RATE-INDEPENDENT PROCESSES IN SOLIDS: combination with rate-dependent processes

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Combination of rate-independent processes vs. rate-dependent processes.

$$\mathcal{T}'\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\mathcal{R}_2'\frac{\mathrm{d}u}{\mathrm{d}t}+\partial_u\mathcal{E}(t,u,z)=0, \tag{1a}$$

$$\partial_{\frac{\mathrm{d}z}{\mathrm{d}t}}\mathcal{R}_1\Big(z,\frac{\mathrm{d}z}{\mathrm{d}t}\Big)+\partial_z\mathcal{E}(t,u,z)\ni 0. \tag{1b}$$

with
$u\in\mathcal{U}$ a "displacement" determined essentially by z
$z\in\mathcal{Z}$ an "internal" variable with activated evolution,
$\mathcal{E}:\mathcal{U}\times\mathcal{Z}\to\mathbb{R}\cup\{\infty\}$ the stored energy,
$\mathcal{R}_1:\mathcal{Z}\times\mathcal{Z}\to\mathbb{R}\cup\{\infty\}$ the dissipation pseudopotential
$\mathcal{R}_1(z,\cdot)$ (positively) homogeneous degree-1
$\mathcal{R}_2:\mathcal{V}\to\mathbb{R}$ the dissipation pseudopotential of viscous forces, quadratic
$\mathcal{T}:\mathcal{H}\to\mathbb{R}$ the kinetic energy, quadratic

Functional-analytical ansatz: $\mathcal{V}$, $\mathcal{U}$, $\mathcal{Z}$ Banach spaces, $\mathcal{H}$ a Hilbert space,
$u:[0,T]\to\mathcal{U}$, $\frac{\mathrm{d}u}{\mathrm{d}t}:[0,T]\to\mathcal{V}$,
$\mathcal{V}\subseteq\mathcal{U}\Subset\mathcal{H}\cong\mathcal{H}^*$ densely,
$\mathcal{R}_2:\mathcal{V}\to\mathbb{R}$ and $\mathcal{T}:\mathcal{H}\to\mathbb{R}$ coercive.

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with
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Treatment of the general ansatz:
General theory of rate-independent processes based on dissipation distance:

$$\mathcal{D}_1(z_0,z_1) := \inf\left\{ \int_0^1 \mathcal{R}_1\Big(\widetilde{z}(t), \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}t}(t)\Big)\mathrm{d}t; \quad \widetilde{z}\in C^1([0,1];\mathcal{V}),\ \widetilde{z}(0)=z_0,\ \widetilde{z}(1)=z_1 \right\}$$

In principle, $\mathcal{D}_1$ the dissipation distance can be treated as itself even without referring to $\mathcal{R}_1$ and without any linear structure on $\mathcal{Z}$.

But we will not pursue this high generality here.

Simplification: $\mathcal{R}_1(z,\frac{\mathrm{d}z}{\mathrm{d}t}) = \mathcal{R}_1(\frac{\mathrm{d}z}{\mathrm{d}t})$. Then $\mathcal{D}_1(z_0,z_1) = \mathcal{R}_1(z_1{-}z_0)$ and we assume $\mathcal{R}_1 : \mathcal{Z} \to \mathbb{R}\cup\{\infty\}$ homogeneous degree-1 and coercive.

The philosophy of a suitable definition of a solution to (1) can be based on the energetic-solution concept of A.Mielke at al. applied, for u considered fixed, to the system $(\mathcal{Z}, \mathcal{I}_u, \mathcal{R}_1)$ with (1b), by

$$\mathcal{I}_u(t,z) := [\mathcal{E}\circ u](t,z) = \mathcal{E}(t,u(t),z),$$

and further combined with a conventional weak-solution concept as far as the "momentum equation" (1a) concerns.

