PDEs for multiphase advanced materials

Cortona, Arezzo, Italy

Existence for the steady problem of a mixture of two power-law fluids[1]

Hermenegildo Borges de Oliveira

FCT - Universidade do Algarve & CMAF - Universidade de Lisboa
w3.ualg.pt/∼holivei
holivei@ualg.pt

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- Isothermal flows of viscous incompressible (and homogeneous) fluids in stationary regime:
  - Conservation of mass

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{1}$$

  - Conservation of linear momentum

$$\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = \mathbf{f} - \nabla p + \mathbf{div}\,\mathbf{S} \quad \text{in} \quad \Omega; \tag{2}$$

  - Deviatoric part of the Cauchy stress tensor

$$\mathbf{S} = \left(\mu_1|\mathbf{D}|^{\gamma-2} + \mu_2|\mathbf{D}|^{q(x)-2}\right)\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}\left(\nabla\,\mathbf{u} + \nabla\,\mathbf{u}^T\right); \tag{3}$$

  - Boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{4}$$

    - Unknowns: $\mathbf{u}\in\mathbb{R}^N$ – velocity field; $p\in\mathbb{R}^N$ – pressure;
    - Problem data: $\mathbf{f}\in\mathbb{R}^N$;
    - Remark: Dimensions of interest in the applications are $N=2,\ N=3$

- The simplest model of Fluid Mech. is the Newtonian fluid: Stokes (1845)

$$\mathbf{S} = 2\mu\mathbf{D}, \quad \mu = Const. > 0;$$

  - Examples: water solutions, gasoline, vegetal and mineral oils, ...;
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  - Real fluids: μ may depend on temperature, shear rate $|\mathbf{D}|$, time, pressure;

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- Ostwald (1925) – de Waele (1923) simplest non-Newtonian model:

$$\mathbf{S} = \mu|\mathbf{D}|^{n-1}\mathbf{D} \equiv \mu|\mathbf{D}|^{\gamma-2}\mathbf{D} \Rrightarrow \begin{cases} \text{Bingham (1921)} & n = 0 \quad \Leftrightarrow \quad \gamma = 1, \\ \text{pseudo-plastic} & 0 < n < 1 \quad \Leftrightarrow \quad 1 < \gamma < 2, \\ \text{Newtonian} & n = 1 \quad \Leftrightarrow \quad \gamma = 2, \\ \text{dilatant} & n > 1 \quad \Leftrightarrow \quad \gamma > 2, \end{cases}$$

  - Examples: **Bingham** toothpaste, mayonnaise; **Pseudo-plastic** milk fluids, varnishes, shampoo, blood; **Dilatant** polar ice, volcano lava, wet sand .
  - The viscosity depends on the shear stress (generalized Newtonian fluids);
  - Proposed for modeling pseudo-plastic fluids; it has been used also for dilatant;
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  - Fails at high shear rates where the viscosity must ultimately be a constant;
- The Sisko (1958) model: rectifies the failure of the Ostwald-de Waele

$$\mathbf{S} = \left(\mu_1 + \mu_2|\mathbf{D}|^{\gamma-2}\right)\mathbf{D}$$

  - It was originally proposed for high shear-rate measurements on some comercial greases (mixtures of petroleum with thickening agents).

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- Applications: automobile industry, e.g. clutches (ERF) and shock absorbers (MRF), and modeling e.g. the cooling process of volcano lava flow (TRF).

- Trembling Sisko model:

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- Justification of the model:
  - The object of superposition of generalized fluids is to produce flow patterns similar to those of practical interest;
  - The best example are polymer solutions in which the polymer segments tend to repel each other, since they prefer contact the solvent molecules rather then among themselves (see e.g. *Rheophysics* by P. Oswald (2009));
  - Superposition of fluids is justified, in the light of theoretical mechanics, as a powerful tool to replace the Boltzman superposition principle[2] in the case of materials with nonlinear behavior (see e.g. *Nonlinear vsicoelasticity* by J.M. Dealy (2009)).
  - Sisko's model has been checked experimentally to fit accurately the viscosity data of many mixtures (see e.g. *An introduction to rheology* by Barnes, Hutton and Walters (1993));

