Long-time behaviour of a simplified Ericksen-Leslie non-autonomous system for nematic liquid crystal flows

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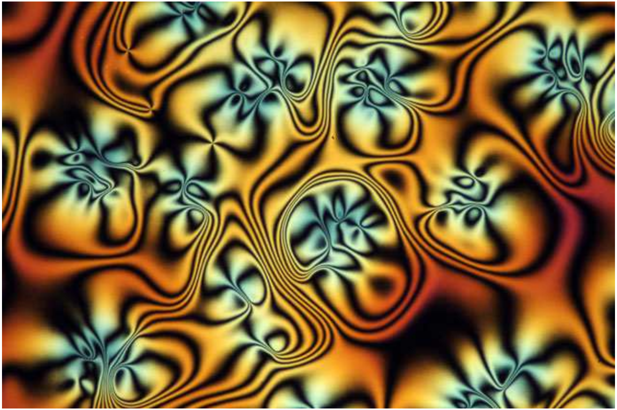
Nematic liquid crystals

- liquid crystals are materials consisting molecules having rod or disc-like shapes
- over tiny temperature ranges multiple phase transitions from solid to liquid occur
- in these transitions anisotropic properties are important
- if the molecules are elongated, usually a nematic phase arises (think of a bunch of toothsticks)

Nematic liquid crystals



Nematic liquid crystal film on glycerin surface

Frank's free energy and dissipativity

The direction of the molecules can be represented by an order parameter $\mathbf{n} \in \mathbb{S}^{n-1}$

We introduce Frank's free energy

$$\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}K_1(\nabla\cdot\mathbf{n})^2+\frac{1}{2}K_2\left(\mathbf{n}\cdot(\nabla\wedge\mathbf{n})\right)^2+\frac{1}{2}K_3\left(\mathbf{n}\wedge(\nabla\wedge\mathbf{n})\right)^2.$$

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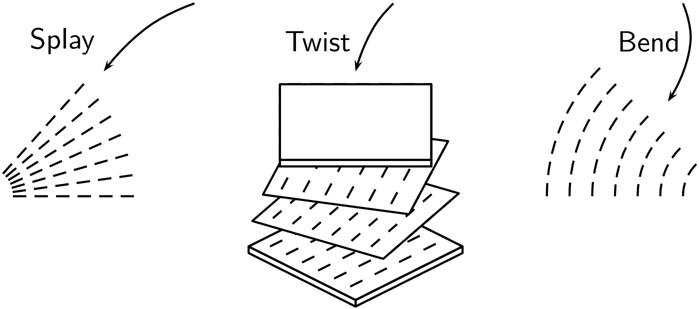

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We assume dissipativity:

$$\Pi_V + \Pi_S = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \sigma_F\right) dV + \int_V D\, dV$$

dove $D \geq 0$.

The resulting system

After some computations we obtain [F.-H. Lin, C. Liu '95, '96]:

$$\begin{cases} \rho\dot{\mathbf{u}} - \nu\Delta\mathbf{u} + \nabla p = -(\nabla\mathbf{n})^T\Delta\mathbf{n} + \rho\mathbf{F} \\ \nabla\cdot\mathbf{u} = 0 \\ \dot{\mathbf{n}} = \Delta\mathbf{n} - \mathbf{f}(\mathbf{n}) \end{cases}$$

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The constraint $\mathbf{n} \in \mathbb{S}^{n-1}$ can be relaxed:

$$\mathcal{F} \doteq \frac{1}{4\epsilon^2}(|\mathbf{n}|^2 - 1)^2 \qquad \mathbf{f}(\mathbf{n}) \doteq \nabla_{\mathbf{n}}\mathcal{F} = \frac{1}{\epsilon^2}(|\mathbf{n}|^2 - 1)\mathbf{n}.$$

We finally get...

The resulting system

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = -\nabla \mathbf{d}^t \Delta \mathbf{d} + \mathbf{g}(t) & \\ \nabla \cdot \mathbf{u} = 0 & \\ \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}) & \text{in } \Omega \times (0, \infty) \\ |\mathbf{d}| \leq 1 & \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \mathbf{d}(\mathbf{x}, 0) = \mathbf{d}_0 & \text{per } \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{d}(\mathbf{x}, t) = \mathbf{h}(\mathbf{x}, t) & \text{su } \partial\Omega \times (0, \infty) \end{cases}$$