We consider still initial conditions:

$$u(0) = u_0, \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = \dot{u}_0, \qquad z(0) = z_0.$$

This leads to:

We call $q=(u,z):[0,T]\to\mathcal{Q}=\mathcal{U}\times\mathcal{Z}$ an energetic solution to the problem (1) with the initial conditions if

$$u\in C_{\rm w}([0,T];\mathcal{U}),$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}\in L^2(I;\mathcal{V})\cap C_{\rm w}([0,T];\mathcal{H}),$$

$$z:[0,T]\to\mathcal{Z}\ \text{ with } z([0,T]) \text{ relatively compact},$$

$$\mathrm{Var}_{\mathcal{R}_1}(z;0,T)=(\text{the variation of } z \text{ over } [0,T] \text{ w.r.t. } \mathcal{R}_1)\ \ <\infty\ ,$$

$$t\mapsto\partial_t\mathcal{E}(t,u(t),z(t))\ \text{ is integrable on } [0,T],$$

and if:

- the "momentum equation" (1a) with the inital condition $\frac{\mathrm{d}u}{\mathrm{d}t}(0) = \dot{u}_0$ holds in the weak sense, i.e.

$$\int_0^T \Big\langle \mathcal{R}_2' \frac{\mathrm{d}u}{\mathrm{d}t} + \partial_u \mathcal{E}\big(t, u(t), z(t)\big)\, ,\, v(t) \Big\rangle - \Big(\mathcal{T}' \frac{\mathrm{d}u}{\mathrm{d}t} \Big| \frac{\mathrm{d}v}{\mathrm{d}t} \Big) \mathrm{d}t + \Big(\mathcal{T}' \frac{\mathrm{d}u}{\mathrm{d}t}(T) \Big| v(T) \Big) = \big(\mathcal{T}' \dot{u}_0 \big| v(0) \big),$$

holds for all $v \in C([0,T];\mathcal{U}) \cap C^1([0,T];\mathcal{V})$,

- the energy inequality holds, i.e.

$$\mathcal{T}\big(\frac{\mathrm{d}u}{\mathrm{d}t}(T)\big) + \mathcal{E}\big(T, u(T), z(T)\big) + \mathrm{Var}_{\mathcal{R}_1}(z; 0, T) + 2\int_0^T \mathcal{R}_2\big(\frac{\mathrm{d}u}{\mathrm{d}t}\big)\mathrm{d}t \leq \mathcal{T}\big(\dot{u}_0\big) + \mathcal{E}\big(0, u_0, z_0\big) + \int_0^T \partial_t \mathcal{E}(t, u(t), z(t))\mathrm{d}t,$$

- the semi-stability holds for all $v \in \mathcal{Z}$ and for a.a. $t \in I$:

$$\mathcal{E}\big(t, u(t), z(t)\big) \leq \mathcal{E}\big(t, u(t), v\big) + \mathcal{R}_1\big(v - z(t)\big)$$

- the remaining initial conditions $u(0) = u_0$ and $z(0) = z_0$ are satisfied.

Discretization in time by a fully implicit formula:

$$\mathcal{T}'\frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2} + \mathcal{R}_2'\frac{u_\tau^k - u_\tau^{k-1}}{\tau} + \partial_u\mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) = 0,$$

$$\partial\mathcal{R}_1\Big(\frac{z_\tau^k - z_\tau^{k-1}}{\tau}\Big) + \partial_z\mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) \ni 0$$

where $\mathcal{E}_\tau^k(u,z) := \mathcal{E}_\tau(k\tau, u, z)$ with $\mathcal{E}_\tau(t,u,z) := \frac{1}{\tau}\int_{-\tau}^0 \mathcal{E}(t{+}\xi, u, z)\mathrm{d}\xi$, for $k = 1, ..., T/\tau$ and using, for $k = 1$,

$$u_\tau^0 = u_0, \qquad u_\tau^{-1} = u_0 - \tau\dot{u}_0, \qquad z_\tau^0 = z_0,$$

The existence of the discrete solution (u_τ^k, z_τ^k):

the direct method,

(u_τ^k, z_τ^k) can be taken as a solution to:

$$\left.\begin{array}{ll}\text{minimize} & \tau^2\mathcal{T}\Big(\frac{u-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big) + \tau\mathcal{R}_1\Big(\frac{z-z_\tau^{k-1}}{\tau}\Big) \\ & \qquad + \tau\mathcal{R}_2\Big(\frac{u-u_\tau^{k-1}}{\tau}\Big) + \mathcal{E}_\tau^k(u,z) \\ \text{subject to} & (u,z)\in\mathcal{Q}=\mathcal{U}\times\mathcal{Z}.\end{array}\right\} \qquad (P_\tau^k)$$

It suggests a conceptually implementable numerical strategy.