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- $\mathcal{P}(\Omega)$ the set of all measurable functions $q:\Omega\to[1,\infty]$;
- $\mathrm{L}^{q(\cdot)}(\Omega)$ the space of all functions $f\in\mathcal{P}(\Omega)$ such that

$$A_{q(\cdot)}(f):=\int_\Omega |f(x)|^{q(x)}dx<\infty,\quad \|f\|_{\mathrm{L}^{q(\cdot)}(\Omega)}:=\inf\left\{\kappa>0: A_{q(\cdot)}\left(\frac{f}{\kappa}\right)\leq 1\right\};$$

- $W^{1,q(\cdot)}(\Omega):=\left\{f\in\mathrm{L}^{q(\cdot)}(\Omega):\mathrm{D}^\alpha f\in\mathrm{L}^{q(\cdot)}(\Omega),\ 0\leq|\alpha|\leq 1\right\}$;
- Inherit almost properties of classical Lebesgue and Sobolev spaces, provided

$$1<\alpha:=\operatorname{ess\,inf} q(\cdot)\leq q(\cdot)\leq\operatorname{ess\,sup} q(\cdot):=\beta<\infty; \tag{7}$$

- Orlicz-Sobolev space with zero boundary values:

$$W_0^{1,q(\cdot)}(\Omega):=\overline{\left\{f\in W^{1,q(\cdot)}(\Omega):\mathrm{supp}\ f\subset\subset\Omega\right\}}^{\|\cdot\|_{W^{1,q(\cdot)}(\Omega)}}$$

- One problem:

$$C_0^\infty(\Omega)\text{ is not necessarily dense in }W_0^{1,q(\cdot)}(\Omega)$$

  - The closure of $\mathrm{C}_0^\infty(\Omega)$ in $\mathrm{W}^{1,q(\cdot)}(\Omega)$ is strictly contained in $\mathrm{W}_0^{1,q(\cdot)}(\Omega)$;
  - A necessary condition for the equality is the globally log-H continuity for $g=\frac{1}{q}$ (locally log-H continuous + log-H decay):

$$|g(\mathbf{x})-g(\mathbf{y})|\leq\frac{C_1}{\ln(e+1/|\mathbf{x}-\mathbf{y}|)},\quad |g(\mathbf{x})-q_\infty|\leq\frac{C_2}{\ln(e+|\mathbf{x}|)}\quad\forall\ \mathbf{x},\ \mathbf{y}\in\Omega. \tag{8}$$

- Spaces of Fluid Mechanics
  - $\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \mathrm{div}\,\mathbf{v} = 0\}$;
  - $\mathbf{V}_\gamma :=$ closure of $\mathcal{V}$ in $\mathbf{W}^{1,\gamma}(\Omega)$. The power-law index $\gamma = Const.$;
  - $\mathbf{V}_{q(\cdot)} :=$ closure of $\mathcal{V}$ in $\mathbf{W}^{1,q(\cdot)}(\Omega)$. Requires (8);
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Definition

Let Ω be a bounded domain of $\mathbb{R}^N$, with $N \geq 2$. Assume that $\mathbf{f} \in \mathbf{L}^1(\Omega)$, γ is a constant such that $1 < \gamma < \infty$ and $q \in \mathcal{P}(\Omega)$ is a variable exponent satisfying to (7). A vector field $\mathbf{u}$ is a (very) weak solution to the problem (1)-(3), if:

- $\mathbf{u} \in \mathbf{W}_{q(\cdot)} \cap \mathbf{V}_\gamma$;
- For every $\varphi \in \mathbf{W}_{q(\cdot)} \cap \mathbf{V}_\gamma$ (For every $\varphi \in \mathcal{V}$)

$$\int_\Omega \left(\mu_1|\mathbf{D}(\mathbf{u})|^{\gamma-2} + \mu_2|\mathbf{D}(\mathbf{u})|^{q(\mathbf{x})-2} - \mathbf{u}\otimes\mathbf{u}\right) : \mathbf{D}(\varphi)\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\varphi\, d\,\mathbf{x}.$$

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- Remark: Note that if $\alpha \geq \gamma$, then $\mathbf{W}_{q(\cdot)} \hookrightarrow \mathbf{V}_\gamma$ and therefore it is enough to look for weak solutions in the class $\mathbf{W}_{q(\cdot)}$ and if $\gamma \geq \beta$, then $\mathbf{V}_\gamma \hookrightarrow \mathbf{W}_{q(\cdot)}$ and therefore it is enough to look for weak solutions in the class $\mathbf{V}_\gamma$.

