \text{ is convex.}$$

The typical example in our applications in swarming: the attractive Morse potential

$$U(x) = 1 - e^{-|x|} \text{ is -1-convex.}$$

Let us denote $\partial^0 U(x) = \nabla U(x)$ for all $x \neq 0$ and $\partial^0 U(0) = 0$.

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 U * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot \partial^0 U(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

Gradient Flow Solutions

Let U be a potential with at most quadratic behavior at infinity such that its only possible singularity is at zero. Moreover, assume that U is λ-convex:

$$U(x) - \frac{\lambda}{2}|x|^2 \text{ is convex.}$$

The typical example in our applications in swarming: the attractive Morse potential

$$U(x) = 1 - e^{-|x|} \text{ is -1-convex.}$$

Let us denote $\partial^0 U(x) = \nabla U(x)$ for all $x \neq 0$ and $\partial^0 U(0) = 0$.

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 U * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot \partial^0 U(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

Gradient Flow Solutions

Let U be a potential with at most quadratic behavior at infinity such that its only possible singularity is at zero. Moreover, assume that U is λ-convex:

$$U(x) - \frac{\lambda}{2}|x|^2 \text{ is convex.}$$

The typical example in our applications in swarming: the attractive Morse potential

$$U(x) = 1 - e^{-|x|} \text{ is -1-convex.}$$

Let us denote $\partial^0 U(x) = \nabla U(x)$ for all $x \neq 0$ and $\partial^0 U(0) = 0$.

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a *weak measure solution* with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 U * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x,\tau)\, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x,0)\, d\mu_0(x) =$$
$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla\varphi(t,x) \cdot \partial^0 U(x-y)\, d\mu(t)(x)\, d\mu(t)(y),$$

for all test functions $\varphi \in C_c^\infty([0,t) \times \mathbb{R}^d)$.

Sub-differential Characterization

Characterization of Sub-differential

Given a potential with the hypotheses above, the vector field

$$\kappa(x) := \int_{y \neq x} \nabla U(x-y)\, d\mu(y) \equiv (\partial^0 U * \mu)(x)$$

is the unique element of the minimal subdifferential of $\mathcal{F}$, i.e. $\partial^0 U * \mu = \partial^0 \mathcal{F}[\mu]$.

The solution obtained by JKO is a gradient flow-type solution:

$$v(t) = -\partial^0 \mathcal{F}[\mu(t)] = -\partial^0 U * \mu(t), \;\; \|v(t)\|_{L^2(\mu(t))} = |\mu'|(t)\; a.e.\, t > 0$$

with $\mu(0) = \mu_0$ and $v(t)$ is the tangent vector to the curve $\mu(t)$ with minimal norm.

Characterization of Sub-differential 2

Recently, in collaboration with S. Lisini and E. Mainini, we extend this to the case of $U(x)$ convex (not only λ-convex) and radial, allowing more Lipschitz points in the potential.

Sub-differential Characterization

Characterization of Sub-differential

Given a potential with the hypotheses above, the vector field

$$\kappa(x) := \int_{y \neq x} \nabla U(x-y)\, d\mu(y) \equiv (\partial^0 U * \mu)(x)$$

is the unique element of the minimal subdifferential of $\mathcal{F}$, i.e. $\partial^0 U * \mu = \partial^0 \mathcal{F}[\mu]$.

The solution obtained by JKO is a gradient flow-type solution:

$$v(t) = -\partial^0 \mathcal{F}[\mu(t)] = -\partial^0 U * \mu(t), \; \|v(t)\|_{L^2(\mu(t))} = |\mu'|(t) \, a.e. \, t > 0$$

with $\mu(0) = \mu_0$ and $v(t)$ is the tangent vector to the curve $\mu(t)$ with minimal norm.

Characterization of Sub-differential 2

Recently, in collaboration with S. Lisini and E. Mainini, we extend this to the case of $U(x)$ convex (not only λ-convex) and radial, allowing more Lipschitz points in the potential.

Sub-differential Characterization

Characterization of Sub-differential

Given a potential with the hypotheses above, the vector field

$$\kappa(x) := \int_{y \neq x} \nabla U(x-y)\, d\mu(y) \equiv (\partial^0 U * \mu)(x)$$

is the unique element of the minimal subdifferential of $\mathcal{F}$, i.e. $\partial^0 U * \mu = \partial^0 \mathcal{F}[\mu]$.