A problem with deriving an energy balance:

the homogeneous-degree-1 term standardly rely on (P_τ^k)
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Knowing already u_τ^k, let us still consider
an auxiliary modified (partly linearized) minimization problem:

$$\left.\begin{array}{ll}\text{minimize} & \Big(\mathcal{T}'\dfrac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big|u\Big)+\mathcal{R}_1\big(z-z_\tau^{k-1}\big)\\ & +\big(1-\sqrt{\tau}\big)\Big\langle\mathcal{R}_2'\dfrac{u_\tau^k-u_\tau^{k-1}}{\tau},u\Big\rangle+\tau^{3/2}\mathcal{R}_2\Big(\dfrac{u-u_\tau^{k-1}}{\tau}\Big)\\ & +\mathcal{E}_\tau^k(u,z)\\ \text{subject to} & (u,z)\in\mathcal{Q}.\end{array}\right\}\ (\widetilde{P}_\tau^k)$$

Let us denote by $(\tilde{u}_\tau^k,\tilde{z}_\tau^k)$ a solution to $(\widetilde{P}_\tau^k)$.

This solution $(\tilde{u}_\tau^k, \tilde{z}_\tau^k)$ to $(\widetilde{P}_\tau^k)$ must satisfy

$$\mathcal{T}'\frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2} + \left(1-\sqrt{\tau}\,\right)\mathcal{R}_2'\frac{u_\tau^k - u_\tau^{k-1}}{\tau} + \partial_u\mathcal{E}_\tau^k(\tilde{u}_\tau^k, \tilde{z}_\tau^k) = 0,$$

$$\partial\mathcal{R}_1\left(\tilde{z}_\tau^k - z_\tau^{k-1}\right) + \partial_z\mathcal{E}_\tau^k(\tilde{u}_\tau^k, \tilde{z}_\tau^k) \ni 0.$$

- subtract these equality and (in fact) inequality respectively from the discrete fomulas for (u_τ^k, z_τ^k),
- test respectively by $\tilde{u}_\tau^k - u_\tau^k$ and $\tilde{z}_\tau^k - z_\tau^k$,
- sum it, and use degree-2 homogeneity of $\mathcal{R}_2$

we get
$$\frac{2}{\sqrt{\tau}}\mathcal{R}_2(u_\tau^k - \tilde{u}_\tau^k) + \mathcal{R}_1(z_\tau^k - \tilde{z}_\tau^k)$$
$$+\left\langle \partial\mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) - \partial\mathcal{E}_\tau^k(\tilde{u}_\tau^k, \tilde{z}_\tau^k), (u_\tau^k, z_\tau^k) - (\tilde{u}_\tau^k, \tilde{z}_\tau^k)\right\rangle \leq 0.$$

Strict convexity of $(u, z) \mapsto \mathcal{E}(u, z) + \ell R_2(u)$ for some large ℓ

$$\Longrightarrow \quad \tilde{u}_\tau^k = u_\tau^k \text{ and } \tilde{z}_\tau^k = z_\tau^k \text{ if } \tau \leq \frac{\ell^2}{4}.$$

Abbreviate $\widetilde{J}_\tau^k :=$ "cost functional of" $\widetilde{P}_\tau^k$.
Then

$$\widetilde{J}_\tau^k(u_\tau^k, z_\tau^k) = \widetilde{J}_\tau^k(\tilde{u}_\tau^k, \tilde{z}_\tau^k) = \min(\widetilde{P}_\tau^k) \le \widetilde{J}_\tau^k(u_\tau^{k-1}, z_\tau^{k-1}).$$

It gives

$$\mathcal{T}\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big) + \mathcal{R}_1\big(z_\tau^k - z_\tau^{k-1}\big) + \tau\big(2-\sqrt{\tau}\big)\mathcal{R}_2\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big) + \mathcal{E}_\tau^k\big(u_\tau^k, z_\tau^k\big) \le \mathcal{T}\Big(\frac{u_\tau^{k-1} - u_\tau^{k-2}}{\tau}\Big) + \mathcal{E}_\tau^k\big(u_\tau^{k-1}, z_\tau^{k-1}\big)$$

We further use:

$$\mathcal{E}_\tau^k\big(u_\tau^{k-1}, z_\tau^{k-1}\big) = \mathcal{E}_\tau^{k-1}\big(u_\tau^{k-1}, z_\tau^{k-1}\big) + \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t \mathcal{E}_\tau\big(t, u_\tau^{k-1}, z_\tau^{k-1}\big)\mathrm{d}t.$$

