ADMAT2012
PDEs for multiphase advanced materials

# An Abstract Existence Theorem for Parabolic Systems

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65th birthday of Gianni Gilardi

Annali Scuola Normale Superiore - Pisa
Classe di Scienze
Serie IV - Vol. IX, n. 4 (1981)

**The Behavior of the Free Boundary for the Dam Problem (*).**

HANS WILHELM ALT - GIANNI GILARDI

**1. – Introduction.**

In this paper we study the behavior of the free boundary for the dam problem near the given boundaries to reservoirs, atmosphere, and impervious layers, where we restrict ourselves to the two dimensional homogeneous case.

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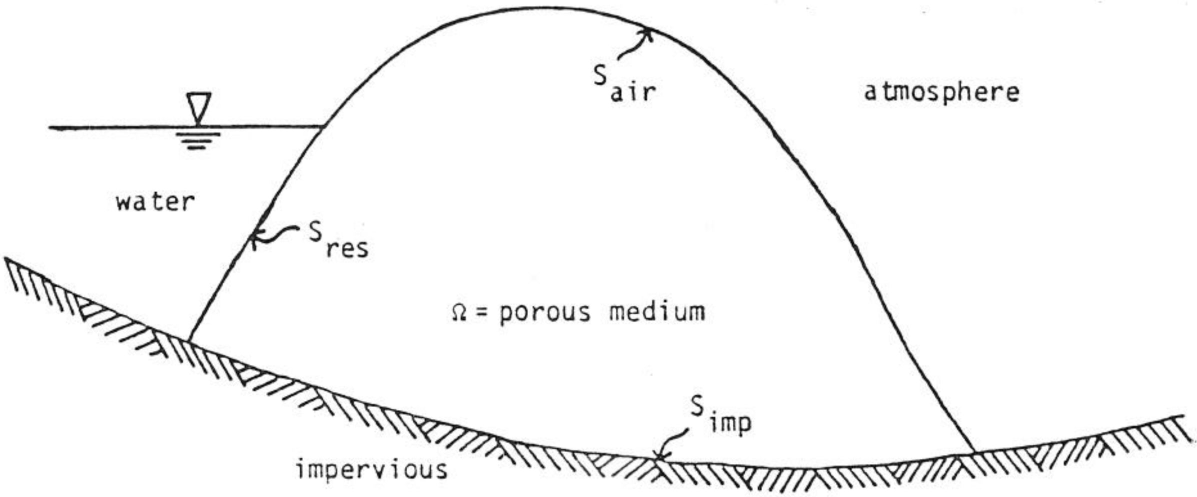


$$M(u^0) := \{v \in H^{1,2}(\Omega) : v = u_0 \text{ on } S_{\text{res}} \text{ and } v \leqslant 0 \text{ on } S_{\text{air}}\} .$$

(1.1) $u \in M(u_0)$ with $u \geqslant 0$, and $\gamma \in L^\infty(\Omega)$ with $\chi_{\{u>0\}} \leqslant \gamma \leqslant 1$.

(1.2) For every $v \in M(u_0)$

$$\int_\Omega \nabla(v-u)(\nabla u + \gamma e) \geqslant 0 .$$

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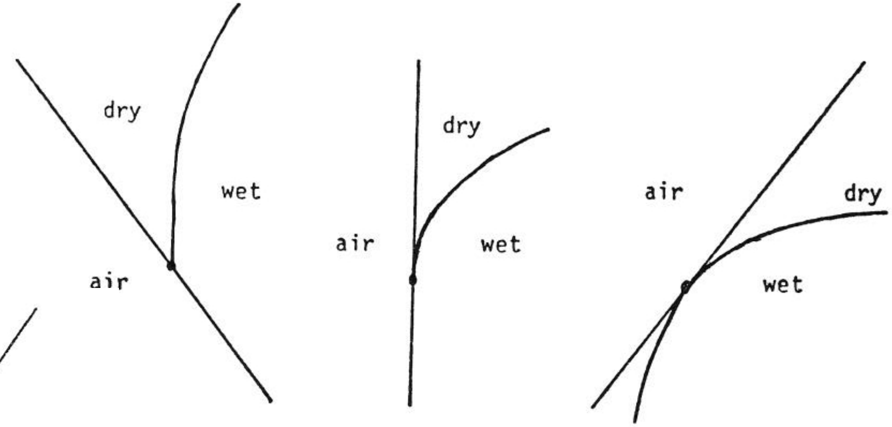