The solution obtained by JKO is a gradient flow-type solution:

$$v(t) = -\partial^0 \mathcal{F}[\mu(t)] = -\partial^0 U * \mu(t), \ \ \|v(t)\|_{L^2(\mu(t))} = |\mu'|(t)\, a.e.\, t > 0$$

with $\mu(0) = \mu_0$ and $v(t)$ is the tangent vector to the curve $\mu(t)$ with minimal norm.

Characterization of Sub-differential 2

Recently, in collaboration with S. Lisini and E. Mainini, we extend this to the case of $U(x)$ convex (not only λ-convex) and radial, allowing more Lipschitz points in the potential.

Well-posedness of Gradient Flow Solutions

Energy equality is satisfied:

$$\int_a^b \int_{\mathbb{R}^d} |v(t,x)|^2 \, d\mu(t)(x) \, dt + \mathcal{F}[\mu(a)] = \mathcal{F}[\mu(b)]$$

holds for all $0 \leq a \leq b < \infty$.

W_2-Expansion

Given two gradient flow solutions $\mu^1(t)$ and $\mu^2(t)$ in the sense of the theorem above, then

$$W_2(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} \, W_2(\mu_0^1, \mu_0^2)$$

for all $t \geq 0$. In particular, we have a unique gradient flow solution for any given $\mu_0 \in \mathcal{P}_2^o(\mathbb{R}^d)$.

Well-posedness of Gradient Flow Solutions

Energy equality is satisfied:

$$\int_a^b \int_{\mathbb{R}^d} |v(t,x)|^2 \, d\mu(t)(x) \, dt + \mathcal{F}[\mu(a)] = \mathcal{F}[\mu(b)]$$

holds for all $0 \leq a \leq b < \infty$.

W_2-Expansion

Given two gradient flow solutions $\mu^1(t)$ and $\mu^2(t)$ in the sense of the theorem above, then

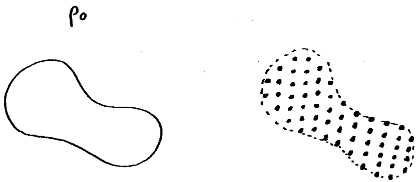
$$W_2(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} \, W_2(\mu_0^1, \mu_0^2)$$

for all $t \geq 0$. In particular, we have a unique gradient flow solution for any given $\mu_0 \in \mathcal{P}_2^o(\mathbb{R}^d)$.

Proof of blowup using the particle model

We want to prove that if $U(x) = k(|x|)$, λ-convex and

$$\boxed{\int_0^L \frac{dr}{k'(r)} < +\infty,} \Longrightarrow \boxed{\rho(t) \to \delta_{x_c} \text{ in finite time}}$$



Find a bound (independent of the nb. of particles) for the time it takes for all the particles to arrive at X_0.

$$\dot{X}_i = -\sum_{j\neq i} m_j \nabla U(X_i - X_j) = -\sum_{j\neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|)$$

Outline

Repulsive Singular Potential in 1D

The nonlocal equation

$$u_t + (H(u)u)_x = 0$$

with general nonnegative initial Borel measures u_0. Here, $H(u)$ denotes the classical Hilbert transform

$$H(u) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z)}{x-z}\, dz\,.$$

Motivations in fluid mechanics as 1D "analogs" of the Euler equation: Constantin, Lax, Majda, Córdoba, Fontelos... and dislocation dynamics in crystals: Head, Biler, Karch, Monneau.

It has the structure of gradient flow with potential:

$$\tilde{U}(x) = \begin{cases} -\frac{1}{\pi}\log|x| & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}.$$

Repulsive Singular Potential in 1D

The nonlocal equation

$$u_t + (H(u)u)_x = 0$$

with general nonnegative initial Borel measures u_0. Here, $H(u)$ denotes the classical Hilbert transform

$$H(u) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z)}{x-z}\, dz\,.$$

Motivations in fluid mechanics as 1D "analogs" of the Euler equation: Constantin, Lax, Majda, Córdoba, Fontelos... and dislocation dynamics in crystals: Head, Biler, Karch, Monneau.

It has the structure of gradient flow with potential:

$$\tilde{U}(x) = \begin{cases} -\frac{1}{\pi}\log|x| & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}.$$

Repulsive Singular Potential in 1D

The nonlocal equation

$$u_t + (H(u)u)_x = 0$$

with general nonnegative initial Borel measures u_0. Here, $H(u)$ denotes the classical Hilbert transform

$$H(u) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z)}{x-z}\, dz\,.$$

Motivations in fluid mechanics as 1D "analogs" of the Euler equation: Constantin, Lax, Majda, Córdoba, Fontelos... and dislocation dynamics in crystals: Head, Biler, Karch, Monneau.