Summing it for $k = 1, ..., T/\tau$, we get the approximate energy balance:

$$\mathcal{T}\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T)\Big) + \mathcal{E}\big(T, u_\tau(T), z_\tau(T)\big)$$

$$+ \mathrm{Var}_{\mathcal{R}_1}\big(z_\tau; 0, T\big) + \big(2-\sqrt{\tau}\,\big)\int_0^T \mathcal{R}_2\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}\Big)\mathrm{d}t$$

$$\leq \mathcal{T}\big(\dot{u}_0\big) + \mathcal{E}\big(0, u_0, z_0\big) + \int_0^T \partial_t\mathcal{E}_\tau\big(t, \underline{u}_\tau(t), \underline{z}_\tau(t)\big)\mathrm{d}t,$$

where

u_τ := piecewise affine interpolation of $\{u_\tau^k\}_{k=0}^{T/\tau}$,

$\bar{u}_\tau$:= "forward" piecewise constant interpolation of $\{u_\tau^k\}_{k=0}^{T/\tau}$,

$\underline{u}_\tau$:= "backward" piecewise constant interpolation of $\{u_\tau^k\}_{k=0}^{T/\tau}$,

and similarly for z_τ, $\bar{z}_\tau$, and $\underline{z}_\tau$.

Taking (u_τ^k, z_τ^k) a solution to (P_τ^k) and fixing u_τ^k, we can see that z_τ^k fulfills

$$\mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) + \mathcal{R}_1(z_\tau^k - z_\tau^{k-1}) \le \mathcal{E}_\tau^k(u_\tau^k, v) + \mathcal{R}_1(v - z_\tau^{k-1})$$

for all $v \in \mathcal{Z}$. Using the triangle inequality of $\mathcal{R}_1$, we also know $\mathcal{R}_1(v - z_\tau^{k-1}) - \mathcal{R}_1(z_\tau^k - z_\tau^{k-1}) \le \mathcal{R}_1(v - z_\tau^k)$. Altogether, we get

$$\mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) \le \mathcal{E}_\tau^k(u_\tau^k, v) + \mathcal{R}_1(v - z_\tau^k).$$

After summation for $k = 1, ..., T/\tau$, we get the "integrated" semi-stability for the discrete solution:

$$\int_0^T \bar{\mathcal{E}}_\tau(t, \bar{u}_\tau(t), \bar{z}_\tau(t))\mathrm{d}t \le \int_0^T \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau(t), v(t)\big) + \mathcal{R}_1\big(v(t) - \bar{z}_\tau(t)\big)\mathrm{d}t$$

holds for all $v \in L^\infty([0, T]; \mathcal{Z})$.

We have also the following discrete analog of the momentum equation:

$$\int_0^T \Big\langle \mathcal{R}_2' \frac{\mathrm{d}u_\tau}{\mathrm{d}t} + \partial_u \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau(t), \bar{z}_\tau(t)\big)\,,\, \bar{v}_\tau(t) \Big\rangle \mathrm{d}t - \int_\tau^T \Big(\mathcal{T}' \frac{\mathrm{d}u_\tau}{\mathrm{d}t}(\cdot - \tau) \Big| \frac{\mathrm{d}v_\tau}{\mathrm{d}t}\Big)\mathrm{d}t + \Big(\mathcal{T}' \frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T) \Big| v_\tau(T)\Big) = \big(\mathcal{T}' \dot{u}_0 \big| v_\tau(\tau)\big),$$

holds for all $v \in C^1([0,T]; \mathcal{U} \cap \mathcal{V})$ where v_τ and $\bar{v}_\tau$ are respectively the piecewise affine and the piecewise constant intepolants of $\{v(k\tau)\}_{k=0}^{T/\tau}$.

Standard assumptions on coercivity, lower semicontinuity, etc.

An essential assumption:
existence of a joint recovery sequence in the sense

$$\forall (t_k, u_k, z_k) \to (t, u, z) \ \ \forall \widetilde{z} \in \mathcal{Z} \quad \exists (\widetilde{z}_k)_{k\in\mathbb{N}} :$$
$$\limsup_{k\to\infty} \big(\mathcal{E}(t_k, u_k, \widetilde{z}_k) + \mathcal{R}_1(\widetilde{z}_k - z_k) - \mathcal{E}(t_k, u_k, z_k)\big)$$
$$\leq \mathcal{E}(t, u, \widetilde{z}) + \mathcal{R}_1(\widetilde{z} - z) - \mathcal{E}(t, u, z).$$

Possibly, we also benefit from assuming a uniform monotonicity of $\partial_u \mathcal{E}(t, \cdot, z)$.

