A temperature-dependent model for adhesive contact with friction

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joint work with
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We consider
a **thermo** viscoelastic body $\Omega \subset \mathbb{R}^3$ which is in

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and we study its evolution, taking into account

- **unilateral contact** (Signorini conditions)
- **adhesive forces** ($\sim$ glue) between the body and the support $\Rightarrow$ **energy and dissipation concentrated** on the contact surface
- **frictional effects** (Coulomb law)
- **thermal effects**: in the bulk domain and on the contact surface

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- **thermal effects**: in the bulk domain and on the contact surface (for the moment, **neglected**)

Related literature

(on static, quasistatic, dynamic **contact problems** with or without friction, with or without adhesion/delamination, in the **isothermal case**):

- ....
- Chau, Fernández, Sofonea, Telega, Han
- Raous, Cangémi, Cocou
- Martins, Oden
- Kočvara, Mielke, Roubiček, Scardia, Thomas, Zanini
- Eck, Jarušek
- Andersson, Andrews, Klarbring, Kuttler, Shillor, Wright
- ....

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    "phase parameter" $\sim$ proportion of active bonds between body & support

  - $\mathbf{u}_{|\Gamma_c}$ (trace of the displacement)
  - $\nabla\chi$ $(\mathbf{u}_{|\Gamma_c} \& \nabla\chi$ account for local interactions between adhesive & body, and in the adhesive)

Modeling

We refer to M. Frémond's theory

[Frémond, '80s–'90s & "Non-Smooth Thermomechanics" 2002]

- Equations for **u** and χ recovered from the **principle of virtual powers**
- The **energy balance** of the system also includes **micro-forces and micro-motions**
  - microscopic bonds are responsible for the adhesion, microscopic motions lead to rupture $\Rightarrow$ evolution of the adhesion
  - account for the **power of the microscopic forces** in the power of the interior forces

Physical constraints

Constitutive relations are recovered from the **volume & surface free energies**

$$\Psi_\Omega = \Psi_\Omega(\varepsilon(\mathbf{u})), \qquad \Psi_{\Gamma_c} = \Psi_{\Gamma_c}(\mathbf{u}_{|_{\Gamma_c}}, \chi, \nabla\chi)$$

and the **volume & surface pseudo-potentials of dissipation**

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$\rightsquigarrow$ **non-smooth (multivalued) operators** in the equations

The adhesion phenomenon

- The "damage parameter" $\chi \sim$ **fraction of active glue fibers** at each point of the contact surface
  - $\chi = 0$ no adhesion (completely broken bonds)
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$$I_{[0,1]}(\chi) \text{ in the surface energy functional } \Psi_c \;\Rightarrow\; \partial I_{[0,1]}(\chi) \text{ in eq. for } \chi$$

- "Damage" of the glue is **irreversible**, hence we enforce $\boxed{\dot\chi \leq 0}$ by the term

$$I_{(-\infty,0]}(\dot\chi) \text{ in the surface dissipation potential } \Phi_c \;\Rightarrow\; \partial I_{(-\infty,0]}(\dot\chi) \text{ in eq. for } \chi$$

The unilateral contact

- Notation for the normal and tangential components of displacement vector $\mathbf{u}$ and stress vector $\sigma\mathbf{n}$

$$\mathbf{u} = u_N\mathbf{n} + \mathbf{u}_T, \quad u_N = u_i n_i, \quad \sigma\mathbf{n} = \sigma_N\mathbf{n} + \sigma_T, \quad \sigma_N = \sigma_{ij} n_i n_j$$

with $\mathbf{n} = (n_i)$ outward normal unit vector to $\partial\Omega$.