It has the structure of gradient flow with potential:

$$\tilde{U}(x) = \begin{cases} -\frac{1}{\pi}\log|x| & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}.$$

Repulsive Singular Potential in 1D

The nonlocal equation

$$u_t + \left(H(u)u\right)_x = 0$$

with general nonnegative initial Borel measures u_0. Here, $H(u)$ denotes the classical Hilbert transform

$$H(u) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z)}{x-z}\, dz\,.$$

Motivations in fluid mechanics as 1D "analogs" of the Euler equation: Constantin, Lax, Majda, Córdoba, Fontelos... and dislocation dynamics in crystals: Head, Biler, Karch, Monneau.

It has the structure of gradient flow with potential:

$$\tilde{U}(x) = \begin{cases} -\frac{1}{\pi}\log|x| & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}\,.$$

Displacement Convexity in 1D

Given the free energy:

$$E_\alpha[\rho] = \begin{cases} \alpha\,\mathcal{V}[\rho] + \mathcal{W}[\rho] & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases},$$

with $\alpha = 0$ or $= 1$ where for $\rho \in \mathcal{P}_2(\mathbb{R})$

$$\mathcal{V}[\rho] := \int_{\mathbb{R}} \frac{x^2}{2}\rho(x)\,dx \qquad \text{and} \qquad \mathcal{W}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2} \tilde{U}(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Convexity and Global Minimum

The functional E_α is displacement convex. The functional E_1 has a unique compactly supported global minimum given by the semicircular law:

$$\bar\rho(x)\,dx = \frac{1}{\pi}\sqrt{(2-x^2)_+}\,dx\,.$$

Saff-Totik, Logarithmic Capacity.

Displacement Convexity in 1D

Given the free energy:

$$E_\alpha[\rho] = \begin{cases} \alpha\,\mathcal{V}[\rho] + \mathcal{W}[\rho] & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases},$$

with $\alpha = 0$ or $= 1$ where for $\rho \in \mathcal{P}_2(\mathbb{R})$

$$\mathcal{V}[\rho] := \int_{\mathbb{R}} \frac{x^2}{2}\rho(x)\,dx \qquad \text{and} \qquad \mathcal{W}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2} \tilde{U}(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Convexity and Global Minimum

The functional E_α is displacement convex. The functional E_1 has a unique compactly supported global minimum given by the semicircular law:

$$\bar{\rho}(x)\,dx = \frac{1}{\pi}\sqrt{(2-x^2)_+}\,dx\,.$$

Saff-Totik, Logarithmic Capacity.

Displacement Convexity in 1D

Given the free energy:

$$E_\alpha[\rho] = \begin{cases} \alpha\,\mathcal{V}[\rho] + \mathcal{W}[\rho] & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases},$$

with $\alpha = 0$ or $= 1$ where for $\rho \in \mathcal{P}_2(\mathbb{R})$

$$\mathcal{V}[\rho] := \int_{\mathbb{R}} \frac{x^2}{2}\rho(x)\,dx \qquad \text{and} \qquad \mathcal{W}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2} \tilde{U}(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Convexity and Global Minimum

The functional E_α is displacement convex. The functional E_1 has a unique compactly supported global minimum given by the semicircular law:

$$\bar{\rho}(x)\,dx = \frac{1}{\pi}\sqrt{(2-x^2)_+}\,dx\,.$$

Saff-Totik, Logarithmic Capacity.

Displacement Convexity in 1D

Given the free energy:

$$E_\alpha[\rho] = \begin{cases} \alpha\,\mathcal{V}[\rho] + \mathcal{W}[\rho] & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases},$$

with $\alpha = 0$ or $= 1$ where for $\rho \in \mathcal{P}_2(\mathbb{R})$

$$\mathcal{V}[\rho] := \int_{\mathbb{R}} \frac{x^2}{2}\rho(x)\,dx \quad \text{and} \quad \mathcal{W}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2} \tilde{U}(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Convexity and Global Minimum

The functional E_α is displacement convex. The functional E_1 has a unique compactly supported global minimum given by the semicircular law:

$$\bar{\rho}(x)\,dx = \frac{1}{\pi}\sqrt{(2-x^2)_+}\,dx\,.$$

Saff-Totik, Logarithmic Capacity.

