A two scale problem as a mathematical model for sulfate attack in sewer pipes

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Concrete Corrosion of a sewer pipe (6∼8mm per year)

Repair pipes in 5 years

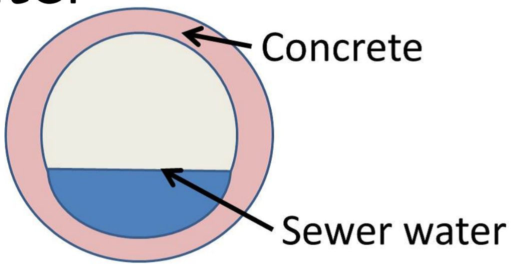
Aim To construct a mathematical model for concrete corrosion

Contents of this talk

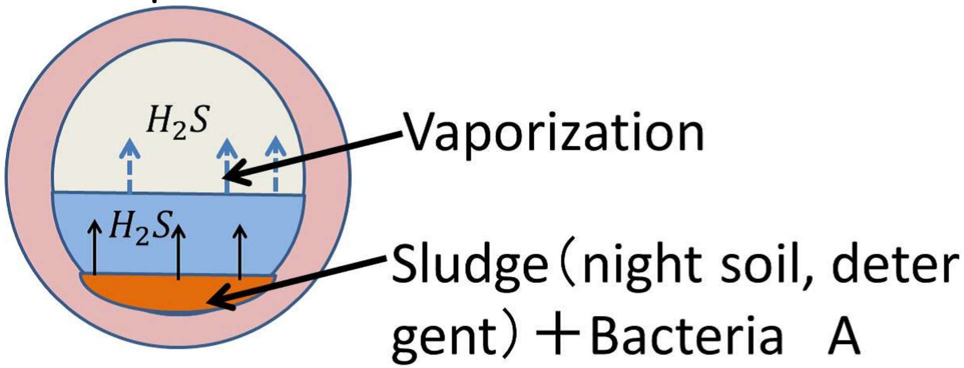
1. Mechanism of concrete corrosion
2. Two-scale modeling
3. Definition of a solution and our main results
4. Large time behavior
5. Future problems

1. Mechanism of concrete corrosion

Initial state:



At Bottom: After a while, sludge is piled up and becomes anaerobic.



Bacteria A: sulfate reducing bacteria

$SO_4^{-2} + 2C + H_2O \rightarrow H_2S(aq) + 2HCO_3^-$

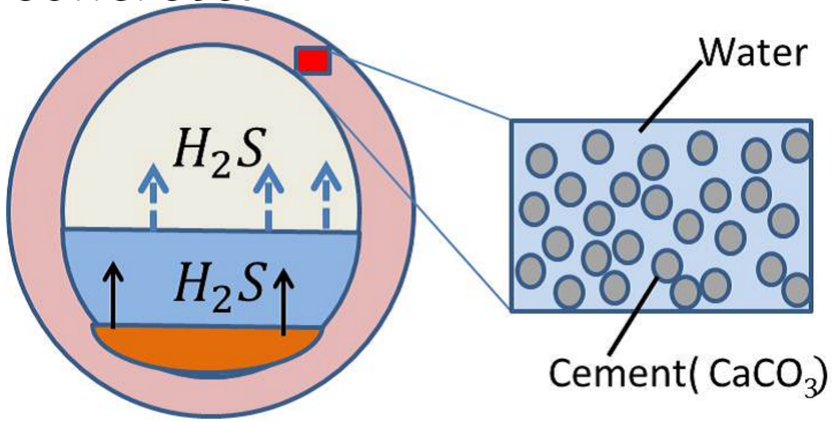
$H_2S(aq) \rightleftharpoons H_2S(g)$

H_2S (hydrogen sulfide)

On Ceiling: Bacteria B (Sulfur oxidizing bacteria) produces sulfuric acid:

$H_2S(aq) + O_2 \rightarrow H_2SO_4$

Gypsum is produced from sulfuric acid and concrete:



$H_2SO_4 \rightarrow 2H^+ + SO_4^{-2}$,

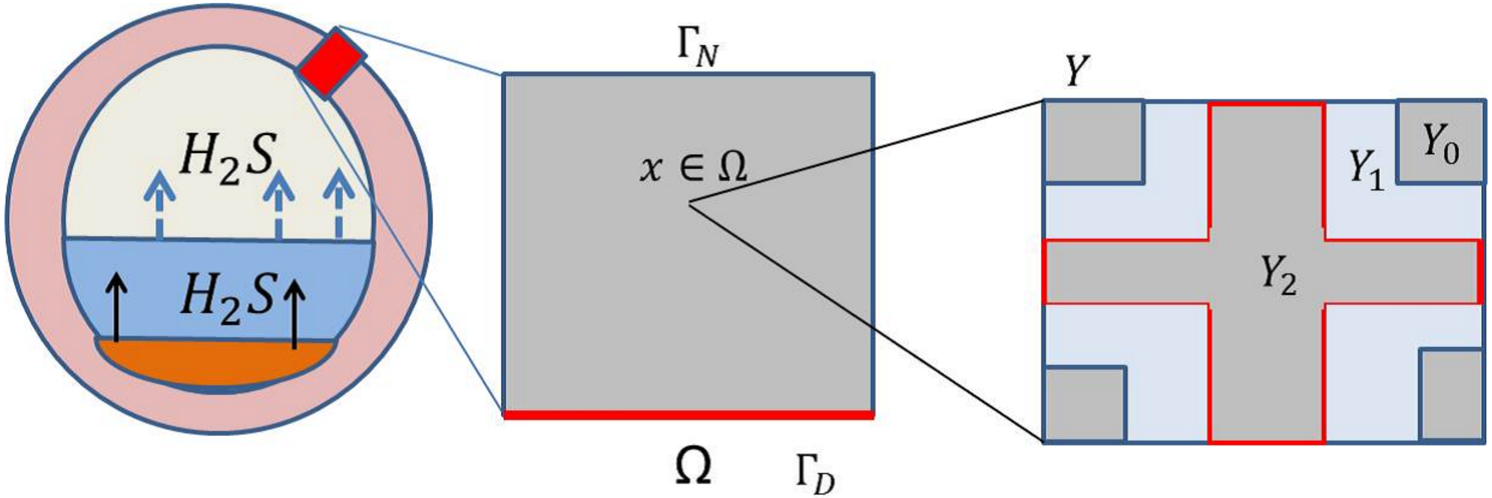
$2H_2O + H^+ + SO_4^{-2} + CaCO_3 \rightarrow$

$CaSO_4 \cdot 2H_2O + HCO_3^-$

$CaSO_4 \cdot 2H_2O$: (Gypsum)

Volume expansion by product of Gypsum $\Rightarrow$ concrete degradation

2. Two-scale modeling



Assumption 1. t: Time variable
macro-domain $\Omega \subset \mathbf{R}^3$
Hydrogen sulfide H_2S(g) diffuses in Ω.
w_3: Concentration of H_2S(g)

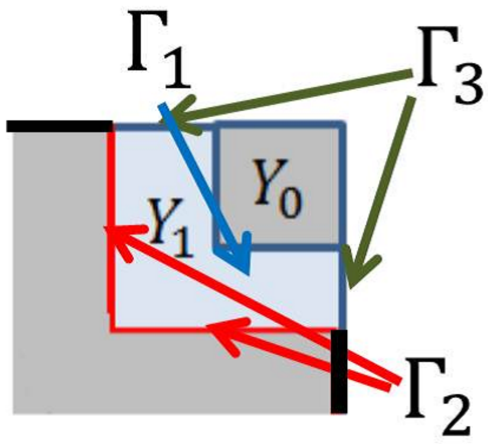
$$w_3 = w_3(t, x) \text{ for } x \in \Omega$$

Assumption 2.
micro-domain $Y \subset \mathbf{R}^3$
$Y = Y_0 \cup Y_1 \cup Y_2$, Y_0: Region of Cement,
Y_1: Water Region, Y_2: Air region

For each $x \in \Omega$, Y corresponds.
H_2S(g) diffuses through Y_2 and
w_3 is a constant in Y_2.
w_1: Concentration of H_2SO_4(aq)
w_2: Concentration of H_2S(aq)

$$w_1 = w_1(t, x, y) \text{ for } (x, y) \in \Omega \times Y_1,$$
$$w_2 = w_2(t, x, y) \text{ for } (x, y) \in \Omega \times Y_1$$

Mass conservation laws

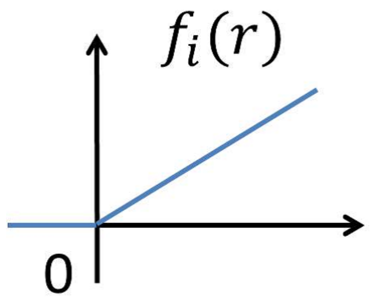


Assumption 3. w_1 [H_2SO_4(aq)] diffuses in Y_1, w_2 [H_2S(aq)] diffuses in Y_1.

(bacteria B) $H_2S(aq) + O_2 \to H_2SO_4$.

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = f_2(w_2) - f_1(w_1) \text{ in } \Omega \times Y_1$$

$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = -f_2(w_2) + f_1(w_1) \text{ in } \Omega \times Y_1,$$

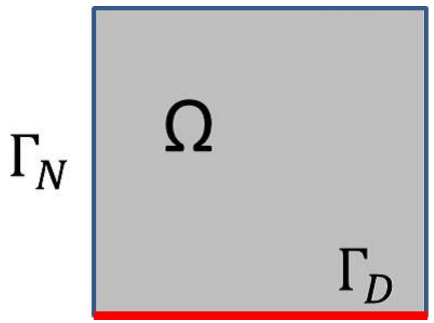


∇_y denotes derivative w.r.t. $y \in Y_1$

f_1, f_2: continuous, increasing $f_1(0) = f_2(0) = 0$

Examples of f_1, f_2: $f_1(r) = a[r]^+$, $f_2(r) = b[r]^+$

Assumption 4. w_3 [H_2S(g)] diffuses in Ω and for each $x \in \Omega$ Henry's law holds.