Fig.3.1 Fig.3.2 Fig.3.3

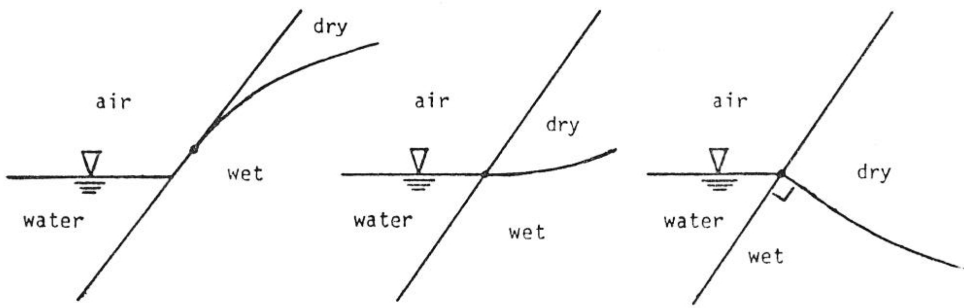


Fig.2.1 Fig.2.2 Fig.2.3

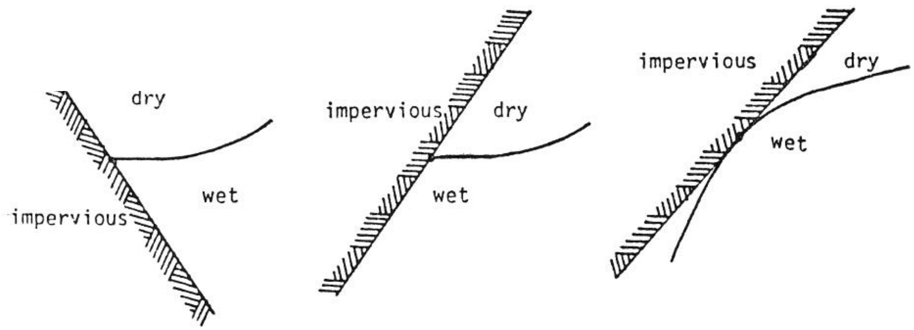


Fig.4.1 Fig.4.2 Fig.4.3

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COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **11**, Number **5**, September **2012**

10.3934/cpaa.2012.11.2079

pp. **2079–2123**

# AN ABSTRACT EXISTENCE THEOREM FOR PARABOLIC SYSTEMS

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(Communicated by Bernd Kawohl)

Abstract. In this paper we prove an abstract existence theorem which can be applied to solve parabolic problems in a wide range of applications. It also applies to parabolic variational inequalities. The abstract theorem is based on a Gelfand triple $(V, H, V^*)$, where the standard realization for parabolic systems of second order is $(W^{1,2}(\Omega), L^2(\Omega), W^{1,2}(\Omega)^*)$. But also realizations to other problems are possible, for example, to fourth order systems. In all applications to boundary value problems the set $M \subset V$ is an affine subspace, whereas for variational inequalities the constraint $M$ is a closed convex set.

The proof is purely abstract and new. The corresponding compactness theorem is based on [5]. The present paper is suitable for lectures, since it relays on the corresponding abstract elliptic theory.

1. **Introduction.** In this paper we give an abstract existence proof for parabolic systems. The abstract theorem has been applied to many boundary value problems. Among other things it includes also cases in which the parabolic part is degenerated, therefore it contains elliptic-parabolic problems. It also includes the case of a convex constraint, therefore it contains variational inequalities. The proof for the combination of both effects is new, and has been presented by me in the lecture about partial differential equations in 2003.

**PDE system given by:**

$$\partial_t v_k + \text{div} q_k = g_k := r_k + \mathbf{f}_k \qquad \text{for } k = 1, \ldots, N$$

Multiply equations by $\lambda_k$ and add them:

$$(*)\ \underbrace{\sum_k \lambda_k \partial_t v_k}_{= \partial_t w} + \underbrace{\sum_k \lambda_k \text{div} q_k}_{= \text{div}(\sum_k \lambda_k q_k) - \sum_k \nabla\lambda_k \bullet q_k} - \sum_k \lambda_k r_k = \sum_k \lambda_k \mathbf{f}_k$$

If $(*)$, then this is

$$\partial_t w + \text{div}(\sum_k \lambda_k q_k) + \underbrace{\Big( \sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k (-r_k) \Big)}_{=: D \geq 0 \text{ Dissipation}} = \sum_k \lambda_k \mathbf{f}_k$$

**Free energy inequality (physical interpretation):**

$$\partial_t w + \text{div}(\sum_k \lambda_k q_k) \leq \sum_k \lambda_k \mathbf{f}_k$$

$$\partial_t w + \operatorname{div}(\sum_k \lambda_k q_k) + \Big( \underbrace{\sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k(-r_k)}_{=:\, D \geq 0} \Big) = \sum_k \lambda_k \mathbf{f}_k$$

## A-priori estimate (mathematical interpretation):

$$\int_\Omega \underbrace{\int_{t_0}^{t_1} \partial_t w \,\mathrm{dL}_1}_{\text{integrate in } t} \,\mathrm{dL}_n + \int_{t_0}^{t_1}\int_\Omega \underbrace{\sum_k \nabla\lambda_k \bullet(-q_k) + \sum_k \lambda_k(-r_k)}_{= D \geq 0} \,\mathrm{dL}_n\,\mathrm{dL}_1$$

$$= -\int_{t_0}^{t_1} \underbrace{\int_\Omega \operatorname{div}(\sum_k \lambda_k q_k)\,\mathrm{dL}_n}_{\text{boundary data}} \,\mathrm{dL}_1 + \underbrace{\int_{t_0}^{t_1}\int_\Omega \sum_k \lambda_k \mathbf{f}_k \,\mathrm{dL}_n\,\mathrm{dL}_1}_{\text{external terms}}$$

leads to

$$\int_\Omega w(t_1,x)\,\mathrm{d}x + \int_{t_0}^{t_1}\int_\Omega \underbrace{\sum_k \nabla\lambda_k \bullet(-q_k) + \sum_k \lambda_k(-r_k)}_{= D \geq 0} \,\mathrm{dL}_n\,\mathrm{dL}_1$$

$$= \underbrace{\int_\Omega w(t_0,x)\,\mathrm{d}x}_{\text{initial data}} + \int_{t_0}^{t_1} \underbrace{\int_{\partial\Omega} (\sum_k \lambda_k \mathbf{f}_k)\bullet\nu \,\mathrm{dH}_{n-1}}_{\text{boundary data}} \,\mathrm{dL}_1 + \text{(external terms)}$$