Global Measure Solutions

Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and the functional E_α. The following assertions hold:

1. **(Existence and Uniqueness)** The JKO discrete interpolated curve ρ_t^τ converges locally uniformly to a locally Lipschitz curve $\rho_t := S_t[\rho_0]$ in $\mathcal{P}_2(\mathbb{R})$ which is the unique gradient flow of E_α with $\lim_{t\to 0+} \rho_t = \rho_0$. Moreover, the curve lies in $\mathcal{P}_2^{ac}(\mathbb{R})$, for all $t > 0$.
2. **(Contractive semigroup)** The map $t \mapsto S_t[\rho_0]$ is a α-contracting semigroup on $\mathcal{P}_2(\mathbb{R})$, i.e.
$$W_2(S_t[\rho_0], S_t[\mu_0]) \le e^{-\alpha t} W_2(\rho_0, \mu_0) \qquad \text{for all } \rho_0, \mu_0 \in \mathcal{P}_2(\mathbb{R}) .$$
3. **(Asymptotic behavior)** For $\alpha = 1$ and for all $0 < t_0 < t < \infty$, we have
$$W_2(\rho_t, \overline{\rho}) \le e^{-(t-t_0)} W_2(\rho_{t_0}, \overline{\rho})$$
and
$$E_1[\rho_t] - E_1[\overline{\rho}] \le e^{-2(t-t_0)} (E_1[\rho_{t_0}] - E_1[\overline{\rho}]) .$$

Global Measure Solutions

Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and the functional E_α. The following assertions hold:

1. **(Existence and Uniqueness)** The JKO discrete interpolated curve ρ_t^τ converges locally uniformly to a locally Lipschitz curve $\rho_t := S_t[\rho_0]$ in $\mathcal{P}_2(\mathbb{R})$ which is the unique gradient flow of E_α with $\lim_{t\to 0+} \rho_t = \rho_0$. Moreover, the curve lies in $\mathcal{P}_2^{ac}(\mathbb{R})$, for all $t > 0$.
2. **(Contractive semigroup)** The map $t \mapsto S_t[\rho_0]$ is a α-contracting semigroup on $\mathcal{P}_2(\mathbb{R})$, i.e.

$$W_2(S_t[\rho_0], S_t[\mu_0]) \le e^{-\alpha t} W_2(\rho_0, \mu_0) \qquad \text{for all } \rho_0, \mu_0 \in \mathcal{P}_2(\mathbb{R}) .$$

3. **(Asymptotic behavior)** For $\alpha = 1$ and for all $0 < t_0 < t < \infty$, we have

$$W_2(\rho_t, \overline{\rho}) \le e^{-(t-t_0)} W_2(\rho_{t_0}, \overline{\rho})$$

and

$$E_1[\rho_t] - E_1[\overline{\rho}] \le e^{-2(t-t_0)} (E_1[\rho_{t_0}] - E_1[\overline{\rho}]) .$$

Global Measure Solutions

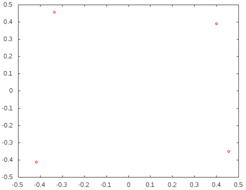
Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and the functional E_α. The following assertions hold:

1. **(Existence and Uniqueness)** The JKO discrete interpolated curve ρ_t^τ converges locally uniformly to a locally Lipschitz curve $\rho_t := S_t[\rho_0]$ in $\mathcal{P}_2(\mathbb{R})$ which is the unique gradient flow of E_α with $\lim_{t\to 0+} \rho_t = \rho_0$. Moreover, the curve lies in $\mathcal{P}_2^{ac}(\mathbb{R})$, for all $t > 0$.
2. **(Contractive semigroup)** The map $t \mapsto S_t[\rho_0]$ is a α-contracting semigroup on $\mathcal{P}_2(\mathbb{R})$, i.e.
$$W_2(S_t[\rho_0], S_t[\mu_0]) \le e^{-\alpha t}\, W_2(\rho_0, \mu_0) \qquad \text{for all } \rho_0, \mu_0 \in \mathcal{P}_2(\mathbb{R})\,.$$
3. **(Asymptotic behavior)** For $\alpha = 1$ and for all $0 < t_0 < t < \infty$, we have
$$W_2(\rho_t, \overline{\rho}) \le e^{-(t-t_0)}\, W_2(\rho_{t_0}, \overline{\rho})$$
and
$$E_1[\rho_t] - E_1[\overline{\rho}] \le e^{-2(t-t_0)}(E_1[\rho_{t_0}] - E_1[\overline{\rho}])\,.$$