- Navier-Stokes : Hopf (1951) (Leray (1934) for the Cauchy problem).
- Ladyzhenskaya (1967), Lions (1969): $\mathbf{f} \in \mathbf{V}'_\gamma$ and

$$\gamma \geq \frac{3N}{N+2}. \tag{9}$$

  - Ladyzhenskaya: $\mathbf{S} = \left(\mu_1 + \mu_2|\mathbf{D}|^{\gamma-2}\right)\mathbf{D}$ and $N = 3$, and Lions: $\mathbf{S} = \mu|\mathbf{D}|^{\gamma-2}\mathbf{D}$
  - Proof: Theory of monotone operators together with compactness arguments,
  - The lower bound: $\gamma \geq \frac{3N}{N+2} \Rightarrow \mathbf{u} \otimes \mathbf{u} : \mathbf{D}(\varphi) \in \mathbf{L}^1(\Omega)$ for $\mathbf{u}$, $\varphi \in \mathbf{V}_\gamma$.

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- Frehse, Málek and Steinhauer (1997): $\mathbf{f} \in \mathbf{L}^{\gamma'}(\Omega)$; Růžička (1997): $\mathbf{f} \in \mathbf{V}'_\gamma$,

$$\gamma \geq \frac{2N}{N+1}. \tag{10}$$

  - Proof: In addition, it was used the L^∞-truncation method.
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- Frehse, Málek and Steinhauer (2003): $\mathbf{f} \in \mathbf{L}^{\gamma'}(\Omega)$ and

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  - Proof: It was used the Lipschitz-truncation method instead.
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  - The strict inequality $\gamma > \frac{2N}{N+2}$ is due to $\mathbf{V}_\gamma \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$.
- Open problem: $1 < \gamma \leq \frac{2N}{N+2}$ and $N > 2$.

- Růžička (2000): $\mathbf{f} \in \mathbf{V}'_\alpha$ and

$$\alpha \geq \frac{3N}{N+2}. \tag{12}$$

  - The test functions $\varphi \in \mathbf{W}_{q(\cdot)}$.
  - Proof: follows the approach of Ladyzhenskaya-Lions and uses $\mathbf{W}_{q(\cdot)} \hookrightarrow \mathbf{V}_\alpha$.

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- Huber (2011): $\mathbf{f} \in (\mathbf{W}_0^{1,q(\cdot)}(\Omega))'$ and

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  - The solution $\mathbf{u} \in \mathbf{V}_{q(\cdot)}$ (requires (8)) and the test function $\varphi \in \mathcal{V}$;
  - Proof: combines Bogowski (1979) results on divergence problems in Orlicz-Sobolev spaces with the L^∞-truncation method.

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- Diening, Málek and Steinhauer (2008): the same assumptions of Huber,

$$\alpha > \frac{2N}{N+2}. \tag{14}$$

  - The solutions satisfies to the energy relation:

$$\int_\Omega (\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{u} \otimes \mathbf{u}) : \mathbf{D}(\varphi)\, d\mathbf{x} = \int_\Omega p \,\mathrm{div}\varphi \, d\mathbf{x} + \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x} \qquad \forall\, \varphi \in \mathbf{W}_0^{1,\infty}(\Omega) \tag{15}$$

  - Proof: uses Lipschitz-truncations of functions in Orlicz-Sobolev spaces.

Theorem

Let Ω *be a bounded domain in* $\mathbb{R}^N$, $N \geq 2$, *with a Lipschitz-continuous boundary* $\partial\Omega$. *Assume that* $1 < \gamma < \infty$, $q \in \mathcal{P}(\Omega)$ *satisfies to (7) and* $\mathbf{f} \in \left(\mathbf{V}_\gamma \cap \mathbf{W}_{q(\cdot)}\right)'$. *Then, if*

$$\min\{\gamma, \alpha\} \geq \frac{3N}{N+2},$$

there exists a weak solution to the problem (1)-(3).