Step 1: a-priori estimates: from the approximate energy balance by Gronwall inequality:

$$\left\|u_\tau\right\|_{L^\infty([0,T];\mathcal{U}))\,\cap\, W^{1,2}([0,T];\mathcal{V})} \le C_1, \tag{5a}$$

$$\left\|\frac{\mathrm{d}u_\tau}{\mathrm{d}t}\right\|_{L^\infty([0,T];\mathcal{H})\,\cap\,\mathrm{BV}([0,T];\mathcal{U}^*+\mathcal{V}^*)} \le C_2, \tag{5b}$$

$$\max_{t\in[0,T]} \bar{\mathcal{E}}_\tau(t,\bar{u}_\tau(t),\bar{z}_\tau(t)) \le C_3, \tag{5c}$$

$$\left\|z_\tau\right\|_{L^\infty([0,T];\mathcal{Z})} \le C_4; \tag{5d}$$

$$\mathrm{Var}_{\mathcal{R}_1}(\bar{z}_\tau;0,T) \le C_5; \tag{5e}$$

note that the BV-estimate in (5b) represents an estimate of the acceleration $\frac{\mathrm{d}^2u_\tau}{\mathrm{d}t^2}$ as a measure $M([0,T];\mathcal{U}^*+\mathcal{V}^*)$.

Step 2: selection of subsequences

weakly* converging (Banach's selection principle) to some u and z,

pointwise converging (Helly's selection principle):

$$z_\tau(t) \to z(t) \text{ weakly in } \mathcal{Z} \text{ for all } t.$$

in case of a uniform monotonicity of $\partial_u\mathcal{E}(t,\cdot,z)$ also

$$u_\tau \to u \text{ strongly in } L^p([0,T];\mathcal{U}).$$

Step 3: limit passage in the stability:
using the joint recovery sequence condition for the approximate semi-stability

$$\int_0^T \bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{z}_\tau(t)\big)\mathrm{d}t \le \int_0^T \bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),v(t)\big)+\mathcal{R}_1\big(v(t)-\bar{z}_\tau(t)\big)\mathrm{d}t$$

to get the limit semi-stability

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for all $v \in L^\infty([0,T];\mathcal{Z})$.

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to get the limit semi-stability and desintegrating it

$$\int_0^T \mathcal{E}\big(t, u(t), z(t)\big)\mathrm{d}t \leq \int_0^T \mathcal{E}\big(t, u(t), v(t)\big) + \mathcal{R}_1\big(v(t) - z(t)\big)\mathrm{d}t$$

for all $v \in \mathcal{Z}$ and a.a. $t \in [0, T]$.

Step 4: limit passage in the upper energy inequality:

$$\mathcal{T}\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T)\Big) + \mathcal{E}\big(T, u_\tau(T), z_\tau(T)\big) + \mathrm{Var}_{\mathcal{R}_1}\big(z_\tau; 0, T\big) + \big(2-\sqrt{\tau}\,\big)\int_0^T \mathcal{R}_2\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}\Big)\mathrm{d}t \le \mathcal{T}\big(\dot{u}_0\big) + \mathcal{E}\big(0, u_0, z_0\big) + \int_0^T \partial_t\mathcal{E}_\tau\big(t, u_\tau(t), z_\tau(t)\big)\mathrm{d}t.$$

by lower semicontinuity in the l.h.s. and continuity in the r.h.s.

Step 4: limit passage in the upper energy inequality:

$$\mathcal{T}\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T)\Big) + \mathcal{E}\big(T, u_\tau(T), z_\tau(T)\big) + \mathrm{Var}_{\mathcal{R}_1}\big(z_\tau; 0, T\big) + \big(2-\sqrt{\tau}\big)\int_0^T \mathcal{R}_2\Big(\frac{\mathrm{d}u_\tau}{\mathrm{d}t}\Big)\mathrm{d}t \leq \mathcal{T}\big(\dot{u}_0\big) + \mathcal{E}\big(0, u_0, z_0\big) + \int_0^T \partial_t \mathcal{E}_\tau\big(t, u_\tau(t), z_\tau(t)\big)\mathrm{d}t.$$

by lower semicontinuity in the l.h.s. and continuity in the r.h.s.