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and to render the **Signorini conditions**

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It is given by

$$\boxed{R_N = -\sigma_N = \chi u_N + \partial I_{]-\infty,0]}(u_N)}$$

$$\Updownarrow$$

$$\boxed{u_N \leq 0, \ \sigma_N + \chi u_N \leq 0, \ u_N\,(\sigma_N + \chi u_N) = 0}$$

The friction effects: the Coulomb law

The **tangential component** of the reaction on Γ_c is

$$\mathbf{R}_T = -\sigma_T = \chi \mathbf{u}_T + \nu |\sigma_N + \chi u_N| \mathbf{d}(\dot{\mathbf{u}}_T)$$

where

- $$\mathbf{d}(\mathbf{v}_T) = \begin{cases} \dfrac{\mathbf{v}_T}{|\mathbf{v}_T|} & \text{if } \mathbf{v}_T \neq \mathbf{0} \\ \mathbf{z}_T \quad |\mathbf{z}| \leq 1 & \text{if } \mathbf{v}_T = \mathbf{0} \end{cases}$$

  $\rightsquigarrow$ if v_T is scalar, then $\mathbf{d} = \mathrm{Sign} : \mathbb{R} \rightrightarrows \mathbb{R}$
- ν **friction coefficient**
- $\sigma_N + \chi u_N = -\partial I_{(-\infty,0]}(u_N)$

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The regularized Coulomb law

The tangential component of the reaction needs to be **regularized**

$$-\sigma_T = \chi \mathbf{u}_T + \nu |\mathcal{R}(\sigma_N + \chi u_N)| \mathbf{d}(\dot{\mathbf{u}}_T)$$

where

- $$\mathbf{d}(\mathbf{v}_T) = \begin{cases} \dfrac{\mathbf{v}_T}{|\mathbf{v}_T|} & \text{if } \mathbf{v}_T \neq \mathbf{0} \\ \mathbf{z}_T \quad |\mathbf{z}| \leq 1 & \text{if } \mathbf{v}_T = \mathbf{0} \end{cases}$$
- $\mathcal{R}$ **nonlocal smoothing operator** (physically meaningful)

For friction problems without adhesion, use of $\mathcal{R}$ first proposed in **[Duvaut,'80]**

The PDE system: the momentum balance

$\Omega \subset R^3$ smooth, bounded and $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_c$

- **the momentum balance** (**quasistatic** evolution)

$$-\operatorname{div} \sigma = \mathbf{f} \quad \text{in } \Omega \times (0, T)$$
$$\sigma = K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}})$$

where: K elasticity tensor, K_v viscosity tensor, $\mathbf{f}$ external mechanical force

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- **boundary conditions**

$$\begin{aligned}
&\mathbf{u} = \mathbf{0} && \text{on } \Gamma_1 \times (0,T),\\
&\sigma\mathbf{n} = \mathbf{g} && \text{on } \Gamma_2 \times (0,T),\\
&-\sigma_N = \chi u_N + \partial I_{]-\infty,0]}(u_N) && \text{on } \Gamma_c \times (0,T),\\
&-\sigma_T = \chi \mathbf{u}_T + \nu|\mathcal{R}(\sigma_N + \chi u_N)|\mathbf{d}(\dot{\mathbf{u}}_T) && \text{on } \Gamma_c \times (0,T).
\end{aligned}$$

The PDE system: the evolution of the adhesion

We consider on the contact surface

$$\dot{\chi} - \Delta\chi + \partial I_{[0,1]}(\chi) + \partial I_{]-\infty,0]}(\dot{\chi}) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \qquad \text{on } \Gamma_c \times (0,T)$$

$$\partial_{n_s}\chi = 0, \qquad \text{on } \partial\Gamma_c \times (0,T)$$

- $\partial I_{[0,1]}(\chi) \Rightarrow \chi \in [0,1]$ **(physical consistency)**
- $\partial I_{]-\infty,0]}(\dot{\chi}) \Rightarrow \dot{\chi} \leq 0$ **(irreversible adhesion)**
- ω constant (coefficient of internal cohesion, neglected in the sequel)
- $-\frac{1}{2}|\mathbf{u}|^2$ source of damage due to displacement

The Problem: variational formulation

- Bilinear forms of linear viscoelasticity

$$\begin{cases} a(\mathbf{u},\mathbf{v}) := \lambda \int_\Omega \mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v}) + 2\mu \sum_{i,j=1}^3 \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) & \lambda,\ \mu \text{ the Lamé consts.,} \\ b(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^3 \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \end{cases}$$

for $\mathbf{u}$, $\mathbf{v} \in \mathbf{W} = \{\mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v} = \mathbf{0}$ a.e. on $\Gamma_1\}$.