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Remarks

- *If $\gamma = 2$, it extends the existence result established by Ladyzhenskaya (1967) to the case of a variable exponent q;*
- *Since $\mathbf{V}_{q(\cdot)} \subsetneqq \mathbf{W}_{q(\cdot)}$, this result is obtained in a larger class.*

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$$\gamma \geq \max\left\{\frac{2N}{N+2} + \delta, \beta\right\}, \tag{16}$$

there exists a very weak solution to the problem (1)-(3).

- For the trembling Sisko model we have existence of very weak solutions

$$\mathbf{S} = \left(\mu_1 + \mu_2|\mathbf{D}|^{q(x)-2}\right)\mathbf{D} \quad \text{for } 1 < \alpha \leq \beta \leq 2. \tag{17}$$

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that $q \in \mathcal{P}(\Omega)$ satisfies to (7) and $\mathbf{f} \in \left(\mathbf{V}_\gamma \cap \mathbf{W}_{q(\cdot)}\right)'$. Then, if for any $\delta > 0$

$$\gamma \geq \max\left\{\frac{2N}{N+2} + \delta, \beta\right\}, \tag{16}$$

there exists a very weak solution to the problem (1)-(3).

- For the trembling Sisko model we have existence of very weak solutions

$$\mathbf{S} = \left(\mu_1 + \mu_2|\mathbf{D}|^{q(\mathbf{x})-2}\right)\mathbf{D} \quad \text{for } 1 < \alpha \leq \beta \leq 2. \tag{17}$$

- In order to make the proof as transparent as possible, we shall assume that

$$\mathbf{f} = -\mathbf{div}\,\mathbf{F}, \quad \mathbf{F} \in \mathbb{M}^N_{\mathrm{sym}}, \quad \mathbf{F} \in \mathbf{L}^{q'(\cdot)}(\Omega). \tag{18}$$

  - The assumption (18) does not restrict the result's extent, because $\mathbf{f} = -\mathbf{div}\,\mathbf{F}$ and $\mathbf{F} \in \mathbf{L}^{q'(\cdot)}(\Omega)$ implies that $\mathbf{f} \in \mathbf{W}'_{q(\cdot)}$, and $\mathbf{W}'_{q(\cdot)} \hookrightarrow \left(\mathbf{V}_\gamma \cap \mathbf{W}_{q(\cdot)}\right)'$.
  - The assumption $\mathbf{F} \in \mathbb{M}^N_{\mathrm{sym}}$ is made in order to avoid unnecessary calculus.

- Let $\Phi \in C^\infty([0,\infty))$ be a non-increasing: $0 \le \Phi \le 1$ in $[0,\infty)$, $\Phi \equiv 1$ in $[0,1]$, $\Phi \equiv 0$ in $[2,\infty)$ and $0 \le -\Phi' \le 2$. For $\epsilon > 0$, we set

$$\Phi_\epsilon(s) := \Phi(\epsilon s), \quad s \in [0,\infty). \tag{19}$$

We consider the following regularized problem in Ω:

$$\operatorname{div} \mathbf{u}_\epsilon = 0, \tag{20}$$

$$\mathbf{div}(\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon \Phi_\epsilon(|\mathbf{u}_\epsilon|)) = \mathbf{f} - \nabla p_\epsilon + \mathbf{div}\left[\left(\mu_1|\mathbf{D}(\mathbf{u}_\epsilon)|^{\gamma-2} + \mu_2|\mathbf{D}(\mathbf{u}_\epsilon)|^{q(\mathbf{x})-2}\right)\mathbf{D}(\mathbf{u}_\epsilon)\right], \tag{21}$$

$$\mathbf{u}_\epsilon = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{22}$$

Proposition

... Then, for each $\epsilon > 0$, there exists a weak solution $\mathbf{u}_\epsilon \in \mathbf{V}_\gamma$ to the problem (20)-(22). In addition, every weak solution satisfies to the following energy equality:

$$\int_\Omega \left(\mu_1|\mathbf{D}(\mathbf{u}_\epsilon)|^\gamma + \mu_2|\mathbf{D}(\mathbf{u}_\epsilon)|^{q(\mathbf{x})}\right) d\mathbf{x} = \int_\Omega \mathbf{F} : \mathbf{D}(\mathbf{u}_\epsilon) d\mathbf{x}. \tag{23}$$

- The proof is based on Schauder's fixed point theorem.