Known results

- no-slip + Dirichlet, autonomous: [F.-H. Lin & C. Liu '95, '96], [S. Shkoller '01], [F. Guillén-Gonzáles et al. '09], [H. Wu '10]
- no-slip + Dirichlet, variable density: [F. Jiang & Z. Tan '09], [X.-G. Liu & Z.-Y. Zhang '09]
- no-slip + Dirichlet non-autonomous, convergence to stationary states: [M. Grasselli, H. Wu '11, preprint]
- free-slip + Neumann: [C. Liu & J. Shen '01], [E. Feireisl, E. Rocca & G. Schimperna '11–non-isothermal case]
- $\Omega = R3$, autonomous: [J. Fan & T. Ozawa '09], [X. Hu & D. Wang '10]
- numerical approximation: [C. Liu & N.J. Walkington '00, '02], [P. Lin & C. Liu '06]

Overview

We will consider

- a non-autonomous bulk force $\mathbf{g}(t)$
- no-slip B.C. on $\mathbf{u}$
- non-autonomous Dirichlet B.C. on the director $\mathbf{d}$.

Our results:

- existence (if $n = 2, 3$) and uniqueness of solutions (for $n = 2$ only)
- global attractor under general non-autonomous ("non-compact") forcing terms
- exponential attractors in the case of periodic forcing terms

Weak solutions

Definition

Let $T > 0$. A couple $(\mathbf{u}, \mathbf{d})$ is a weak solution if $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; \mathbf{V} \times \mathbf{H}^2)$, $(\partial_t \mathbf{u}, \partial_t \mathbf{d}) \in L^p(0, T; \mathbf{V}^*) \times L^2(0, T; \mathbf{L}^2)$ (with $p = 2$ if $n = 2$ and $p = 4/3$ if $n = 3$), if it satisfies the B.C. and the initial datum and if:

- $\langle \partial_t \mathbf{u}, \mathbf{v} \rangle + \langle (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v} \rangle + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\Delta \mathbf{d}, \nabla \mathbf{d} \mathbf{v}) = \langle \mathbf{g}(t), \mathbf{v} \rangle$ $\forall \mathbf{v} \in \mathbf{V}$, a.e. in $(0, T)$
- $\partial_t \mathbf{d}(t) + (\mathbf{u}(t) \cdot \nabla) \mathbf{d}(t) = \Delta \mathbf{d}(t) - \mathbf{f}(\mathbf{d}(t))$ e $|\mathbf{d}(\mathbf{x}, t)| \leq 1$ q.o.

Definition

The solution is strong if, in addition, $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; (\mathbf{H} \cap \mathbf{H}^2) \times \mathbf{H}^3)$. In this case it satisfies the system a.e.

Existence and uniqueness

Theorem

Let $n = 2, 3$, if

$$\mathbf{g} \in L^2(0, T; \mathbf{V}^*)$$
$$\mathbf{h} \in L^2(0, T; \mathbf{H}^{3/2}(\partial\Omega))$$
$$\partial_t \mathbf{h} \in L^2(0, T; \mathbf{H}^{-1/2}(\partial\Omega)), \quad |\mathbf{h}| \leq 1 \textit{a.e. on } \partial\Omega \times (0, T)$$
$$\mathbf{u}_0 \in \mathbf{H}$$
$$\mathbf{d}_0 \in \mathbf{H}^1, \quad |\mathbf{d}_0| \leq 1 \textit{a.e. in } \Omega,$$

then there exists a weak solution

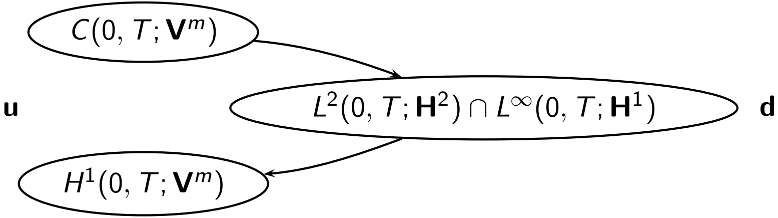
Theorem

For $n = 2$ this solution is also unique. If the data are regular, it is also a strong solution

Sketch of proof (existence)

Lifting of the non-autonomous boundary data for **d**

Schauder theorem on the approximating Galerkin scheme



The general theory

Definition

A family $\{U(t,\tau)\}$, $t > \tau$, $U(t,\tau) : X \to X$ is a process if

- $U(t,s)U(s,\tau) = U(t,\tau) \qquad \forall t, s \geq 0, \forall \tau \in \mathbb{R}$
- $U(\tau,\tau) = \mathbb{I} \qquad \forall \tau \in \mathbb{R}$

Definition

A set $K \subset X$ is uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the process $\{U_\sigma(t,\tau)\}$, if $\forall \tau \in \mathbb{R}$ and $\forall B$ bounded:

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_X(U_\sigma(t,\tau)B, K) = 0$$

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Definition

A closed set $\mathcal{A}_\Sigma \subset X$ is the global attractor of $\{U_\sigma(t,\tau)\}$ if:

- $\mathcal{A}_\Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) attracting
- $\mathcal{A}_\Sigma$ is contained in every other closed uniformly attracting set

Definition

$\{U_\sigma(t,\tau)\}$ has uniformly compact ω-limit if $\forall \tau \in \mathbb{R}$ and $\forall B$ bounded:

$$B_t = \bigcup_{\sigma\in\Sigma} \bigcup_{s\geq t} U_\sigma(s,\tau)B$$

is bounded for all t and if $\lim_{t\to\infty} \alpha(B_t) = 0$.