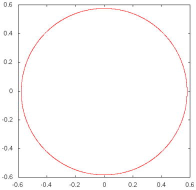
Outline

Some numerics: Particle Simulations $d = 2$

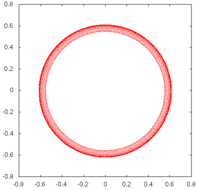
Potential $a = 4$, $b = 2.1$



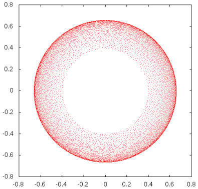
Potential $a = 4$, $b = 1.85$



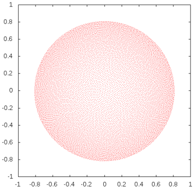
Potential $a = 4$, $b = 1.1$



Potential $a = 4$, $b = 0.85$



Potential $a = 4$, $b = 0.05$

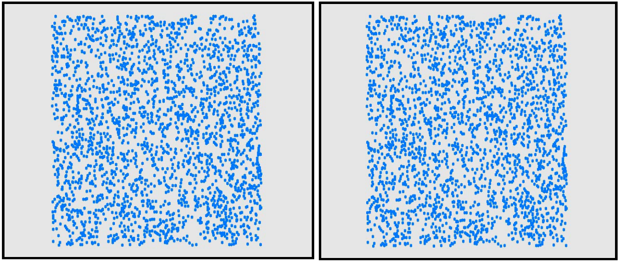


$$\dot{X}_i = -\sum_{j\neq i} m_j \, \nabla U(X_i - X_j)$$

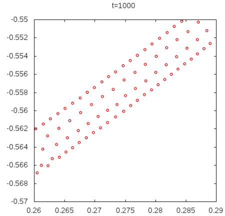
$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$$

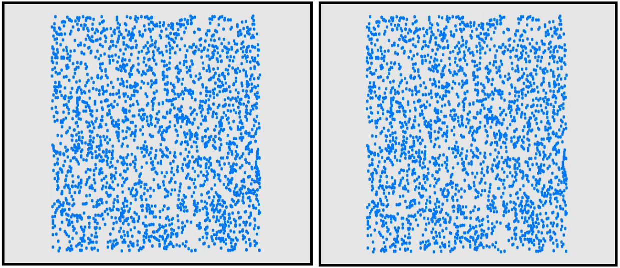
$$2 - d \leq b < a$$

Some numerics: Particle Simulations $d = 2$

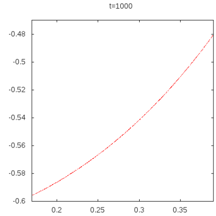


Potential $a = 4$, $b = 1.1$





Potential $a = 4$, $b = 1.85$



Outline

Existence of Spherical Shells Steady States

Spherical Shells Stationary States

Given a radially symmetric potential $U(x) = k(|x|)$ belonging to $C^2(\mathbb{R}^d \backslash \{0\})$ such that $k'(r)r^{d-2}$ is integrable on $(0,1)$. Let us assume that the potential is repulsive-attractive in the following sense: there exists $R_a > 0$ such that

$$k'(r) > 0 \text{ for } r > R_a, \qquad \text{and} \qquad 0 > k'(r) > -C_W \text{ for } 0 < r < R_a$$

and $\lim_{r\to\infty} r^{d-1}k'(r) = +\infty$.

Then there exists at least a $R > 0$ such that the spherical shell $\delta_R \in \mathcal{P}(\mathbb{R}^d)$ is a steady state to $\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$.

Existence of Spherical Shells Steady States

Spherical Shells Stationary States

Given a radially symmetric potential $U(x) = k(|x|)$ belonging to $C^2(\mathbb{R}^d \backslash \{0\})$ such that $k'(r) r^{d-2}$ is integrable on $(0,1)$. Let us assume that the potential is repulsive-attractive in the following sense: there exists $R_a > 0$ such that

$$k'(r) > 0 \text{ for } r > R_a, \qquad \text{and} \qquad 0 > k'(r) > -C_W \text{ for } 0 < r < R_a$$

and $\lim_{r\to\infty} r^{d-1} k'(r) = +\infty$.

Then there exists at least a $R > 0$ such that the spherical shell $\delta_R \in \mathcal{P}(\mathbb{R}^d)$ is a steady state to $\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$.