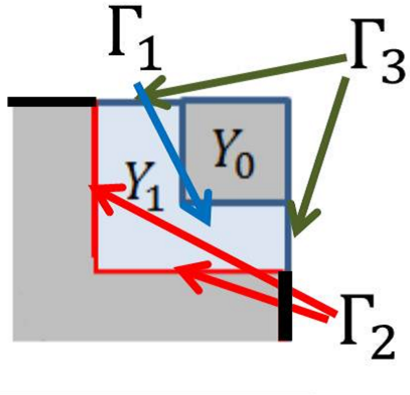


$$\partial_t w_3 - \nabla \cdot (d_3 \nabla w_3) = -\alpha \int_{\Gamma_2} (h_0 w_3 - w_2) d\gamma_y \text{ in } \Omega,$$

$$d_3 \nabla w_3 \cdot \nu(x) = 0 \text{ on } \Gamma_N, w_3 = w_3^D \text{ on } \Gamma_D.$$

∇ denotes derivative w.r.t. $x \in \Omega$

Boundary conditions for w_1 and w_2



Assumption 5. On Γ_1 H_2SO_4(aq) and $CaCO_3$ react and produce Gypsum.
η: rate of this reaction
w_4: Concentration of Gypsum

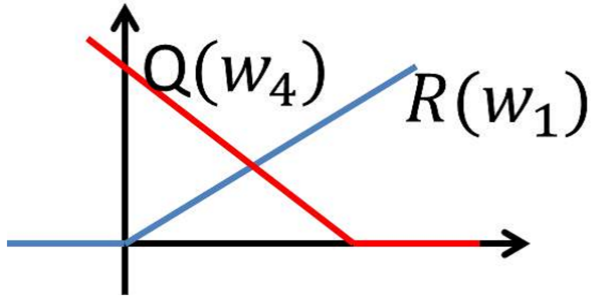
$$d_1 \nabla_y w_1 \cdot \nu(y) = -\eta(w_1, w_4) \text{ on } \Gamma_1,$$
$$\partial_t w_4 = \eta(w_1, w_4) \text{ on } \Gamma_1.$$

Moreover, (Gypsum inhibits the product)

$$\eta(w_1, w_4) = R(w_1)Q(w_4),$$

$R' \geq 0, Q' \leq 0,$ $R > 0$ on $(0, \infty)$,
$Q = 0$ on $(\beta_{\max}, \infty)$
($\beta_{\max}$ is a positive constant.)

Example of R and Q:



Assumption 6. H_2SO_4(aq) can not move over Γ_2, Γ_3.

$$d_1 \nabla_y w_1 \cdot \nu(y) = 0 \text{ on } \Gamma_2 \cup \Gamma_3$$

Assumption 7. H_2S(aq) can not move over Γ_1, Γ_3.

$$d_2 \nabla_y w_2 \cdot \nu(y) = 0 \text{ on } \Gamma_1 \cup \Gamma_3$$

Assumption 8. H_2S(aq) w_2 satisfies Henry's law on Γ_2.

$$d_2 \nabla_y w_2 \cdot \nu(y) = \alpha(h_0 w_3 - w_2) \text{ on } \Gamma_2$$

Assumption 9. The boundaries of Ω and Y_1 are Lipschitz continuous.

Our model We denote by P the following system:

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -f_1(w_1) + f_2(w_2) \quad \text{in } (0,T)\times\Omega\times Y_1,$$
$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = f_1(w_1) - f_2(w_2) \quad \text{in } (0,T)\times\Omega\times Y_1,$$
$$\partial_t w_3 - \nabla \cdot (d_3 \nabla w_3) = -\alpha \int_{\Gamma_2} \Big(h_0 w_3 - w_2\Big) d\gamma_y \quad \text{in } (0,T)\times\Omega,$$
$$\partial_t w_4 = \eta(w_1, w_4) \quad \text{on } (0,T)\times\Omega\times\Gamma_1.$$

$$\begin{cases} w_j(0,x,y) = w_{j0}(x,y), \ \ j\in\{1,2\} & \text{in } \Omega\times Y_1,\\ w_3(0,x) = w_{30}(x) \quad \text{in } \Omega, \qquad w_4(0,x,y) = w_{40}(x,y) & \text{on } \Omega\times\Gamma_1, \end{cases}$$

$$\begin{cases} d_1\nabla_y w_1\cdot\nu(y) = -\eta(w_1,w_4) \quad \text{on } (0,T)\times\Omega\times\Gamma_1,\\ d_1\nabla_y w_1\cdot\nu(y) = 0 \quad \text{on } (0,T)\times\Omega\times\Gamma_2 \text{ and } (0,T)\times\Omega\times\Gamma_3,\\ d_2\nabla_y w_2\cdot\nu(y) = 0 \quad \text{on } (0,T)\times\Omega\times\Gamma_1 \text{ and } (0,T)\times\Omega\times\Gamma_3,\\ d_2\nabla_y w_2\cdot\nu(y) = \alpha(h_0 w_3 - w_2\Big) \quad \text{on } (0,T)\times\Omega\times\Gamma_2,\\ d_3\nabla w_3\cdot\nu(x) = 0 \quad \text{on } (0,T)\times\Gamma_N,\\ w_3 = w_3^D \quad \text{on } (0,T)\times\Gamma_D, \end{cases}$$