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The general theory

Theorem (S. Lu et al. '05)

Let $\{U_\sigma(t,\tau)\}$ be a process $(X \times \Sigma, X)$-weakly continuous having uniformly compact ω-limit. Let B_0 be bounded and uniformly weakly attracting. Then the extended semigroup has the global attractor $\mathfrak{A} = \omega(B_0 \times \Sigma)$ which is compact (in the weak topology). Moreover

- $\Pi_X \mathfrak{A} = \mathcal{A}_\Sigma$ *is the uniform attractor of* $\{U_\sigma(t,\tau)\}$ *(in the strong topology!)*
- $\Pi_\Sigma \mathfrak{A} = \Sigma$
- $\mathfrak{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\}$

Normal functions

Definition

$f \in L^2_{loc}(\mathbb{R}; E)$ is normal if $\forall \epsilon > 0\ \exists \eta > 0$:

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} |\varphi(s)|_E^2\, ds \leq \epsilon.$$

$L^2_n(\mathbb{R}; E)$ will be the space of E-valued normal functions

In general, the translation hull of normal functions is non-compact
Example: the translation hull of

$$f(t) = \sum_n e_n \chi_{[n,n+1]}(t), \quad \{e_n\} \text{ basis for } E$$

is non-compact in $L^p_{\mathrm{loc}}(0, \infty; E)$

Attractors

Theorem

Let $n = 2$ and

$$\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{V}^*)$$
$$\mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{3/2}(\partial\Omega))$$
$$\partial_t \mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{-1/2}(\partial\Omega))$$

Then the compact, uniform (w.r.t. $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$) in $\mathbf{H} \times \mathbf{H}^1$) attractor exists. In particular, it attracts bounded subsets of $\mathbf{H} \times \mathbf{H}^1$ in $\mathbf{H} \times \mathbf{H}^1$ uniformly (w.r.t. $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$)

Exponential attractors

Definition

A compact set $\mathcal{M} \subset X$ is an exponential attractor for the semigroup $\{S(t)\}$ if it has finite fractal dimension, it is positively invariant and it attracts bounded subsets exponentially fast:

$$\text{dist}_X(S(t)B, \mathcal{M}) \leq Q(|B|_X)e^{-\alpha t}, t \geq 0, \alpha > 0, Q \text{ monotonic.}$$

Definition

Let $X_1 \Subset X$. S has the smoothing property on B if $\exists C(B) > 0$:

$$|Su - Sv|_{X_1} \leq C|u - v|_X \qquad \forall u, v \in B.$$

Theorem (M. Efendiev, A. Miranville, S. Zelik '00)

If S has the smoothing property and $S\mathcal{O}_\delta(B) \subset B$, then there exists an exponential attractor $\mathcal{M}_S$ in the X_1-topology for the discrete semigroup.

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Quasi-periodic functions

Definition

Let $(\alpha^1, \ldots, \alpha^k)$ be incommensurable and let $\phi : \mathbb{R}^k \to \Xi$ be continuous and 2π-periodic in every argument. Then $\sigma(s) \doteq \phi(\alpha s)$ is a Ξ-valued quasi-periodic function

The translation hull of a quasi-periodic function is homomorphic to $\mathbb{T}^k$.

Theorem

Let $\mathbf{g}$, $\mathbf{h}$ and $\partial_t \mathbf{h}$ be $\mathbf{L}^2$, $\mathbf{H}^{5/2}(\partial\Omega)-$ and $\mathbf{H}^{1/2}(\partial\Omega)$-valued quasi-periodic functions. Then there exists an exponential attractor $\mathcal{M}$ for the extended semigroup $\{S(t)\}$ on $\mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k$. Moreover, $\Pi_1\mathcal{M}$ is the uniform exponential attractor (w.r.t. $\theta \in \mathbb{T}^k$) for the process and $\Pi_2\mathcal{M} = \mathbb{T}^k$.

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Further developments

- Estimate of the fractal dimension of the global attractor
- Study of more general Ericksen-Leslie-type models
- Numerics
- Reduced equations for moving singularities

Some useful references

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