- From (23), we can prove that

$$\int_\Omega \left(|\mathbf{D}(\mathbf{u}_\epsilon)|^\gamma + |\mathbf{D}(\mathbf{u}_\epsilon)|^{q(\mathbf{x})}\right) d\mathbf{x} \leq C. \tag{24}$$

- By Sobolev's inequality, and due to the definition of Φ_ϵ, we have

$$\|\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon \Phi_\epsilon(|\mathbf{u}_\epsilon|)\|_{\mathbf{L}^{\frac{\gamma^*}{2}}(\Omega)} \leq C. \tag{25}$$

- From (24)-(25), there exists $\epsilon_m > 0$ such that $\epsilon_m \to 0$, as $m \to \infty$, and

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{weakly in } \mathbf{V}_\gamma, \quad \text{as } m \to \infty, \tag{26}$$

$$|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) \to \mathbf{S}_1 \quad \text{weakly in} \quad \mathbf{L}^{\gamma'}(\Omega), \quad \text{as} \quad m \to \infty, \tag{27}$$

$$|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) \to \mathbf{S}_2 \quad \text{weakly in} \quad \mathbf{L}^{\gamma'}(\Omega), \quad \text{as} \quad m \to \infty, \tag{28}$$

$$\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m}\Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) \to \mathbf{G} \quad \text{weakly in } \mathbf{L}^{\frac{\gamma^*}{2}}(\Omega), \quad \text{as } m \to \infty. \tag{29}$$

- Using (27)-(29), we can pass to the limit $m \to \infty$ in

$$\int_\Omega \left(\mu_1|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2} + \mu_2|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\right)\mathbf{D}(\mathbf{u}_{\epsilon_m}) : \mathbf{D}(\varphi)\, d\mathbf{x} = \int_\Omega [\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m}\Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) + \mathbf{F}] : \mathbf{D}(\varphi) \tag{30}$$

  valid for all $\varphi \in \mathcal{V}$, to obtain

$$\int_\Omega (\mu_1\mathbf{S}_1 + \mu_2\mathbf{S}_2 - \mathbf{G} - \mathbf{F}) : \mathbf{D}(\varphi)\, d\mathbf{x} = 0 \quad \forall\, \varphi \in \mathcal{V}. \tag{31}$$

- Due to (26), the application of Sobolev's compact imbedding theorem implies

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^{\kappa}(\Omega), \quad \text{as } m \to \infty, \qquad \text{for any } \kappa : 1 \leq \kappa < \gamma^*. \tag{32}$$

- Since (16) implies $2 < \gamma^*$, it follows from (32) that

$$\mathbf{u}_{\epsilon_m} \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^{2}(\Omega), \quad \text{as } m \to \infty. \tag{33}$$

- Using (19)) and the result (33), we can prove that

$$\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) \to \mathbf{u} \otimes \mathbf{u} \quad \text{strongly in } \mathbf{L}^{1}(\Omega), \quad \text{as } m \to \infty. \tag{34}$$

- Then gathering the information of (29) and (34), we see that $\mathbf{G} = \mathbf{u} \otimes \mathbf{u}$.
- Now in the limit $(m \to \infty)$ of the integral equation, we have

$$\int_\Omega (\mu_1 \mathbf{S}_1 + \mu_2 \mathbf{S}_2 - \mathbf{u} \otimes \mathbf{u} - \mathbf{F}) : \mathbf{D}(\varphi)\, d\mathbf{x} = 0 \quad \forall\, \varphi \in \mathcal{V}. \tag{35}$$

- Since we shall use test functions which are not divergence free, we first have to determine the approximative pressure from the weak formulation (30).
- First, let ω' be a fixed but arbitrary open bounded subset of Ω such that

$$\omega' \subset\subset \Omega \quad \text{and} \quad \partial\omega' \text{ is Lipschitz} \tag{36}$$