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- **The problem:** Find $(\mathbf{u}, \chi, \eta)$ such that

$$b(\dot{\mathbf{u}},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v}\cdot\mathbf{n} + \int_{\Gamma_c} \nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)\cdot\mathbf{v}_T \ni \langle \mathbf{F},\mathbf{v}\rangle$$

$$\forall \mathbf{v} \in \mathbf{W} \quad \text{a.e. in } (0,T)$$

$$\eta \in \partial I_{(-\infty,0]}(u_N) \quad \text{on } \Gamma_c \times (0,T)$$

$$\dot{\chi} - \Delta\chi + \partial I_{(-\infty,0]}(\dot{\chi}) + \partial I_{[0,1]}(\chi) \ni -\frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0,T),$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \qquad + \textit{Cauchy conditions}$$

Analytical difficulties

$$b(\dot{\mathbf{u}},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v}\cdot\mathbf{n} +$$

$$+ \int_{\Gamma_c} \nu |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \text{ a.e. in } (0,T)$$

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$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \qquad + \textit{Cauchy conditions}$$

$\rightsquigarrow$ **double multivalued constraint** on χ and $\dot{\chi}$

$\Rightarrow$ **doubly nonlinear** character

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$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \qquad + \textit{Cauchy conditions}$$

$\rightsquigarrow$ **(quadratic) coupling** terms on the **boundary**

$\Rightarrow$ (we need **sufficient regularity** for $\mathbf{u}$ and $\dot{\mathbf{u}}$ to control their traces)

Analytical difficulties

$$
\begin{aligned}
& b(\dot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v} \cdot \mathbf{n} \\
& + \int_{\Gamma_c} \nu |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T) \\
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& \partial_{\mathbf{n}_s} \chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \qquad + \textit{Cauchy conditions}
\end{aligned}
$$

$\rightsquigarrow$ **double multivalued constraint** on u_N and $\dot{\mathbf{u}}_T$ on the boundary.

$\Rightarrow$ **main difficulty!**

A regularization of the boundary term $|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$ is **crucial!**

A global-in-time existence theorem

$$
\begin{aligned}
& b(\dot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v}\cdot\mathbf{n} + \\
& + \int_{\Gamma_c} \nu |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T) \\
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& \partial_{\mathbf{n}_s} \chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \quad + \text{Cauchy conditions}
\end{aligned}
$$

Theorem **[Bonetti, B., Rossi, JDE, 2012]**

There **exists a solution** $(\mathbf{u}, \chi, \eta)$

$$
\begin{aligned}
& \mathbf{u} \in H^1(0,T; H^1(\Omega)) \\
& \chi \in W^{1,\infty}(0,T; L^2(\Gamma_c)) \cap H^1(0,T; H^1(\Gamma_c)) \cap L^\infty(0,T; H^2(\Gamma_c)) \\
& \eta \in L^2(0,T; H^{-1/2}(\Gamma_c))
\end{aligned}
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Outline of the proof of existence

- Moreau-Yosida regularization of non-smooth operators
- Time discretization scheme (time-incremental minimization)
- Existence result for the discretized system
- Uniform estimates
- Passage to the limit
- Identification (of nonlinearities)

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- **Passage to the limit**
- **Identification (of nonlinearities)**

Passage to the limit

- by compactness and monotonicity-semicontinuity arguments
  - ....
  - **Main difficulty**: the terms

$$\nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T) \qquad \& \qquad \eta \in \partial I_{(-\infty,0]}(u_N)$$

simultaneously present in the first equation.

A crucial a priori estimate

- By comparison in the first equation

$$|\partial I_{(-\infty,0]}(u_N)\mathbf{n} + \nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C$$

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$\nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$ & $\partial I_{(-\infty,0]}(u_N)\mathbf{n}$ are **orthogonal**, hence

$$\begin{cases} |\partial I_{(-\infty,0]}(u_N)\mathbf{n}|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C, \\ |\nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C \end{cases}$$

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- In addition (from its definition), $|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^\infty((0,T)\times(\Gamma_c))} \leq 1$

- First step: identification of $\partial I_{(-\infty,0]}(u_N)$

  $\rightsquigarrow$ by semicontinuity, passing to the limit weakly in the first equation

- Second step: identification of $|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$