**Multipliers $\lambda_k = u_k$ as independent variables $u = (u_k)_{k=1,\dots,N}$**

$\partial_t v_k + \mathrm{div} q_k = g_k$ for $k = 1, \ldots, N$

**Multipliers $\lambda_k = u_k$:** $(*)$ $\partial_t w = \sum_k u_k \partial_t v_k$

**If multipliers $u_k$ or parabolic terms $v_k$ are independent variables:**

**Proposition** $(*)$ is satisfied, if and only if

$$v_k = \beta_k(u) = \psi_{'u_k}(u)\,, \quad w = f(u)\,, \quad w + \psi = u \bullet v$$

Proof: $\sum_l f_{'u_l} \partial_t u_l = \partial_t w = \sum_k u_k \partial_t v_k = \sum_k u_k \beta_{u_k 'u_l} \partial_t u_l \quad \Leftrightarrow \quad f_{'u_l} = \sum_k u_k \beta_{u_k 'u_l}$ $(\Rightarrow \beta_k = \psi_{'u_k})$

$\Leftrightarrow \quad f_{'u_l} = (u \bullet \beta - \psi)_{'u_l}$

$\Leftrightarrow \quad f = u \bullet \beta - \psi + \text{const}$

**Proposition** $(*)$ is satisfied, if and only if

$$u_k = \beta_k^*(v)\,, \quad w = \psi^*(v)\,, \quad \beta_k^* = \psi^*_{'v_k}$$

Proof: $\sum_k \psi^*_{'v_k} \partial_t v_k = \partial_t \psi^*(v) = \partial_t w = \sum_k u_k \partial_t v_k = \sum_k \beta_k^*(v) \partial_t v_k \quad \Leftrightarrow \quad \psi^*_{'v_k}(v) = \beta_k^*(v)$

**Conclusion** $(*)$ is equivalent to

$$\psi^*(v) + \psi(u) = u \bullet v$$

**Under convexity assumption:** $v$ conjuate variable of $u$, $\psi^*$ is the conjuate convex function to $\psi$

and $v_k = \psi_{'u_k}(u) = \beta_k(u)\,, \quad u_k = \psi^*_{'v_k}(v) = \beta_k^*(v)$

**Remark**

- $p + f = \varrho\mu$ ($p$ pressure, $f \equiv f(\varrho)$ free energy, $\mu := f_{'\varrho}$)
- $\eta + \frac{1}{\theta} f = \frac{1}{\theta} e$ ($\theta$ temperature, $e$ internal energy)

that is $f = e - \theta\eta$ ($\eta \equiv \eta(e)$ entropy, $f \equiv f(\theta)$ free energy, $\eta_{'e} = \frac{1}{\theta}$, $f_{'\theta} = -\eta$)

$$\partial_t v_k + \mathrm{div} q_k = g_k \text{ for } k = 1, \ldots, N, \quad \text{multipliers } u_k\text{: } (*)\ \partial_t w = \textstyle\sum_k u_k \partial_t v_k$$

$$v_k = \psi_{'u_k}(u) = \beta_k(u)\,, \quad u_k = \psi^*_{'v_k}(v) = \beta^*_k(v)\,, \quad w = f(u) = \psi^*(v)$$

**Hence the parabolic system reads** for $k = 1, \ldots, N$

$$\partial_t \beta_k(u) + \mathrm{div} q_k(u, \nabla u) = g_k$$

that is in a domain $]0, T[\times\Omega$ for $\zeta \in C_0^\infty$

$$\int_0^T \int_\Omega \sum_k \zeta_k \Big(\partial_t \beta_k(u) + \mathrm{div} q_k(u, \nabla u) - g_k\Big)\, \mathrm{d}x\, \mathrm{d}t = 0$$

or in a weak version

$$\int_0^T \Big( - \int_\Omega \sum_k \partial_t \zeta_k \beta_k(u)\, \mathrm{d}x + \int_\Omega \sum_k \big(\nabla \zeta_k \bullet (-q_k(u, \nabla u)) - \zeta_k g_k\big)\, \mathrm{d}x \Big)\, \mathrm{d}t = 0$$

**6.3 Existence theorem.** Let $H$ be a Hilbert space and $V$ a separable reflexive Banach space as in (6.1) and with a compact embedding $\mathrm{Id}_V : V \to H$. Further, let $M \subset V$ be a nonempty closed affine set, $\mathcal{M}$ and $\mathcal{A}$ as in (6.3) and (6.5), and $b : H \to H$ as in (6.4), with the assumptions 6.2(1)–6.2(3).

Then there exist solutions of the "evolution equation", that is if $u_0 \in \mathrm{clos}_H(M)$ there is a $u \in L^p([0,T];V)$ with

$$u(t) \in M \quad \text{for almost all } t,$$

$$\left\{ \begin{aligned} \int_0^T \Big( - (\, \partial_t \xi(t) \,,\, b(u(t)) - b(u_0) \,)_H \\ + \langle\, \xi(t) \,,\, A(t, u(t)) \,\rangle_V \Big)\, \mathrm{d}t = 0 \end{aligned} \right\} \tag{6.8}$$

for all $\xi \in C_0^\infty([0,T[;V)$ with $\xi(t) \in M_1$ for all $t$.