- We use a version of de Rham's Theorem[3] (Bogovskiĭ (1980) and Pileckas (1983)) to prove the existence of a unique function

$$p_{\epsilon_m} \in \mathbf{L}^{r'}(\omega'), \quad 1 < r \le r_0 := \min\left\{\gamma', \frac{\gamma^*}{2}\right\}, \quad \text{with} \quad \int_{\omega'} p_{\epsilon_m}\, d\mathbf{x} = 0 \tag{37}$$

and such that (for all $\varphi \in \mathbf{W}_0^{1,r'}(\omega')$)

$$\int_{\omega'} \left(\mu_1 |\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2} + \mu_2 |\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\right) \mathbf{D}(\mathbf{u}_{\epsilon_m}) : \mathbf{D}(\varphi)\, d\mathbf{x} =$$
$$\int_{\omega'} \mathbf{F} : \mathbf{D}(\varphi)\, d\mathbf{x} + \int_{\omega'} \mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) : \mathbf{D}(\varphi)\, d\mathbf{x} + \int_{\omega'} p_{\epsilon_m} \operatorname{div}\varphi\, d\mathbf{x}\,.$$

- Passing to the limit $m \to \infty$, we obtain (for all $\varphi \in \mathbf{W}_0^{1,r'}(\omega')$)

$$\int_{\omega'} (\mu_1 \mathbf{S}_1 + \mu_2 \mathbf{S}_2 - \mathbf{u} \otimes \mathbf{u} - \mathbf{F}) : \mathbf{D}(\varphi)\, d\mathbf{x} = \int_{\omega'} p_0 \operatorname{div}\varphi\, d\mathbf{x}\,. \tag{38}$$

[3] de Rham (1931): $\mathbf{g} = \nabla p$ for some p iff $\int_{Q_T} \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x} dt = 0$ for all $\mathbf{v}$: $\operatorname{div}\mathbf{v} = 0$.

- Let ω be a fixed but arbitrary domain such that

$$\omega \subset\subset \omega' \subset\subset \Omega \quad \text{and} \quad \partial\omega \text{ is } C^2. \tag{39}$$

- By Simader and Sohr (1996) and Wolf (2007), there exist unique functions

$$p^1_{\epsilon_m} \in \mathrm{A}^{\gamma'}(\omega), \quad p^2_{\epsilon_m} \in \mathrm{A}^{\frac{\gamma^*}{2}}(\omega), \tag{40}$$

where $\mathrm{A}^s(\omega) := \{a \in \mathrm{L}^s(\omega) : a = \triangle u, \quad u \in \mathrm{W}^{2,s}_0(\omega)\}$ such that

$$\begin{aligned} \|p^1_{\epsilon_m}\|_{\mathrm{L}^{\gamma'}(\omega)} \leq & C_1 \||\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - \mathbf{S}_1\|_{\mathbf{L}^{\gamma'}(\omega)} + \\ & C_2 \||\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - \mathbf{S}_2\|_{\mathbf{L}^{\gamma'}(\omega)}, \end{aligned} \tag{41}$$

$$\|p^2_{\epsilon_m}\|_{\mathrm{L}^{\frac{\gamma^*}{2}}(\omega)} \leq C_3 \|\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) - \mathbf{u} \otimes \mathbf{u}\|_{\mathbf{L}^{\frac{\gamma^*}{2}}(\omega)}. \tag{42}$$

and

$$p_{\epsilon_m} - p_0 = p^1_{\epsilon_m} + p^2_{\epsilon_m}.$$

- Then

$$\left.\begin{aligned} &\mathbf{div}\left(|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - \mathbf{S}_1 + |\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - \mathbf{S}_2\right) - \\ &\mathbf{div}\left(\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m}\Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) - \mathbf{u} \otimes \mathbf{u}\right) = \nabla\left(p^1_{\epsilon_m} + p^2_{\epsilon_m}\right) \end{aligned}\right\} \text{ in } \mathcal{D}'(\omega). \tag{43}$$