Step 5: the lower energy inequality:

semistability (a.e.) and upper-energy inequality allows

by Riemann-sum approximation of Lebesgue integral to show

the opposite inequality $\Rightarrow$ the energy equality!

Step 6: Improved convergence.

$$\forall\, t \in [0, T]: \ \mathrm{Var}_{\mathcal{R}_1}(z_\tau; [0, t]) \to \mathrm{Var}_{\mathcal{R}_1}(z; [0, t]);$$

$$\forall\, t \in [0, T]: \ \mathcal{E}(t, u_\tau(t), z_\tau(t)) \to \mathcal{E}(t, u(t), z(t));$$

$$\partial_t\mathcal{E}(\cdot, u_\tau(\cdot), z_\tau(\cdot)) \to \partial_t\mathcal{E}(\cdot, u(\cdot), z(\cdot)) \text{ in } L^1((0, T)).$$

Step 7: Convergence in the approximate momentum equation

$$\int_0^T \Big\langle \mathcal{R}_2' \frac{\mathrm{d}u_\tau}{\mathrm{d}t} + \partial_u \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau(t), \bar{z}_\tau(t)\big)\,,\ \bar{v}_\tau(t)\Big\rangle \mathrm{d}t - \int_\tau^T \Big(\mathcal{T}'\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(\cdot-\tau)\Big|\frac{\mathrm{d}v_\tau}{\mathrm{d}t}\Big) + \Big(\mathcal{T}'\frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T)\Big|v_\tau(T)\Big)$$

The only delicate point is to ensure

$$\partial_u \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau, \bar{z}_\tau\big) \to \partial_u \mathcal{E}\big(t, u, z\big) \ \text{ weakly in } L^1(0,T;\mathcal{V}^*).$$

Step 7: Convergence in the approximate momentum equation

$$\int_0^T \Big\langle \mathcal{R}_2' \frac{\mathrm{d}u_\tau}{\mathrm{d}t} + \partial_u \mathcal{E}_\tau\big(t, u_\tau(t), z_\tau(t)\big)\,,\, v_\tau(t) \Big\rangle \mathrm{d}t - \int_\tau^T \Big(\mathcal{T}' \frac{\mathrm{d}u_\tau}{\mathrm{d}t}(\cdot - \tau) \Big| \frac{\mathrm{d}v_\tau}{\mathrm{d}t} \Big) + \Big(\mathcal{T}' \frac{\mathrm{d}u_\tau}{\mathrm{d}t}(T) \Big| v_\tau(T) \Big)$$

The only delicate point is to ensure

$$\partial_u \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau, \bar{z}_\tau\big) \to \partial_u \mathcal{E}(t, u, z) \ \text{ weakly in } L^1(0, T; \mathcal{V}^*).$$

Stability under data perturbation:

Γ-convergence of $\mathcal{E}$'s and $\mathcal{R}$'s and joint-recovery-sequence condition
= a modification of [A.Mielke, T.R., U.Stefanelli].

No convexity of $\mathcal{E} + \ell\mathcal{R}_2$ for large ℓ needed now.

Thermodynamical expansion possible:

$\mathcal{E}$ temperature dependent,

- fully implicit time discretization does not yield an incremental problem with a variational structure (existence by Schauder fixed point only)
- energetic-solution concept important (weak convergence of the dissipative heat source)
- L^1-theory for heat equation (Boccardo, Galouët, et al.) and interpolation of the adiabatic-heat term (Gagliardo, Nirenberg)

Linearized visco-plasticity with hardening at small strains:

$\Omega \subset \mathbb{R}^d$ a bounded domain,

$u =$ displacement,

$z = (\pi, \eta) =$ the plastic deformation and the hardening parameter,

$\mathcal{U} = W^{1,2}(\Omega; \mathbb{R}^d)$,

$\mathcal{Z} = L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym},0} \times \mathbb{R})$,

with $\mathbb{R}^{d\times d}_{\mathrm{sym},0} := \big\{A \in \mathbb{R}^{d\times d};\ A^\top = A,\ \mathrm{tr}(A) = 0\big\}$,

$$\mathcal{E}(t, u, \pi, \eta) = \int_\Omega \frac{1}{2}\mathbb{C}\big(e(u) - \pi\big) : \big(e(u) - \pi\big) + b\eta^2 - f(t) \cdot u \,\mathrm{d}x,$$

with $b > 0$, $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$,

$$\mathcal{R}_1(\dot\pi, \dot\eta) = \int_\Omega \delta^*_P(\dot\pi) + \delta_S(\dot\pi, \dot\eta)\,\mathrm{d}x,$$