Here $M_1 \subset V$ is the subspace, such that $M = u_1 + M_1$ for every $u_1 \in M$.

$H = L^2(\Omega)$

$V = W^{1,2}(\Omega)$

$\xi(t)(x) = \zeta(t,x)$

$u(t)(x) = u(t,x)$

$b(u(t))(x) = \beta(u(t,x))$

## General setting:

Spaces: $H$ Hilbert space, $V$ separable reflexive Banach space
$(V,H,V^*)$ Gelfand tripel: $V \subset H$ $(\Longrightarrow H^* \subset V^*)$
Constraint $M \subset V$ nonempty closed convex

Parabolic: $b : H \to H$ and $\psi : H \to \mathbb{R}$ convex continuously differentiable
$(*)$ $b = \nabla\psi$

Elliptic: $\mathcal{A} : \mathcal{M} \to L^{p^*}([0,T];V^*)$
$\mathcal{M} := \{u \in L^p([0,T];M)\,;\; B(u) \in L^\infty([0,T])\}$
$\mathcal{A}(u)(t) = A(t,u(t))$

## Conclusion:

Under assumption **(1)**-**(3)** there exist solutions of the "evolution equation", that is for given $u_0 \in \text{clos}_H(M)$ there is an $u \in L^p([0,T];V)$ with

$$u(t) \in M \qquad \text{for almost all } t,$$

$$\left\{\begin{aligned} & B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) + (\,\bar u - v(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\ & + \int_0^{\bar t} \Big( -(\,\partial_t(\bar u - v)(t)\,,\, b(u(t)) - b(u_0)\,)_H \\ & \qquad\qquad + \langle\, u(t) - v(t)\,,\, A(t,u(t))\,\rangle_V \Big)\,\mathrm{d}t \le 0 \\ & \text{for almost all } \bar t \in ]0,T[, \end{aligned}\right\}$$

for all $v \in C^\infty([0,T];V)$ with $v(t) \in M$ for almost all $t$.

Consider the parabolic part:

$$\Phi_{\bar u}(u,v)(\bar t) := B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) + (\, \bar u - v(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H - \int_0^{\bar t} (\, \partial_t(\bar u - v)(t)\,,\, b(u(t)) - b(u_0)\,)_H \, \mathrm{d}t$$

$$= \int_0^{\bar t} (\, u(t) - v(t)\,,\, \partial_t b(u(t))\,)_H \, \mathrm{d}t \qquad \text{(this uses formally } (*)\text{)}$$

Valid version, if $\partial_t b(u)$ exists:

$b(u(0)) = b(u_0)$ and for almost all $t > 0$

$$(\, u(t) - v(t)\,,\, \partial_t b(u(t))\,)_H + \langle\, u(t) - v(t)\,,\, A(t, u(t))\,\rangle_V \le 0$$

for all $v \in C^\infty([0,T];V)$ with $v(t) \in M$ for almost all $t$.

Time discrete approximation:

$$\left(\, u^i - v\,,\, \frac{1}{h}(b(u^i) - b(u^{i-1}))\,\right)_H + \left\langle\, u^i - v\,,\, A^i(u^i)\,\right\rangle_V \le 0$$

for all $v \in M$

Use elliptic theory (pseudo-monotone operators): Find $u \in M$ with

$$\langle\, u - v\,,\, F(u)\,\rangle_V \le 0$$

for all $v \in M$

$V$ separable reflexive Banach space

Constraint $M \subset V$ nonempty closed convex, $F : M \to V^*$

**Assumptions:**

**(1) Boundedness**

$F$ maps bounded sets of $M$ into bounded sets of $V^*$

**(2) Continuity condition**

Let $u_m, u \in V$ and $v^* \in V^*$ with

$$\left\{ \begin{array}{c} u_m, u \in M \text{ and } u_m \to u \text{ weakly in } V \text{ for } m \to \infty, \\ F(u_m) \to v^* \text{ weakly}^* \text{ in } V^* \text{ for } m \to \infty, \text{ and} \\ \limsup_{m\to\infty} \langle\, u_m \,,\, F(u_m)\,\rangle_V \leq \langle\, u \,,\, v^* \,\rangle_V \,, \end{array} \right\}$$

then

$$\left\{ \begin{array}{c} \langle\, u - v \,,\, F(u) - v^* \,\rangle_V \leq 0 \text{ for all } v \in M, \text{ and} \\ \limsup_{m\to\infty} \langle\, u_m \,,\, F(u_m)\,\rangle_V = \langle\, u \,,\, v^* \,\rangle_V \,. \end{array} \right\}$$

**(3) Coerciveness**

For some $\bar{u} \in M$ there holds

$$\frac{\langle\, u - \bar{u} \,,\, F(u)\,\rangle_V}{\|u - \bar{u}\|_V} \to \infty \qquad \text{for } u \in M,\ \|u - \bar{u}\|_V \to \infty.$$

**Conclusion:** Under these assumptions there exists $u \in M$, so that

$$\langle\, u - v \,,\, F(u)\,\rangle_V \leq 0 \qquad \text{for all } v \in M.$$

Proof of (2) for monotone operators:
$F = A$ monotone

$$0 \le \langle u_m - v \, , \, A(u_m) - A(v) \rangle_V$$

$$= \underbrace{\langle u_m \, , \, A(u_m) \rangle_V}_{\substack{\text{limit} \\ \le \langle u \, , \, v^* \rangle_V}} - \underbrace{\langle v \, , \, A(u_m) \rangle_V}_{\to \langle v \, , \, v^* \rangle_V} - \underbrace{\langle u_m - v \, , \, A(v) \rangle_V}_{\to \langle u - v \, , \, A(v) \rangle_V}$$