- Using the Hardy-Littlewood maximal operator, we can prove that for all $m \in \mathbb{N}$ and all $j \in \mathbb{N}_0$ there exists $\lambda_{m,j} \in \left[2^{2^j}, 2^{2^{j+1}}\right)$ such that

$$\mathcal{L}_N\left(F_{m,j}\right) \leq 2^{-j}\lambda_{m,j}^{-\kappa}\left\|\mathbf{w}_{\epsilon_m}\right\|_{\mathbf{L}^\kappa(\mathbb{R}^N)}, \quad \text{for any } \kappa : 1 \leq \kappa < \gamma^*, \tag{44}$$

$$\mathcal{L}_N\left(G_{m,j}\right) \leq 2^{-j}\lambda_{m,j}^{-\gamma}\left\|\nabla\mathbf{w}_{\epsilon_m}\right\|_{\mathbf{L}^\gamma(\mathbb{R}^N)}, \tag{45}$$

where

$$F_{m,j} := \left\{\mathbf{x} \in \mathbb{R}^N : \mathcal{M}(|\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) > 2\lambda_{m,j}\right\},$$
$$G_{m,j} := \left\{\mathbf{x} \in \mathbb{R}^N : \mathcal{M}(|\nabla\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) > 2\lambda_{m,j}\right\}.$$

$$\mathcal{M}(|\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) := \sup_{0<R<\infty} \frac{1}{\mathcal{L}_N(B_R(\mathbf{x}))}\int_{B_R(\mathbf{x})} |\mathbf{w}_{\epsilon_m}(\mathbf{y})|\, d\mathbf{y},$$
$$\mathcal{M}(|\nabla\mathbf{w}_{\epsilon_m}|)(\mathbf{x}) := \sup_{0<R<\infty} \frac{1}{\mathcal{L}_N(B_R(\mathbf{x}))}\int_{B_R(\mathbf{x})} |\nabla\mathbf{w}_{\epsilon_m}(\mathbf{y})|\, d\mathbf{y}.s$$

- Setting $R_{m,j} := F_{m,j} \cup G_{m,j}$, we can prove that

$$\limsup_{m\to\infty} \mathcal{L}_N\left(R_{m,j}\right) \leq \limsup_{m\to\infty} C 2^{-j}\lambda_{m,j}^{-\gamma}. \tag{46}$$

- By Acerbi and Fusco (1988), there exists

$$\mathbf{z}_{m,j} \in \mathbf{W}^{1,\infty}(\mathbb{R}^N), \qquad \mathbf{z}_{m,j} = \begin{cases} \mathbf{w}_{\epsilon_m} & \text{in } \omega \setminus A_{m,j} \\ 0 & \mathbb{R}^N \setminus \omega \end{cases}, \tag{47}$$

$$A_{m,j} := \{\mathbf{x} \in \omega : \mathbf{z}_{m,j}(\mathbf{x}) \neq \mathbf{w}_{\epsilon_m}(\mathbf{x})\}, \tag{48}$$

such that

$$\|\mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)} \leq 2\lambda_{m,j}, \tag{49}$$

$$\|\nabla \mathbf{z}_{m,j}\|_{\mathbf{L}^\infty(\omega)} \leq C\lambda_{m,j}, \quad C = C(N,\omega). \tag{50}$$

- By Landes (1996),

$$A_{m,j} \subset \omega \cap R_{m,j}. \tag{51}$$

- As a consequence,

$$\limsup_{m\to\infty} \mathcal{L}_N\left(A_{m,j}\right) \leq \limsup_{m\to\infty} C2^{-j}\lambda_{m,j}^{-\gamma}. \tag{52}$$

- We can prove, successively, that for any $j \in \mathbb{N}_0$

$$\mathbf{z}_{m,j} \to \mathbf{0} \quad \text{weakly in } \mathbf{W}_0^{1,\gamma}(\omega), \quad \text{as } m \to \infty, \tag{53}$$

$$\mathbf{z}_{m,j} \to \mathbf{0} \text{ strongly in } \mathbf{L}^\kappa(\omega), \quad \text{as } m \to \infty, \quad \text{for any } \kappa : 1 \leq \kappa < \gamma^*,$$

$$\mathbf{z}_{m,j} \to \mathbf{0} \quad \text{strongly in } \mathbf{L}^s(\omega), \quad \text{as } m \to \infty, \quad \text{for any } s : 1 \leq s < \infty, \tag{54}$$

$$\mathbf{z}_{m,j} \to \mathbf{0} \quad \text{weakly in } \mathbf{W}_0^{1,s}(\omega), \quad \text{as } m \to \infty, \quad \text{for any } s : 1 \leq s < \infty. \tag{55}$$