Related topic 1 Friedman-Tzavaras (1987) (Catalytic reactor with bed):
$u(t,x)$, $v(t,x)$ in Ω: macro, $u'(t,x,x')$, $v'(t,x,x')$ in Ω': micro

$$
\begin{aligned}
&u_t = \nabla\cdot(\alpha(u)\nabla u) - V_1\cdot\nabla u - \int_{\partial\Omega'}\beta_1(u-u') \text{ in } \Omega,\\
&v_t = \nabla\cdot(\beta(v)\nabla v) - V_2\cdot\nabla v - \int_{\partial\Omega'}\beta_2(v-v') \text{ in } \Omega,\\
&u'_t = \nabla'\cdot(\alpha'(u')\nabla' u') - \gamma(u')\phi(v') \text{ in } \Omega',\\
&v'_t = \nabla'\cdot(\beta'(u')\nabla' v') + \gamma(u')\phi(v') \text{ in } \Omega',\\
&\alpha\frac{\partial u}{\partial n} + \mu u = F, \beta\frac{\partial v}{\partial n} + \nu v = G \text{ on } \partial\Omega,\\
&\alpha'(u')\frac{\partial u'}{\partial n'} + u'(u'-u) = 0, \beta'(v')\frac{\partial v'}{\partial n'} + v'(v'-v) = 0 \text{ on } \partial\Omega',
\end{aligned}
$$

Existence, uniqueness and Large time behavior
$\gamma(r) = cr^p$, $0 < p \le 1$, V_1 and V_2 are constants
$|F_t| \le \frac{C}{t^{1+\varepsilon}}$, $|F| \le \frac{C}{t^{1+\varepsilon}}$, $|G_t| \le \frac{C}{t^{1+\varepsilon}}$
Then $(u,v,u',v') \to (0,\tilde{v},0,\tilde{v}')$ uniformly.
(Hölder continuity of solutions of parabolic equations)

Related topic 2.

A. Muntean - M. Neuss-Radu (2010):

$$
\begin{aligned}
&U_t(t,x) - D\Delta U(t,x) = -\int_{\Gamma_k} b(U(t,x) - u(t,x,y))d\gamma_y \text{ in } \Omega,\\
&u_t(t,x,y) - d_1\Delta_y u(t,x,y) = -\kappa\eta(u(t,x,y), v(t,x,y)) \text{ in } \Omega \times Y,\\
&v_t(t,x,y) - d_2\Delta_y v(t,x,y) = -\alpha\kappa\eta(u(t,x,y), v(t,x,y)) \text{ in } \Omega \times Y,\\
&U = U^D \text{ on } \partial\Omega,\\
&\nabla_y u \cdot n_y = 0 \text{ on } \Gamma_N,\\
&-d_1\nabla_y u \cdot n_y = -b(U(t,x) - u(t,x,y)) \text{ on } \Gamma_R,\\
&\nabla_y v \cdot n_y = 0 \text{ on } \partial\Omega,
\end{aligned}
$$

$b : \mathbf{R} \to \mathbf{R}$, $\eta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are Lipschitz continuous.

Assumptions: $U_0 \in H^2(\Omega)$, $u_0, v_0 \in L^2(\Omega; H^2(Y)) \cap H^1(\Omega \times Y)$

Existence, uniqueness and positivity of a solution

Related topics 3.

T. Fatima, N. Arab, E. P. Zemskov, A. Muntean (2011):

Derivation by homogenization

V. Chalupecký, T. Fatima, A. Muntean (2011):

Existence and uniqueness, numerical simulation with constants d_i

f_1, f_2 are linear.

$\eta(r_1, r_2) = c r_1^p (a - r_2)^q$

$w_{10}, w_{20} \in L^2(\Omega; H^2(Y_1)) \cap H^1(\Omega \times Y_1)$

$w_{30} \in H^2(\Omega)$

Aims of this talk

1. Existence, uniqueness and positivity under
   $w_{10}, w_{20} \in L^2(\Omega; H^1(Y_1)) \cap L^\infty(\Omega \times Y_1)$
   $w_{30} \in H^1(\Omega) \cap L^\infty(\Omega)$
2. Large time behavior
   $w_1(t) \to w_{1\infty}$ weakly in L^2,
   $w_2(t) \to w_{2\infty}$ weakly in L^2,
   $w_3(t) \to w_{3\infty}$ in L^2,
   $w_4(t) \to w_{4\infty}$ in L^1,
   if $f_1(r_1) - f_2(r_2) = \psi(r_1 - \gamma r_2)$,
   ψ: proper, l.s.c. convex on $\mathbf{R}$.

Analytical tools

1. Theory of evolution equations governed by sub-differential without compactness
2. Maximum principle

3. Definition of a solution and main results

$$X = \{z \in H^1(\Omega) | z = 0 \text{ on } \Gamma_D\}.$$

Definition 3.1 For $T > 0$ (w_1, w_2, w_3, w_4) is a solution of P on $[0, T]$, if (S1) $\sim$ (S5) hold.