For $v = u$ it follows $\lim_{m\to\infty} \langle u_m \, , \, A(u_m) \rangle_V = \langle u \, , \, v^* \rangle_V$ and

$$0 \le \langle u - v \, , \, v^* - A(v) \rangle_V$$

Minty lemma ($v \rightsquigarrow (1-\varepsilon)u + \varepsilon v = u - \varepsilon(u-v) \in M$ and $\varepsilon \searrow 0$)

$$0 \le \langle u - v \, , \, v^* - A(u) \rangle_V$$

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Proof of (2) for compact perturbations of monotone operators:
$F(u) = A(u,u)$, $\quad u \mapsto A(v,u)$ monotone

$$0 \le \langle u_m - v \, , \, A(u_m, u_m) - A(u_m, v) \rangle_V$$

$$= \underbrace{\langle u_m \, , \, F(u_m) \rangle_V}_{\substack{\text{limit} \\ \le \langle u \, , \, v^* \rangle_V}} - \underbrace{\langle v \, , \, F(u_m) \rangle_V}_{\to \langle v \, , \, v^* \rangle_V} - \underbrace{\langle u_m - v \, , \, A(u_m, v) \rangle_V}_{\to \langle u - v \, , \, A(u,v) \rangle_V}$$

Rest of argumentation similar as above.

**(1) Boundedness**

$\mathcal{A}$ maps "bounded" sets of $\mathcal{M}$ into bounded sets of $L^{p^*}([0,T];V^*)$
$b$ maps bounded sets of $H$ into bounded sets of $H$

**(2) Continuity condition**

Let $u_m, u \in L^p([0,T];V)$ and $v^* \in L^{p^*}([0,T];V^*)$ with

$$\left\{ \begin{array}{c} u_m(t), u(t) \in M \text{ and } u_m \to u \text{ weakly in } L^p([0,T];V) \text{ for } m \to \infty, \\ \{B(u_m)\,;\ m \in \mathbb{N}\} \text{ bounded in } L^\infty([0,T]) \text{ and} \\ b(u_m) \to b(u) \text{ strongly in } L^1([0,T];H), \\ \mathcal{A}(u_m) \to v^* \text{ weakly in } L^{p^*}([0,T];V^*) \text{ for } m \to \infty, \text{ and} \\ \limsup\limits_{m\to\infty} \int_0^T \langle\, u_m(t)\,,\, \mathcal{A}(u_m)(t)\,\rangle_V \,\mathrm{d}t \le \int_0^T \langle\, u(t)\,,\, v^*(t)\,\rangle_V \,\mathrm{d}t, \end{array} \right\}$$

then

$$\left\{ \begin{array}{c} \int_0^T \langle\, u(t) - v(t)\,,\, \mathcal{A}(u)(t) - v^*(t)\,\rangle_V \,\mathrm{d}t \le 0 \text{ for all } v \in L^p([0,T];V), \text{ and} \\ \limsup\limits_{m\to\infty} \int_0^T \langle\, u_m(t)\,,\, \mathcal{A}(u_m)(t)\,\rangle_V \,\mathrm{d}t = \int_0^T \langle\, u(t)\,,\, v^*(t)\,\rangle_V \,\mathrm{d}t. \end{array} \right\}$$

**(3) Coerciveness** There is an $\bar{u} \in M$ so, that for almost all $t \in ]0,T[$

$$\langle\, u - \bar{u}\,,\, A(t,u)\,\rangle_V \ge c_0 \|u - \bar{u}\|_V^p - C_0 B_{\bar{u}}(u) - G_0(t)$$

for all $u \in M$. Here $G_0 \in L^1([0,T])$.

Let the assumptions **(1)**-**(3)** be satisfied.

**Conclusion:**

Then there exist solutions of the "evolution equation", that is for given $u_0 \in \mathrm{clos}_H(M)$ there is an $u \in L^p([0,T];V)$ with

$$u(t) \in M \qquad \text{for almost all } t,$$

$$\left\{ \begin{array}{l} B_{\bar{u}}(u(\bar{t})) - B_{\bar{u}}(u_0) + (\,\bar{u} - v(\bar{t})\,,\, b(u(\bar{t})) - b(u_0)\,)_H \\ + \displaystyle\int_0^{\bar{t}} \Big( -(\,\partial_t(\bar{u} - v)(t)\,,\, b(u(t)) - b(u_0)\,)_H \\ \qquad\qquad\qquad + \langle\, u(t) - v(t)\,,\, A(t,u(t))\,\rangle_V \Big)\,\mathrm{d}t \le 0 \\ \text{for almost all } \bar{t} \in ]0,T[, \end{array} \right\}$$

for all $v \in C^\infty([0,T];V)$ with $v(t) \in M$ for almost all $t$.

**Conclusion, if $M \subset V$ is a closed affine set:**

There is an $u \in L^p([0,T];V)$ with

$$u(t) \in M \qquad \text{for almost all } t,$$

$$\left\{ \int_0^T \Big( -(\,\partial_t\varphi(t)\,,\, b(u(t)) - b(u_0)\,)_H + \langle\, \varphi(t)\,,\, A(t,u(t))\,\rangle_V \Big)\,\mathrm{d}t = 0 \right\}$$

for all $\varphi \in C_0^\infty([0,T[;V)$ with $\varphi(t) \in M_0$ for almost all $t$.