- We prove that

$$
\begin{aligned}
&\mu_1 \int_\omega \left(|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - |\mathbf{D}(\mathbf{u})|^{\gamma-2}\mathbf{D}(\mathbf{u})\right) : \mathbf{D}(\mathbf{z}_{m,j})\, d\mathbf{x} + \\
&\mu_2 \int_\omega \left(|\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - |\mathbf{D}(\mathbf{u})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u})\right) : \mathbf{D}(\mathbf{z}_{m,j})\, d\mathbf{x} = \\
&\mu_1 \int_\omega \left(\mathbf{S}_1 - |\mathbf{D}(\mathbf{u})|^{\gamma-2}\mathbf{D}(\mathbf{u})\right) : \mathbf{D}(\mathbf{z}_{m,j})\, d\mathbf{x} + \\
&\mu_2 \int_\omega \left(\mathbf{S}_2 - |\mathbf{D}(\mathbf{u})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u})\right) : \mathbf{D}(\mathbf{z}_{m,j})\, d\mathbf{x} + \\
&\int_\omega p^1_{\epsilon_m} \operatorname{div} \mathbf{z}_{m,j}\, d\mathbf{x} + \\
&\int_\omega \left(\mathbf{u}_{\epsilon_m} \otimes \mathbf{u}_{\epsilon_m} \Phi_{\epsilon_m}(|\mathbf{u}_{\epsilon_m}|) - \mathbf{u} \otimes \mathbf{u} + p^2_{\epsilon_m}\mathbf{I}\right) : \mathbf{D}(\mathbf{z}_{m,j})\, d\mathbf{x} \\
&:= J^1_{m,j} + J^2_{m,j} + J^3_{m,j} + J^4_{m,j} \leq C 2^{-\frac{j}{\gamma}} \ (\text{as } m \to \infty).
\end{aligned}
\tag{56}
$$

- Using an argument of Dal Maso and Murat (1998), we start by proving, for any $\theta \in (0,1)$, that

$$\limsup_{m\to\infty} \int_\omega g_{\epsilon_m}^\theta \, d\mathbf{x} \leq C_1 2^{-\theta \frac{j}{\gamma}} + C_2 2^{-\theta \frac{j}{\gamma} - (1-\theta)j} \to 0 \text{ (as } j \to \infty), \qquad (57)$$

$$g_{\epsilon_m} := \mu_1 \left| |\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{\gamma-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - |\mathbf{D}(\mathbf{u})|^{\gamma-2}\mathbf{D}(\mathbf{u}) \right| + \mu_2 \left| |\mathbf{D}(\mathbf{u}_{\epsilon_m})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}_{\epsilon_m}) - |\mathbf{D}(\mathbf{u})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u}) \right|,$$

- Then for any $\theta \in (0,1)$

$$\limsup_{m\to\infty} \int_\omega g_{\epsilon_m}^\theta \, d\mathbf{x} = 0.$$

- Passing to a subsequence,

$$g_{\epsilon_m} \to 0 \quad \text{a.e. in } \omega, \quad \text{as } m \to \infty. \qquad (58)$$

- Using the monotonicity and continuity on $\mathbf{D}(\mathbf{u})$,

$$\mathbf{D}(\mathbf{u}_{\epsilon_m}) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \omega, \quad \text{as } m \to \infty. \qquad (59)$$

- Vitali's theorem allow us to conclude that $\mathbf{S}_1 = |\mathbf{D}(\mathbf{u})|^{\gamma-2}\mathbf{D}(\mathbf{u})$ and $\mathbf{S}_2 = |\mathbf{D}(\mathbf{u})|^{q(\mathbf{x})-2}\mathbf{D}(\mathbf{u})$.