(S1) $w_1, w_2 \in H^1(0,T; L^2(\Omega \times Y_1)) \cap L^\infty(0,T; L^2(\Omega; H^1(Y_1))) \cap L^\infty((0,T) \times \Omega \times Y_1)$,
$w_3 \in H^1(0,T; L^2(\Omega)) \cap L^\infty((0,T) \times \Omega)$, $w_3 - w_3^D \in L^\infty(0,T; X)$,
$w_4 \in H^1(0,T; L^2(\Omega \times \Gamma_1)) \cap L^\infty((0,T) \times \Omega \times \Gamma_1)$,
$w_1(0) = w_{10}$, $w_2(0) = w_{20}$, $w_3(0) = w_{30}$, $w_4(0) = w_{40}$.

(S2) It holds that

$$\begin{aligned} & \int_{\Omega \times Y_1} \partial_t w_1 v_1 dxdy + \int_{\Omega \times Y_1} d_1 \nabla_y w_1 \cdot \nabla_y v_1 dxdy + \int_{\Omega \times \Gamma_1} Q(w_4) R(v_1) dxd\gamma_y \\ = & \int_{\Omega \times Y_1} (-f_1(w_1) + f_2(w_2)) v_1 dxdy \end{aligned}$$

for $v_1 \in L^2(\Omega; H^1(Y_1))$ a.e. on $[0, T]$.

(S3) It holds that

$$
\begin{aligned}
&\int_{\Omega\times Y_1}(\partial_t w_2 v_2 + d_2\nabla_y w_2\cdot\nabla_y v_2)dxdy - \alpha\int_{\Omega\times\Gamma_2}(h_0 w_3 - w_2)v_2 dxd\gamma_y \\
= &\int_{\Omega\times Y_1}(f_1(w_1) - f_2(w_2))v_2 dxdy \quad \text{for } v_2\in L^2(\Omega; H^1(Y_1)) \text{ a.e. on } [0,T].
\end{aligned}
$$

(S4) It holds that

$$
\begin{aligned}
&\int_{\Omega}\partial_t w_3 v_3 dx + \int_{\Omega} d_3\nabla w_3\cdot\nabla v_3 dx \\
= &-\alpha\int_{\Omega\times\Gamma_2}(h_0 w_3 - w_2)v_3 dxd\gamma_y \quad \text{for } v_3\in X \text{ a.e. on } [0,T].
\end{aligned}
$$

(S5) $\partial_t w_4 = \eta(w_1, w_4)$ holds a.e. on $(0,T)\times\Omega\times\Gamma_1$.

Assumptions

(A1) $d_i \in L^\infty(\Omega \times Y_1), i = 1, 2,$ $d_3 \in L^\infty(\Omega)$ satisfies
$d_i(x, y) \geq d_i^0$ for a.e. $(x, y) \in \Omega \times Y_1$ and $i \in \{1, 2\}$,
$d_3(x) \geq d_3^0$ for a.e. $x \in \Omega$,
where $d_i^0 > 0$ is a constant for each $i = 1, 2, 3$.

(A2) $\eta(\alpha, \beta) := R(\alpha)Q(\beta)$, where R, Q are locally Lipschitz continuous and satisfy
$R' \geq 0$ and $Q' \leq 0$ a.e. on $\mathbf{R}$, $R > 0$ on $(0, \infty)$ and $R = 0$ on $(-\infty, 0]$,
$Q > 0$ on $(-\infty, \beta_{\max})$ and $Q = 0$ on $[\beta_{\max}, \infty)$, where β_{max} is a positive constant.

(A3) For $i = 1, 2$ f_i is locally Lipschitz continuous and increasing, $f_i(0) = 0$.

(A4) For $i = 1, 2,$ $w_{i0} \in L^2(\Omega; H^1(Y_1)) \cap L^\infty(\Omega \times Y_1),$ $w_{i0} \geq 0$ on $\Omega \times Y_1$,
$w_{30} \in H^1(\Omega)$, $w_{30} - w_3^D(0, \cdot) \in X$, $w_{30} \geq 0$ on Ω,
$w_{40} \in L^\infty(\Omega \times \Gamma_1)$ with $w_{40} \geq 0$ on $\Omega \times \Gamma_1$.

(A5) $w_3^D \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap L^\infty((0, T) \times \Omega))$
with $\nabla w_3^D \cdot \nu = 0$ on $(0, T) \times \Gamma_N$, $w_3^D \geq 0$ on $(0, T) \times \Omega$.

Theorem 3.1 (existence, uniqueness). (T. Fatima- A.Muntean - A, 2012) Let $T > 0$. If (A1)∼(A5) hold, then P has a unique solution (w_1, w_2, w_3, w_4) on $[0, T]$ and

$$0 \le w_1 \le M_1, 0 \le w_2 \le M_2 \text{ on } (0, T) \times \Omega \times Y_1,$$

$$0 \le w_3 \le M_3 \text{ on } (0, T) \times \Omega, 0 \le w_4 \le M_4 \text{ on } (0, T) \times \Omega \times \Gamma_1.$$

where a positive constant M_i depends only on maximum values of initial and boundary functions, and β_{max}.