Here $M_0 \subset V$ is the subspace, such that $M = u_1 + M_0$ for every $u_1 \in M$.

$$\left( u^i - v \,,\, \frac{1}{h}(u^{*i} - u^{*i-1}) \right)_H + \langle\, u^i - v \,,\, w^{*i} \,\rangle_V \leq 0$$

at time $t^i = ih$ where $u^{*i} := b(u^i),\ \ w^{*i} := A^i(u^i)$

It is assumed that $t$ and $s$ are multiple of $h$, say,

$$t = kh, \qquad t + s = (k+j)h$$

We set $v = u^k$ and sum over $i = k+1, \ldots, k+j$. The result is

$$\sum_{i=k+1}^{k+j} \left( u^i - u^k \,,\, u^{*i} - u^{*i-1} \right)_H \leq \sum_{i=k+1}^{k+j} h \,\langle\, u^k - u^i \,,\, w^{*i} \,\rangle_V$$

The left side is

$$\sum_{i=k+1}^{k+j} \left( u^i - u^k \,,\, u^{*i} - u^{*i-1} \right)_H$$

$$= \sum_{i=k+1}^{k+j} \left( \left( u^i \,,\, u^{*i} \right)_H - \left( u^i \,,\, u^{*i-1} \right)_H \right) - \sum_{i=k+1}^{k+j} \left( u^k \,,\, u^{*i} - u^{*i-1} \right)_H$$

Now by Young's inequality it holds

$$\left( u^i \,,\, u^{*i} \right)_H = \psi^*(u^{*i}) + \psi(u^i), \qquad \left( u^i \,,\, u^{*i-1} \right)_H \leq \psi^*(u^{*i-1}) + \psi(u^i)$$

we compute for the left side

$$
\begin{aligned}
&\sum_{i=k+1}^{k+j} \left( u^i - u^k \,,\, u^{*i} - u^{*i-1} \right)_H \\
&= \sum_{i=k+1}^{k+j} \left( \left( u^i \,,\, u^{*i} \right)_H - \left( u^i \,,\, u^{*i-1} \right)_H \right) - \sum_{i=k+1}^{k+j} \left( u^k \,,\, u^{*i} - u^{*i-1} \right)_H \\
&\geq \sum_{i=k+1}^{k+j} \left( \psi^*(u^{*i}) - \psi^*(u^{*i-1}) \right) - \left( u^k , \sum_{i=k+1}^{k+j} (u^{*i} - u^{*i-1}) \right)_H \\
&= \psi^*(u^{*k+j}) - \psi^*(u^{*k}) - \left( u^k \,,\, u^{*k+j} - u^{*k} \right)_H \\
&= E_{\psi^*}(u^{*k+j}, u^{*k}, u^k) = E_{\psi^*}(b(u^{k+j}), b(u^k), u^k) \\
&\geq 0 \qquad \text{(Weierstraß } E\text{-function)}
\end{aligned}
$$

Thus we have shown

$$
\begin{aligned}
0 &\leq E_{\psi^*}(b(u_h(t+s)), b(u_h(t)), u_h(t)) \\
&= E_{\psi^*}(u^{*k+j}, u^{*k}, u^k) \leq \sum_{i=k+1}^{k+j} \left( u^i - u^k \,,\, u^{*i} - u^{*i-1} \right)_H \\
&\leq \sum_{i=k+1}^{k+j} h \left\langle u^k - u^i \,,\, w^{*i} \right\rangle_V = s \cdot \frac{1}{j} \sum_{i=1}^{j} \left\langle u^k - u^{k+i} \,,\, w^{*k+i} \right\rangle_V
\end{aligned}
$$

$$\int_0^t (\, u(s) - v(s)\,,\, \partial_t b(u(s))\,)_H \,\mathrm{d}s =$$

$$\begin{aligned}\Phi_{\bar u}(u,v)(t) := \;& B_{\bar u}(u(t)) - B_{\bar u}(u_0) + (\,\bar u - v(t)\,,\, b(u(t)) - b(u_0)\,)_H \\ & - \int_0^t (\,\partial_t(\bar u - v)(s)\,,\, b(u(s)) - b(u_0)\,)_H \,\mathrm{d}s\end{aligned}$$

We have proved the inequality

$$\Phi_{\bar u}(u,v)(\bar t) + \int_0^{\bar t} \langle\, u(t) - v(t)\,,\, A(t,u(t)\,\rangle_V \,\mathrm{d}t \le 0$$

for all $v \in C^\infty([0,T];V)$ which satisfies $v(t) \in M$. Here $M$ now is an affine subspace. It follows that this inequality then also holds for all $v \in W^{1,p}(]0,T[;M) \subset W^{1,1}(]0,T[;H) \cap L^p(]0,T[;M)$. Now $u_0 \in \mathrm{clos}_H(M)$, hence there is $u_{0\varepsilon} \in M$ so that $u_{0\varepsilon} \to u_0$ in $H$ as $\varepsilon \to 0$. Then define $u_{\delta\varepsilon}$ for $\delta > 0$ and $\varepsilon > 0$ as