Radial Setting

The velocity field generated by a spherical shell of radius η is given by:

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla U(re_1 - \eta y)\cdot e_1 \, d\sigma(y),$$

Under some conditions on the potential U, the function $\omega \in C^1(\mathbb{R}^2_+)$.

The equation $\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$ written in radial coordinates is

$$\partial_t \hat{\mu} + \partial_r(\hat{\mu}\hat{v}) = 0$$

$$\hat{v}(t,r) = \int_0^{+\infty} \omega(r,\eta)\, d\hat{\mu}_t(\eta)\,.$$

Radial Setting

The velocity field generated by a spherical shell of radius η is given by:

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla U(re_1 - \eta y)\cdot e_1 \, d\sigma(y),$$

Under some conditions on the potential U, the function $\omega \in C^1(\mathbb{R}^2_+)$.

The equation $\frac{\partial \rho}{\partial t}(t,x) = \mathrm{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$ written in radial coordinates is

$$\partial_t \hat\mu + \partial_r(\hat\mu \hat v) = 0$$

$$\hat v(t,r) = \int_0^{+\infty} \omega(r,\eta)\, d\hat\mu_t(\eta)\,.$$

Radial Setting

The velocity field generated by a spherical shell of radius η is given by:

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla U(re_1 - \eta y)\cdot e_1 \, d\sigma(y),$$

Under some conditions on the potential U, the function $\omega \in C^1(\mathbb{R}^2_+)$.

The equation $\frac{\partial\rho}{\partial t}(t,x) = \mathrm{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$ written in radial coordinates is

$$\partial_t\hat{\mu} + \partial_r(\hat{\mu}\hat{v}) = 0$$

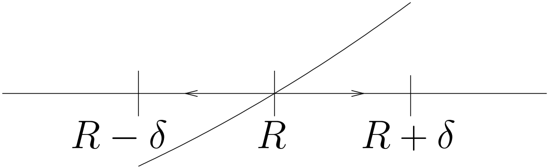
$$\hat{v}(t,r) = \int_0^{+\infty} \omega(r,\eta)d\hat{\mu}_t(\eta)\,.$$

Radial Setting: Instability Result

Instability of Spherical Shells

Assume that the spherical shell δ_R is a steady state, that is, $\omega(R,R) = 0$, and that $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1 \omega(R,R) > 0$.

Then it is not possible for an L^p radially symmetric solution to converge weakly-$*$ as measures to δ_R as $t \to \infty$.

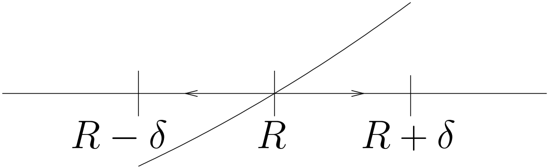


Radial Setting: Instability Result

Instability of Spherical Shells

Assume that the spherical shell δ_R is a steady state, that is, $\omega(R,R) = 0$, and that $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1\omega(R,R) > 0$.

Then it is not possible for an L^p radially symmetric solution to converge weakly-$*$ as measures to δ_R as $t \to \infty$.



Radial Setting: Instability Result

Local Stability of Spherical Shells

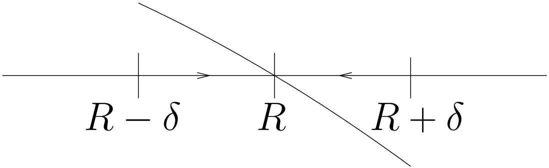
Assume $\omega \in C^1(\mathbb{R}_+^2)$ and that δ_R is a stationary solution, $\omega(R,R) = 0$. Let us assume that

$$\partial_1 \omega(R,R) < 0 \qquad \text{and} \qquad \partial_1 \omega(R,R) + \partial_2 \omega(R,R) < 0\,.$$

Then there exists $\varepsilon_0 > 0$ such that if the initial data $\mu_0 \in \mathcal{P}_2^r(\mathbb{R}^N)$ satisfies $\mathrm{supp}(\hat{\mu}_0) \subset [R - \varepsilon_0, R + \varepsilon_0]$, then the solution satisfies

$$W_2(\hat{\mu}_t, \delta_R) \le C e^{-\gamma t},$$

for any $0 < \gamma < -\max\left(\partial_1 \omega(R,R), \frac{d}{dR}\omega(R,R)\right)$ for suitable C.