$P \subset \mathbb{R}^{n\times n}_{\mathrm{sym},0}$ be a convex closed neighbourhood of the origin,
δ_P is its indicator function, and δ^*_P the conjugate functional to δ_P,
$S := \{z = (\pi, \eta);\ \eta \geq \delta^*_P(\pi)\}$,

$$\mathcal{R}_2(\dot u) = \int_\Omega \frac{1}{2}\mathbb{D}e(\dot u) : e(\dot u)\,\mathrm{d}x,$$

$$\mathcal{T}(\dot u) = \int_\Omega \frac{\varrho}{2}|\dot u|^2\,\mathrm{d}x.$$

Main features:

$\mathcal{R}_1$ discontinous but $\mathcal{E}(t,\cdot,\cdot,\cdot)$ convex and quadratic.

Joint recovery sequence by the "binominal trick":

$$\limsup_{k\to\infty}\Big(\mathcal{E}(t_k,u_k,\widetilde{z}_k)+\mathcal{R}_1(\widetilde{z}_k-z_k)-\mathcal{E}(t_k,u_k,z_k)\Big)$$

$$=\limsup_{k\to\infty}\bigg(\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi_k+\widetilde{\pi}_k)-\mathbb{C}e(u_k)\Big):(\pi_k-\widetilde{\pi}_k)$$

$$+\frac{1}{2}b(\eta_k+\widetilde{\eta}_k)(\eta_k-\widetilde{\eta}_k)\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}_k-\pi_k,\widetilde{\eta}_k-\eta_k)\bigg)$$

$$=\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi+\widetilde{\pi})-\mathbb{C}e(u)\Big):(\pi-\widetilde{\pi})+\frac{1}{2}b(\eta+\widetilde{\eta})(\eta-\widetilde{\eta})\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}-\pi,\widetilde{\eta}-\eta)$$

$$=\mathcal{E}(t,u,\widetilde{z})+\mathcal{R}_1(\widetilde{z}-z)-\mathcal{E}(t,u,z),$$

if we choose $\widetilde{\pi}_k:=\widetilde{\pi}-\pi+\pi_k$ and $\widetilde{\eta}_k:=\widetilde{\eta}-\eta+\eta_k$.

Similar results by

H.-D.Alber, C.Carstensen, C.Chelminski, W.Han & D.Reddy, A.Mielke, at al.

Thermodynamical expansion for thermally dilatable materials:

S.Bartels & T.R. (in preparation)

Main features:
$\mathcal{R}_1$ discontinous but $\mathcal{E}(t,\cdot,\cdot,\cdot)$ convex and quadratic.
Joint recovery sequence by the "binominal trick":

$$\limsup_{k\to\infty}\Big(\mathcal{E}(t_k,u_k,\widetilde{z}_k)+\mathcal{R}_1(\widetilde{z}_k-z_k)-\mathcal{E}(t_k,u_k,z_k)\Big)$$

$$=\limsup_{k\to\infty}\bigg(\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi_k+\widetilde{\pi}_k)-\mathbb{C}e(u_k)\Big):(\pi_k-\widetilde{\pi}_k)$$

$$+\frac{1}{2}b(\eta_k+\widetilde{\eta}_k)(\eta_k-\widetilde{\eta}_k)\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}_k-\pi_k,\widetilde{\eta}_k-\eta_k)\bigg)$$

$$=\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi+\widetilde{\pi})-\mathbb{C}e(u)\Big):(\pi-\widetilde{\pi})+\frac{1}{2}b(\eta+\widetilde{\eta})(\eta-\widetilde{\eta})\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}-\pi,\widetilde{\eta}-\eta)$$

$$=\mathcal{E}(t,u,\widetilde{z})+\mathcal{R}_1(\widetilde{z}-z)-\mathcal{E}(t,u,z),$$