$$u_{\delta\varepsilon}(t) := \frac{1}{\delta}\int_{t-\delta}^t \widetilde{u}_\varepsilon(s)\,\mathrm{d}s, \qquad \text{where} \qquad \widetilde{u}_\varepsilon(t) := \begin{cases} u(t) & \text{for } t > 0, \\ u_{0\varepsilon} & \text{for } t < 0, \end{cases}$$

and let $\quad v := u_{\delta\varepsilon} - \varphi \in W^{1,p}(]0,T[;M)$

with $\varphi \in W^{1,\infty}(]0,T[;V)$ and $\varphi(t) \in M_0$. Since $u_{\delta\varepsilon} \to u$ in $L^p(]0,T[;V)$ as $\delta \to 0$

$$\int_0^{\bar t} \langle\, u(t) - (u_{\delta\varepsilon}(t) - \varphi(t))\,,\, A(t,u(t))\,\rangle_V \,\mathrm{d}t \longrightarrow \int_0^{\bar t} \langle\, \varphi(t)\,,\, A(t,u(t))\,\rangle_V \,\mathrm{d}t$$

$$
\begin{aligned}
\Phi_{\bar u}(u,v)(\bar t) = \;& \Phi_{\bar u}(u,u_{\delta\varepsilon}-\varphi)(\bar t) \\
= \;& B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) \\
& - \int_0^{\bar t} (\, \partial_t(\bar u - u_{\delta\varepsilon} + \varphi)(t) \,,\, b(u(t)) - b(u_0) \,)_H \,\mathrm{d}t \\
& + (\, \bar u - u_{\delta\varepsilon}(\bar t) + \varphi(\bar t) \,,\, b(u(\bar t)) - b(u_0) \,)_H \\
= \;& \Phi_{\bar u}(u,u_{\delta\varepsilon})(\bar t) \\
& - \int_0^{\bar t} (\, \partial_t\varphi(t) \,,\, b(u(t)) - b(u_0) \,)_H \,\mathrm{d}t \\
& + (\, \varphi(\bar t) \,,\, b(u(\bar t)) - b(u_0) \,)_H
\end{aligned}
$$

Since $\liminf_{\varepsilon\to 0} \liminf_{\delta\to 0} \Phi_{\bar u}(u,u_{\delta\varepsilon})(\bar t) \geq 0$ for almost all $\bar t > 0$ we obtain

$$
\begin{aligned}
& (\, \varphi(\bar t) \,,\, b(u(\bar t)) - b(u_0) \,)_H \\
& - \int_0^{\bar t} (\, \partial_t\varphi(t) \,,\, b(u(t)) - b(u_0) \,)_H \,\mathrm{d}t + \int_0^{\bar t} \langle\, \varphi(t) \,,\, A(t,u(t)) \,\rangle_V \,\mathrm{d}t \leq 0
\end{aligned}
$$

This is obviously equivalent to the assertion. Now we can replace $\varphi$ by $-\varphi$ to obtain that the left side equals zero. If we now restrict $\varphi \in C_0^\infty([0,T[;M)$, one chooses $\bar t$ close to $T$ in order to get

$$
\int_0^T \big( (\, -\partial_t\varphi(t) \,,\, b(u(t)) - b(u_0) \,)_H \,\mathrm{d}t + \int_0^T \langle\, \varphi(t) \,,\, A(t,u(t)) \,\rangle_V \big) \,\mathrm{d}t = 0
$$

# Sketch of proof of final theorem

Energy estimate is

$$B_{\bar{u}}(u_h(\bar{t})) + c_0 \int_0^{\bar{t}} \|u_h(t) - \bar{u}\|_V^p \,\mathrm{d}t \le C$$

The compactness lemma implies that

$$\{b(u_h)\,;\ 0 < h < h_0\} \text{ is compact in } L^1([0,T];H)$$

From energy estimate for a subsequence $h \to 0$

$$u_h \to u \text{ weakly in } L^p([0,T];V)$$

Then there is a convergent subsequence $h \to 0$ so that

$$b(u_h) \to b(u) \text{ strongly in } L^1([0,T];H)$$

By the boundedness condition (1) there is a subsequence $h \to 0$

$$\mathcal{A}(u_h) = A(t,u_h) \to u^* \text{ weakly}^* \text{ in } L^{p*}([0,T];V^*)$$

Hence all convergence properties of the "continuity condition" (2) are satisfied with one exception. Now the time discrete inequality reads

$$\Phi_{\bar{u}}^h(u_h,v)(\widetilde{t}) + \int_0^{\widetilde{t}} \langle\, u_h(t) - v(t)\,,\ A_h(t,u_h(t)\,\rangle_V \,\mathrm{d}t \le 0$$

$$\Phi_{\bar{u}}^h(u_h,v)(\widetilde{t}) := \int_0^{\widetilde{t}} \big(\, u_h(t) - v(t)\,,\ \partial_t^{-h} b(u_h(t))\,\big)_H \,\mathrm{d}t$$

We know

$$\Phi^h_{\bar u}(u_h,v)(\bar t_h) + \int_0^{\bar t_h} \langle\, u_h(t)\,,\, A(t,u_h(t)\,\rangle_V \,\mathrm{d}t \le \int_0^{\bar t_h} \langle\, v(t)\,,\, A_h(t,u_h(t)\,\rangle_V \,\mathrm{d}t$$