(Sketch of the proof).

- Solve the problem with given functions in right hand sides by the Galerkin approximation.
- Solve the problem with Lipschitz continuous f_1, f_2, R, Q by Banach's fixed point theorem.
- Estimate maximum values of solutions by choosing $[w_i - M_i]^+$ as a test function.
- Show existence of a solution, even if f_1, f_2, R, Q are locally Lipschitz continuous.

4. Large time behavior

Let ψ be a locally Lipschitz continuous and increasing function with $\psi(0) = 0$ and substitute $\psi(r_1 - \gamma r_2)$ instead of $f_1(r_1) - f_2(r_2)$. Thus we consider

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -\psi(r_1 - \gamma r_2) \quad \text{in } (0,T) \times \Omega \times Y_1,$$
$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = \psi(r_1 - \gamma r_2) \quad \text{in } (0,T) \times \Omega \times Y_1,$$

Example. If $f_1(r_1) = b_1[r_1]^+$ and $f_2(r_2) = b_2[r_2]^+$, then we put $\psi(r) = b_1 r$ for $r \in \mathbf{R}$ and $\gamma = \frac{b_2}{b_1}$.
Then we can have $w_i \geq 0$ for $i = 1, 2$ and $f_1(w_1) - f_2(w_2) = \psi(w_1 - \gamma w_2)$

Moreover, we put $W_3 = w_3 - w_3^D$ and $H = L^2(\Omega \times Y_1) \times L^2(\Omega \times Y_1) \times L^2(\Omega)$ in order to apply the abstract theory.

Theorem 4.1 (A.Muntean - A, 2012)
(A3'): ψ is a locally Lipschitz continuous and increasing function with $\psi(0) = 0$.
(A5') $w_3^D \in L^2_{loc}(0, \infty; H^2(\Omega)) \cap H^1_{loc}(0, \infty; L^2(\Omega)) \cap L^\infty((0, \infty) \times \Omega)$
with $w_3^D \geq 0$ and $\nabla w_3^D \cdot \nu = 0$ on $(0, \infty) \times \Gamma_N$.
If (A1), (A2), (A3'), (A4), (A5') hold, $\partial_t w_3^D - \nabla d_3 \nabla w_3^D \in L^\infty(0, \infty; L^1(\Omega))$,
$(\partial_t(\partial_t w_3^D - \nabla d_3 \nabla w_3^D) \in L^1(0, \infty; L^1(\Omega))$, $\partial_t w_3^D \in L^1(0, \infty; L^1(\Omega))$, then

(1) P has a unique solution (w_1, w_2, W_3, w_4) on $[0, \infty)$ with $0 \leq w_i \leq M_i$,
$0 \leq W_3 \leq M_3$.

(2) $w_4(t) \to w_{4\infty}$ in $L^1(\Omega \times \Gamma_1)$ as $t \to \infty$ and $\partial_t w_4 \in L^1(0, \infty; L^1(\Omega \times \Gamma_1))$

(3) There exists a subsequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$w(t_n) \to w_\infty \text{ weakly in } H \text{ as } n \to \infty$$

for some $w_\infty = (w_{1\infty}, w_{2\infty}, W_{3\infty}) \in H$ and w_∞ is a solution of the stationary problem, where $w(t) = (w_1(t), w_2(t), W_3(t)) \in H$. Moreover, if $(\psi(r) - \psi(r'))(r - r') \geq \mu |r - r'|^{p+1}$ for $r, r' \in \mathbf{R}$, where $\mu > 0$ and $p \geq 1$, then

$$w(t) \to w_\infty \text{ weakly in } H, W_3(t) \to W_{3\infty} \text{ in } L^2(\Omega) \text{ as } t \to \infty.$$

Main idea of the proof of Theorem 4.1

Furuya, Miyashiba, Kenmochi (1986)

Asymptotic behavior of solutions to a class of nonlinear evolution equations, Journal of Differential Equations, 62, 1986, 73-94.

$$H = L^2(\Omega \times Y_1) \times L^2(\Omega \times Y_1) \times L^2(\Omega),$$

$$(u,v)_H = (u_1,v_1)_{L^2(\Omega\times Y_1)} + \gamma(u_2,v_2)_{L^2(\Omega\times Y_1)} + h_0(u_3,v_3)_{L^2(\Omega)},$$

for $u = (u_1,u_2,u_3), v = (v_1,v_2,v_3) \in H$.