  $\rightsquigarrow$ by compactifying character of $\mathcal{R}$

  - $\mathcal{R} : L^2(0,T;H^{-1/2}(\Gamma_c)) \to L^2(0,T;L^2(\Gamma_c))$
  - for all $\eta_\varepsilon, \eta \in L^2(0,T;H^{-1/2}(\Gamma_c))$

$$\eta_\varepsilon \rightharpoonup \eta \ \textbf{weakly} \text{ in } L^2(0,T;H^{-1/2}(\Gamma_c))$$
$$\Rightarrow \ \mathcal{R}(\eta_\varepsilon) \to \mathcal{R}(\eta) \ \textbf{strongly} \text{ in } L^2(0,T;L^2(\Gamma_c))$$

The full model: the nonisothermal case

To account for **thermal effects: [Bonetti-B.-Rossi, 2012, submitted]**

- in the **bulk domain** Ω:
  - $\varepsilon(\mathbf{u})$
  - **thermal effects (θ absolute temperature)**
- on the **contact surface** Γ_c:
  - χ
  - **thermal effects (θ_s absolute temperature)**

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- **friction** *coefficient* **depends on the thermal gap** $(\theta - \theta_s)$
  - $$\nu \rightsquigarrow \nu(\theta - \theta_s)$$
  - **contributions due to friction as source of heat on** Γ_c **(***heat generated by friction***)**.

The equations for θ and θ_{s}

Let θ be the temperature in the bulk domain, solving the following entropy equation (rescaled energy balance, under small perturbation assumption)

$$\partial_t(\log\theta) - \operatorname{div} \mathbf{u}_t - \Delta\theta = h \quad \text{on } \Omega \times (0, T),$$

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The entropy equation for θ_s on the contact surface is

$$\begin{aligned} &\partial_t(\log\theta_s) - \lambda'(\chi)\chi_t - \Delta\theta_s = \\ &\qquad = \chi(\theta-\theta_s)+\nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T| \quad \text{in } \Gamma_c\times(0,T) \\ &\partial_n\theta_s = 0 \quad \text{on } \partial\Gamma_c\times(0,T)\,. \end{aligned}$$

The full system

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}}) \qquad\right) = \mathbf{f} \quad \text{in } \Omega \times (0,T),$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \quad \left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}}) \qquad\right)\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_2 \times (0,T),$$

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$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni \qquad -\frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T),$$

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highly nonlinear PDE system!!!

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Main difficulty: **boundary coupling terms** (thermal & frictional effects)

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Main difficulty: **boundary coupling terms** (thermal & frictional effects)

- friction *coefficient* depends on the thermal gap $(\theta-\theta_s)$
- frictional contributions as source of heat on Γ_c in the eqs. for θ and θ_s

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- **singular character of the θ, θ_s-equations** (**θ-equation is coupled with a third type boundary condition**)

A key a priori estimate

- Testing the equation for θ by $v = \theta\, w\,, w \in W^{1,q}(\Omega)$, with $q > 3$, we prove

$$\|\partial_t \theta\|_{L^1(0,T;(W^{1,q}(\Omega))')} \leq C$$

- In addition

$$\|\theta\|_{L^2(0,T;H^1(\Omega))} \leq C$$

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...careful estimates + assumptions on the regularizing operator $\mathcal{R}$ + conditions on the friction *coefficient* $\nu(\theta - \theta_s)$...

$$\Downarrow$$

passage to the limit in the approximate problem $\rightsquigarrow$ **Existence result** for the full system.

A global-in-time existence theorem

Theorem **[Bonetti, B., Rossi, 2012, submitted]**

There **exists a solution** $(\theta, \theta_s, \mathbf{u}, \chi, \eta)$

$$
\begin{aligned}
&\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)),\\
&\log(\theta) \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))'),\\
&\theta_s \in L^2(0,T;H^1(\Gamma_c)) \cap L^\infty(0,T;L^1(\Gamma_c)),\\
&\log(\theta_s) \in L^\infty(0,T;L^2(\Gamma_c)) \cap H^1(0,T;(H^1(\Gamma_c))'),\\
&\mathbf{u} \in H^1(0,T;H^1(\Omega))\\
&\chi \in H^1(0,T;L^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap L^2(0,T;H^2(\Gamma_c))\\
&\eta \in L^2(0,T;H^{-1/2}(\Gamma_c))
\end{aligned}
$$