For the parabolic part

$$\begin{aligned}
\Phi^h_{\bar u}(u_h,v)(\bar t_h) &= \int_0^{\bar t_h} \Big(\, u_h(t) - v(t)\,,\, \partial_t^{-h} b(u_h(t))\,\Big)_H \,\mathrm{d}t \\
&\ge \int_0^{\bar t_h} \partial_t^{-h} B_{\bar u}(u_h(t))\,\mathrm{d}t + \frac{1}{h}\int_0^{\bar t_h} (\,\bar u - v(t)\,,\, b(u_h(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&\qquad - \frac{1}{h}\int_{-h}^{\bar t_h - h} (\,\bar u - v(t+h)\,,\, b(u_h(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&= B_{\bar u}(u_h(\bar t_h)) - B_{\bar u}(u_0) + (\,\bar u - v_h(\bar t_h)\,,\, b(u_h(\bar t_h)) - b(u_0)\,)_H \\
&\qquad - \int_0^{\bar t_h - h} \Big(\, \partial_t^{+h}(\bar u - v(t))\,,\, b(u_h(t)) - b(u_0)\,\Big)_H \,\mathrm{d}t \\
&\ge\longrightarrow B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) + (\,\bar u - v(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\
&\qquad - \int_0^{\bar t} (\,\partial_t(\bar u - v(t))\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&= \Phi_{\bar u}(u,v)(\bar t)
\end{aligned}$$

Therefore we have proved that

$$\liminf_{h\to 0} \Phi^h_{\bar u}(u_h,v)(\bar t_h) \ge \Phi_{\bar u}(u,v)(\bar t)$$

Since equation reads

$$\begin{aligned}
&\Phi^h_{\bar u}(u_h,v)(\bar t_h) + \int_0^{\bar t_h} \langle\, u_h(t)\,,\; A(t,u_h(t)\,\rangle_V\ \mathrm{d}t \\
&\le \int_0^{\bar t_h} \langle\, v(t)\,,\; A_h(t,u_h(t)\,\rangle_V\ \mathrm{d}t = \int_0^{\bar t_h} \langle\, v_h(t)\,,\; A(t,u_h(t)\,\rangle_V\ \mathrm{d}t \\
&\longrightarrow \int_0^{\bar t} \langle\, v(t)\,,\; u^*(t)\,\rangle_V\ \mathrm{d}t
\end{aligned}$$

for $h \to 0$, we obtain

$$\begin{aligned}
&\liminf_{h\to 0} \Phi^h_{\bar u}(u_h,v)(\bar t_h) + \limsup_{h\to 0} \int_0^{\bar t_h} \langle\, u_h(t)\,,\; A(t,u_h(t)\,\rangle_V\ \mathrm{d}t \\
&\le \int_0^{\bar t} \langle\, v(t)\,,\; u^*(t)\,\rangle_V\ \mathrm{d}t,
\end{aligned}$$

that is

$$\Phi_{\bar u}(u,v)(\bar t) + \limsup_{h\to 0} \int_0^{\bar t_h} \langle\, u_h(t)\,,\; A(t,u_h(t)\,\rangle_V\ \mathrm{d}t \le \int_0^{\bar t} \langle\, v(t)\,,\; u^*(t)\,\rangle_V\ \mathrm{d}t$$

Now we come to the remaining property for the sequences in (2).
We set $v = u_\delta$:

$$\Phi_{\bar{u}}(u, u_\delta)(\bar{t}) + \limsup_{h \to 0} \int_0^{\bar{t}_h} \langle\, u_h(t)\,,\; A(t, u_h(t)\,\rangle_V \;\mathrm{d}t$$
$$\leq \int_0^{\bar{t}} \langle\, u_\delta(t)\,,\; u^*(t)\,\rangle_V \;\mathrm{d}t \longrightarrow \int_0^{\bar{t}} \langle\, u(t)\,,\; u^*(t)\,\rangle_V \;\mathrm{d}t$$

as $\delta \to 0$. Since $\Phi_{\bar{u}}(u, u_\delta)(\bar{t})$ in the limit $\delta \to 0$ was nonnegative, we arrive at

$$\limsup_{h \to 0} \int_0^{\bar{t}_h} \langle\, u_h(t)\,,\; A(t, u_h(t)\,\rangle_V \;\mathrm{d}t \leq \int_0^{\bar{t}} \langle\, u(t)\,,\; u^*(t)\,\rangle_V \;\mathrm{d}t$$

This was the last property in the assumption of (2), and therefore

$$\left\{ \begin{array}{c} \displaystyle\int_0^{\bar{t}} \langle\, u(t) - v(t)\,,\; A(t, u(t)) - u^*(t)\,\rangle_V \;\mathrm{d}t \leq 0, \qquad \text{and} \\ \displaystyle\limsup_{h \to 0} \int_0^{\bar{t}_h} \langle\, u_h(t)\,,\; A(t, u_h(t))\,\rangle_V \;\mathrm{d}t = \int_0^{\bar{t}} \langle\, u(t)\,,\; u^*(t)\,\rangle_V \;\mathrm{d}t \end{array} \right\}$$

Plugging in the equality one gets

$$\Phi_{\bar{u}}(u, v)(\bar{t}) + \int_0^{\bar{t}} \langle\, u(t) - v(t)\,,\; u^*(t)\,\rangle_V \;\mathrm{d}t \leq 0$$

and then the inequality, therefore the assertion

$$\Phi_{\bar{u}}(u, v)(\bar{t}) + \int_0^{\bar{t}} \langle\, u(t) - v(t)\,,\; A(t, u(t)\,\rangle_V \;\mathrm{d}t \leq 0$$