For given $w_4 \in L^2(\Omega \times \Gamma_1)$ we define $\varphi^t(w_4;\cdot) : H \to (-\infty,\infty]$ in the following way:

$$\begin{aligned}
&\varphi^t(w_4;w) \\
= \ & \frac{1}{2}\int_{\Omega\times Y_1} d_1|\nabla_y w_1|^2 dxdy + \int_{\Omega\times\Gamma_1} Q(w_4)\hat{R}(w_1)dxd\gamma_y + \frac{\gamma}{2}\int_{\Omega\times Y_1} d_2|\nabla_y w_2|^2 dxdy \\
&+ \int_{\Omega\times Y_1}\hat{\psi}(w_1-\gamma w_2)dxdy + \frac{\gamma}{2}\alpha\int_{\Omega\times\Gamma_2}|h_0(W_3+w_3^D)-w_2|^2 dxd\gamma_y \\
&+\frac{h_0}{2}\int_\Omega d_3|\nabla W_3|^2dx - h_0\int_\Omega f(t)W_3dx \qquad \text{if } w=(w_1,w_2,W_3)\in K,
\end{aligned}$$

where $K = L^2(\Omega;H^1(Y_1)) \times L^2(\Omega;H^1(Y_1)) \times X$, $\hat{R}$ and $\hat{\psi}$ are primitives of R and ψ, $f(t) = \partial_t w_3^D(t) - \nabla d_3 \nabla w_3^D(t)$.

Lemma 4.1. $\varphi^t(w_4(t);w)$ is proper. l.s.c. and convex on H. $\partial\varphi^t(w_4(t);w)$ is single-valued and $w^* = (w_1^*, w_2^*, w_3^*) = \partial\varphi^t(w_4(t);w)$ if and only if $w^* \in H$ and

$$
\begin{aligned}
(w_1^*, v_1)_{L^2(\Omega\times Y_1)} &= \int_{\Omega\times Y_1} d_1 \nabla_y w_1 \cdot \nabla_y v_1 dxdy + \int_{\Omega\times\Gamma_1} Q(w_4) R(w_1) v_1 dxd\gamma_y \\
&\quad + \int_{\Omega\times Y_1} \psi(w_1 - \gamma w_2) v_1 dxdy,
\end{aligned}
$$

$$
\begin{aligned}
&(w_2^*, v_2)_{L^2(\Omega\times Y_1)} \\
= &\int_{\Omega\times Y_1} d_2 \nabla_y w_2 \cdot \nabla_y v_2 dxdy - \alpha \int_{\Omega\times\Gamma_2} (h_0(W_3 + w_3^D) - w_2) v_2 dxd\gamma_y \\
&+ \int_{\Omega\times Y_1} \psi(w_1 - \gamma w_2) v_2 dxdy,
\end{aligned}
$$

$$
\begin{aligned}
(w_3^*, v_3)_{L^2(\Omega)} &= \int_{\Omega} d_3 \nabla w_3 \cdot \nabla v_3 dxdy - \int_{\Omega} f(t) v_3 dx \\
&\quad + \alpha \int_{\Omega\times\Gamma_2} (h_0(W_3 + w_3^D) - w_2) v_3 dxd\gamma_y
\end{aligned}
$$

for $(v_1, v_2, v_3) \in L^2(\Omega; H^1(\Omega)) \times L^2(\Omega; H^1(\Omega)) \times X$.

Lemma 4.2. For $w \in K$

$$\begin{aligned} &\varphi^t(w_4(t); w) - \varphi^s(w_4(s); w) \\ \leq\ & C(|w_4(t) - w_4(s)|_{L^2(\Omega\times\Gamma_1)} + |f(t) - f(s)|_{L^2(\Omega)})(1 + \varphi^s(w_4(s); w)) \end{aligned}$$

Sketch of the proof of Theorem 4.1

1st step. For given $w_4 \in W^{1,1}(0, T; L^2(\Omega \times \Gamma_1))$ we solve

$$w_t + \partial\varphi^t(w_4(t); w) = 0,\ w(0) = w_0.$$

2nd step. By Banach's fixed point theorem we show existence and uniqueness of a solution of P. $\partial_t w_4 = \eta(w_1, w_4)$

3rd step. We obtain positivity and maximum values of a solution.

4th step. Since $w_{4t} = R(w_1)Q(w_4) \geq 0$, Lebesgue monotone convergence theorem implies

$$w_4(t) \to w_{4\infty} \text{ in } L^1(\Omega \times \Gamma_1) \text{ as } t \to \infty, w_{4t} \in L^1(0, \infty; L^1(\Omega \times \Gamma_1)).$$

5th step. We show $w_t \in W^{1,2}(0, \infty; H)$, $\varphi^t(w_4(t), w(t)) \in L^\infty(0, \infty)$.

6th step.

$$\omega_w(w_0) = \{z \in H | w(t_n) \to z \text{ weakly in } H \text{ for some } \{t_n\}\},$$

$$\begin{aligned}
& \varphi^\infty(w_{4\infty}; w) \\
= \; & \frac{1}{2}\int_{\Omega\times Y_1} d_1 |\nabla_y w_1|^2 dxdy + \int_{\Omega\times\Gamma_1} Q(w_{4\infty})\widehat{R}(w_1) dxd\gamma_y \\
& + \frac{\gamma}{2}\int_{\Omega\times Y_1} d_2 |\nabla_y w_2|^2 dxdy \\
& + \int_{\Omega\times Y_1} \widehat{\psi}(w_1 - \gamma w_2) dxd\gamma_y + \frac{\gamma}{2}\alpha \int_{\Omega\times\Gamma_2} |h_0(W_3 + w_{3\infty}) - w_2|^2 dxdy \\
& + \frac{h_0}{2}\int_\Omega d_3 |\nabla W_3|^2 dx - h_0 \int_\Omega f_\infty W_3 dx \qquad \text{if } w = (w_1, w_2, W_3) \in K.
\end{aligned}$$

$$F(\varphi^\infty) = \{z \in H | \varphi^\infty(w_{4\infty}; z) = \min \varphi^\infty(w_{4\infty}; \cdot)\}.$$