Radial Setting: Instability Result

Local Stability of Spherical Shells

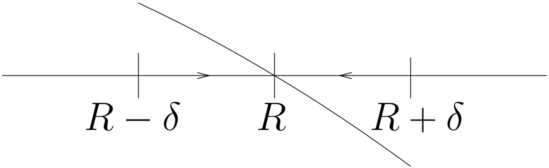
Assume $\omega \in C^1(\mathbb{R}^2_+)$ and that δ_R is a stationary solution, $\omega(R,R)=0$. Let us assume that

$$\partial_1\omega(R,R) < 0 \qquad \text{and} \qquad \partial_1\omega(R,R) + \partial_2\omega(R,R) < 0\,.$$

Then there exists $\varepsilon_0 > 0$ such that if the initial data $\mu_0 \in \mathcal{P}_2^r(\mathbb{R}^N)$ satisfies $\mathrm{supp}(\hat{\mu}_0) \subset [R-\varepsilon_0, R+\varepsilon_0]$, then the solution satisfies

$$W_2(\hat{\mu}_t, \delta_R) \le Ce^{-\gamma t},$$

for any $0 < \gamma < -\max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)$ for suitable C.

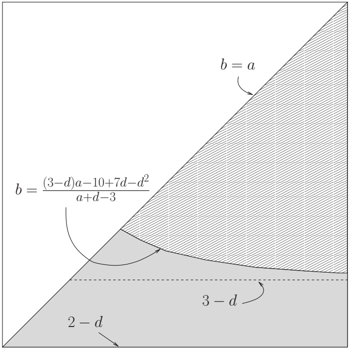


Power-Law Case

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - d < b < a$$

Theorem: Ins/Stability of Delta Rings with respect to radial perturbations.

- There is a computable value of R such that the uniform distribution on the sphere of radius R, δ_R is an steady state.
- If the velocity field generated by δ_R is strictly increasing at R then it is unstable.
- If the velocity field generated by δ_R is strictly decreasing at R then it is locally asymptotically stable.

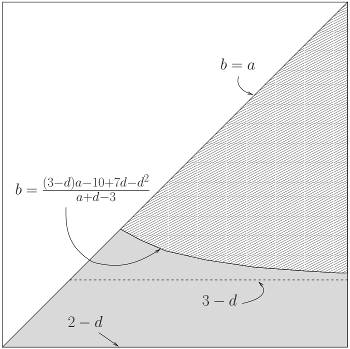


Power-Law Case

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - d < b < a$$

Theorem: Ins/Stability of Delta Rings with respect to radial perturbations.

- There is a computable value of R such that the uniform distribution on the sphere of radius R, δ_R is an steady state.
- If the velocity field generated by δ_R is strictly increasing at R then it is unstable.
- If the velocity field generated by δ_R is strictly decreasing at R then it is locally asymptotically stable.

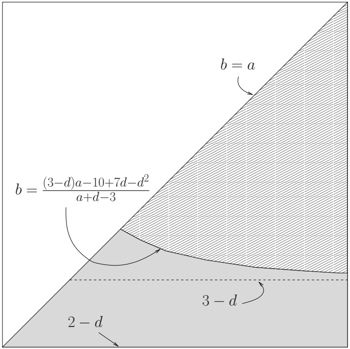


Power-Law Case

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - d < b < a$$

Theorem: Ins/Stability of Delta Rings with respect to radial perturbations.

- There is a computable value of R such that the uniform distribution on the sphere of radius R, δ_R is an steady state.
- If the velocity field generated by δ_R is strictly increasing at R then it is unstable.
- If the velocity field generated by δ_R is strictly decreasing at R then it is locally asymptotically stable.

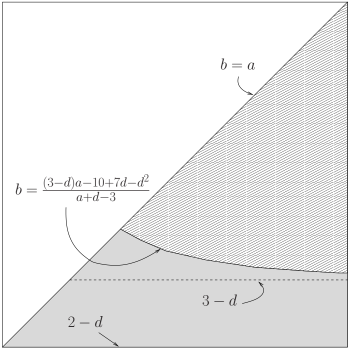


Power-Law Case

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - d < b < a$$

<u>Theorem</u>: Ins/Stability of Delta Rings with respect to radial perturbations.

- There is a computable value of R such that the uniform distribution on the sphere of radius R, δ_R is an steady state.
- If the velocity field generated by δ_R is strictly increasing at R then it is unstable.
- If the velocity field generated by δ_R is strictly decreasing at R then it is locally asymptotically stable.



Outline

Mild Repulsive potentials: $b > 2$

Support is essentially 0-dimensional.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy with respect to W_∞ cannot have a k-dimensional component for any $1 \leq k \leq d$.