7th step.

$$\lim_{t\to\infty} \varphi^t(w_4(t); w(t)) = m_0.$$

8th step. There exists $w_\infty \in H$ such that $w(t_n) \to w_\infty$ weakly in H, namely, $w_\infty \in \omega_w(w_0)$

Moreover, since $\varphi^\infty(w_{4\infty}; w_\infty) \le m_0$, $w_\infty \in F(\varphi^\infty)$. Hence, w_∞ is a solution of the stationary problem.

9th step. Under $(\psi(r) - \psi(r'))(r - r') \ge \mu|r - r'|^{p+1}$ for $r, r' \in \mathbf{R}$

for each $w_{4\infty}$ the stationary problem has a unique solution.

Let $(w_{1\infty}^{(1)}, w_{2\infty}^{(1)}, W_{3\infty}^{(1)})$, $(w_{1\infty}^{(2)}, w_{2\infty}^{(2)}, W_{3\infty}^{(2)})$ be solutions of the stationary problem.

Put $w_{1\infty} = w_{1\infty}^{(1)} - w_{1\infty}^{(2)}$, $w_{2\infty} = w_{2\infty}^{(1)} - w_{2\infty}^{(2)}$, $W_{3\infty} = W_{3\infty}^{(1)} - W_{3\infty}^{(2)}$.

$$-\nabla_y \cdot (d_1 \nabla_y w_{1\infty}) = -(\psi(w_{1\infty}^{(1)} - \gamma w_{2\infty}^{(1)}) - \psi(w_{1\infty}^{(2)} - \gamma w_{2\infty}^{(2)})) \text{ in } \Omega \times Y_1, \quad (1)$$

$$-\nabla_y \cdot (d_2 \nabla_y w_{2\infty}) = \psi(w_{1\infty}^{(1)} - \gamma w_{2\infty}^{(1)}) - \psi(w_{1\infty}^{(2)} - \gamma w_{2\infty}^{(2)}) \text{ in } \Omega \times Y_1, \quad (2)$$

$$-\nabla \cdot (d_3 \nabla W_{3\infty}) = -\alpha \int_{\Gamma_2} \Big(h_0 W_{3\infty} - w_{2\infty}\Big) d\gamma_y \quad \text{in } \Omega. \quad (3)$$

By (1) $\times w_{1\infty}$, (2) $\times \gamma w_{2\infty}$, (3) $\times \gamma h_0 W_{3\infty}$

$$
\begin{aligned}
0 \;\ge\; & d_1^0 \int_{\Omega\times Y_1} |\nabla_y w_{1\infty}|^2 dxdy + \gamma d_2^0 \int_{\Omega\times Y_1} |\nabla_y w_{2\infty}|^2 dxdy \\
& + \int_{\Omega\times Y_1} (\psi(w_{1\infty}^{(1)} - \gamma w_{2\infty}^{(1)}) - \psi(w_{1\infty}^{(2)} - \gamma w_{2\infty}^{(2)}))(w_{1\infty} - \gamma w_{2\infty}) dxdy \\
& + \gamma\alpha \int_{\Omega\times\Gamma_2} |h_0 W_{3\infty} - w_{2\infty}|^2 dx d\gamma_y + \gamma d_3^0 \int_\Omega |\nabla W_{3\infty}|^2 dx.
\end{aligned}
$$

By $\nabla W_{3\infty} = 0$ on Ω and $W_{3\infty} = 0$ on Γ_D we have $W_{3\infty} = 0$ on Ω.

Hence, $\int_{\Omega\times\Gamma_2} |h_0 W_{3\infty} - w_{2\infty}|^2 dx d\gamma_y$, $\int_{\Omega\times Y_1} |\nabla_y w_{2\infty}|^2 dxdy = 0$ imply $w_{2\infty} = 0$ on $\Omega \times Y_1$. By the assumption

$$
(\psi(w_{1\infty}^{(1)} - \gamma w_{2\infty}^{(1)}) - \psi(w_{1\infty}^{(2)} - \gamma w_{2\infty}^{(2)}))(w_{1\infty} - \gamma w_{2\infty}) \ge \mu |w_{1\infty} - \gamma w_{2\infty}|^{p+1}.
$$

Then $w_{1\infty} = 0$ on $\Omega \times Y_1$.

final step Since the stationary solution is unique, $w(t) \to w_\infty$ weakly in H as $t \to \infty$

Moreover, by $W_3 \in L^\infty(0,\infty; X)$ we have $W_3(t) \to W_{3\infty}$ in $L^2(\Omega)$.

5. Future problems

1. Strong convergence of w_1, w_2, the difference of the convergence rates between macro and micro parameters.

2. No neglect of change of water mass.

3. To find natural boundary condition on Γ_3.