Assumptions are really that the convexity properties near the origin are equal to a power-law with $b > 2$.

Strategy: By Contradiction we built a better competitor locally by sending part of the mass to a Dirac Delta.

Mild Repulsive potentials: $b > 2$

Support is essentially 0-dimensional.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy with respect to W_∞ cannot have a k-dimensional component for any $1 \le k \le d$.

Assumptions are really that the convexity properties near the origin are equal to a power-law with $b > 2$.

Strategy: By Contradiction we built a better competitor locally by sending part of the mass to a Dirac Delta.

Mild Repulsive potentials: $b > 2$

Support is essentially 0-dimensional.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy with respect to W_∞ cannot have a k-dimensional component for any $1 \leq k \leq d$.

Assumptions are really that the convexity properties near the origin are equal to a power-law with $b > 2$.

Strategy: By Contradiction we built a better competitor locally by sending part of the mass to a Dirac Delta.

Mild Repulsive potentials: $b > 2$

Support is essentially 0-dimensional.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy with respect to W_∞ cannot have a k-dimensional component for any $1 \le k \le d$.

Assumptions are really that the convexity properties near the origin are equal to a power-law with $b > 2$.

Strategy: By Contradiction we built a better competitor locally by sending part of the mass to a Dirac Delta.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \backslash \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \ldots, x_k)$, or equivalently, if $g(r)r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \backslash \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \dots, x_k)$, or equivalently, if $g(r)r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \backslash \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \ldots, x_k)$, or equivalently, if $g(r)r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \setminus \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \ldots, x_k)$, or equivalently, if $g(r)r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \setminus \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \ldots, x_k)$, or equivalently, if $g(r)r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Strongly Repulsive potentials $2 - d \leq b < 2$

A radially symmetric function $g \in C(\mathbb{R}^N \backslash \{0\})$ is said to be locally integrable on k-dimensional manifolds if

$$\int_{[0,1]^k} |g(\hat{x}, 0)| \, d\hat{x} < +\infty$$

where $\hat{x} = (x_1, \ldots, x_k)$, or equivalently, if $g(r) r^{k-1}$ is integrable on $(0, 1)$.

Dimension of the Support depends on b.

Assume that μ is a local minimizer of the interaction energy with respect to W_∞ and that U is radial.

If the divergence of the velocity field created by μ, i.e., $-\Delta U * \mu$ is not integrable on k-dimensional manifolds, then μ cannot contain k-dimensional manifolds in its support.

Remark: For $U(x) \sim -|x|^b$ near zero, ΔW is locally integrable on k-dimensional manifold iff $2 - b < k$.

Strategy: Look for 2nd-order optimality conditions for μ to be local minimizer.

Conclusions

- Optimal Transportation Tools can deal with evolutions by PDEs leading to concentration of measures happening at finite or infinite time.
- The dimensionality of the support of local minimizers of the interaction potential with respect to the d_∞ topology can be studied in terms of the repulsion strength of the potential near zero.
- References:

1. C.-DiFrancesco-Figalli-Laurent-Slepcev (DukeMJ 2011 & Nonlinear Analysis TMA 2012).
2. C.-Ferreira-Precioso (Adv. in Math. 2012).
3. Balagué-C.-Laurent-Raoul (preprint 2011 and work in preparation).

Conclusions

- Optimal Transportation Tools can deal with evolutions by PDEs leading to concentration of measures happening at finite or infinite time.
- The dimensionality of the support of local minimizers of the interaction potential with respect to the d_∞ topology can be studied in terms of the repulsion strength of the potential near zero.
- References:
  1. C.-DiFrancesco-Figalli-Laurent-Slepcev (DukeMJ 2011 & Nonlinear Analysis TMA 2012).
  2. C.-Ferreira-Precioso (Adv. in Math. 2012).
  3. Balagué-C.-Laurent-Raoul (preprint 2011 and work in preparation).

Conclusions

- Optimal Transportation Tools can deal with evolutions by PDEs leading to concentration of measures happening at finite or infinite time.
- The dimensionality of the support of local minimizers of the interaction potential with respect to the d_∞ topology can be studied in terms of the repulsion strength of the potential near zero.
- References:
  1. C.-DiFrancesco-Figalli-Laurent-Slepcev (DukeMJ 2011 & Nonlinear Analysis TMA 2012).
  2. C.-Ferreira-Precioso (Adv. in Math. 2012).
  3. Balagué-C.-Laurent-Raoul (preprint 2011 and work in preparation).