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$$\limsup_{k\to\infty}\Big(\mathcal{E}(t_k,u_k,\widetilde{z}_k)+\mathcal{R}_1(\widetilde{z}_k-z_k)-\mathcal{E}(t_k,u_k,z_k)\Big)$$

$$=\limsup_{k\to\infty}\bigg(\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi_k+\widetilde{\pi}_k)-\mathbb{C}e(u_k)\Big):(\pi_k-\widetilde{\pi}_k)$$

$$+\frac{1}{2}b(\eta_k+\widetilde{\eta}_k)(\eta_k-\widetilde{\eta}_k)\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}_k-\pi_k,\widetilde{\eta}_k-\eta_k)\bigg)$$

$$=\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi+\widetilde{\pi})-\mathbb{C}e(u)\Big):(\pi-\widetilde{\pi})+\frac{1}{2}b(\eta+\widetilde{\eta})(\eta-\widetilde{\eta})\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}-\pi,\widetilde{\eta}-\eta)$$

$$=\mathcal{E}(t,u,\widetilde{z})+\mathcal{R}_1(\widetilde{z}-z)-\mathcal{E}(t,u,z),$$

if we choose $\widetilde{\pi}_k:=\widetilde{\pi}-\pi+\pi_k$ and $\widetilde{\eta}_k:=\widetilde{\eta}-\eta+\eta_k$.

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Main features:
$\mathcal{R}_1$ discontinous but $\mathcal{E}(t,\cdot,\cdot,\cdot)$ convex and quadratic.
Joint recovery sequence by the "binominal trick":

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(\mathcal{E}(t_k,u_k,\widetilde{z}_k)+\mathcal{R}_1(\widetilde{z}_k-z_k)-\mathcal{E}(t_k,u_k,z_k)\Big)\\
&\quad=\limsup_{k\to\infty}\bigg(\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi_k+\widetilde{\pi}_k)-\mathbb{C}e(u_k)\Big):(\pi_k-\widetilde{\pi}_k)\\
&\qquad\qquad+\frac{1}{2}b(\eta_k+\widetilde{\eta}_k)(\eta_k-\widetilde{\eta}_k)\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}_k-\pi_k,\widetilde{\eta}_k-\eta_k)\bigg)\\
&\quad=\int_\Omega\Big(\frac{1}{2}\mathbb{C}(\pi+\widetilde{\pi})-\mathbb{C}e(u)\Big):(\pi-\widetilde{\pi})+\frac{1}{2}b(\eta+\widetilde{\eta})(\eta-\widetilde{\eta})\mathrm{d}x+\mathcal{R}_1(\widetilde{\pi}-\pi,\widetilde{\eta}-\eta)\\
&\quad=\mathcal{E}(t,u,\widetilde{z})+\mathcal{R}_1(\widetilde{z}-z)-\mathcal{E}(t,u,z),
\end{aligned}$$

if we choose $\widetilde{\pi}_k:=\widetilde{\pi}-\pi+\pi_k$ and $\widetilde{\eta}_k:=\widetilde{\eta}-\eta+\eta_k$.

Similar results by
H.-D.Alber, C.Carstensen, C.Chelminski, W.Han & D.Reddy, A.Mielke, at al.
Thermodynamical expansion for thermally dilatable materials:
S.Bartels & T.R. (in preparation)

Gradient damage (partial) at small strains:

$\Omega\subset\mathbb{R}^d$ a bounded domain,
$u=$displacement,
$z=$a scalar damage parameter,
$\mathcal{U}=W^{1,2}(\Omega;\mathbb{R}^d)$,
$\mathcal{Z}=W^{1,p}(\Omega)$,

$$\mathcal{E}(t,u,z)=\int_\Omega\frac{z}{2}\mathbb{C}e(u){:}e(u)+\delta_{[0,1]}(z)+b|\nabla z|^p+\frac{1}{2}\mathbb{C}_0e(u){:}e(u)-f(t)\cdot u\,\mathrm{d}x,$$

with $b>0$ and $\mathbb{C}_0$ positive definite,

$$\mathcal{R}_1(\dot z)=\int_\Omega\delta_{(-\infty,0]}(\dot z)-\kappa\dot z\,\mathrm{d}x,$$

with $\kappa>0$ the energy per d-dimensional volume dissipated by damage,

$$\mathcal{R}_2(\dot u)=\int_\Omega\frac{1}{2}\mathbb{D}e(\dot u):e(\dot u)\,\mathrm{d}x,$$

$$\mathcal{T}(\dot u)=\int_\Omega\frac{\varrho}{2}|\dot u|^2\,\mathrm{d}x.$$

Main features:

$\mathcal{R}_1$ discontinous and $\mathcal{E}(t,\cdot,\cdot)$ nonconvex
but $\partial_u\mathcal{E}(t,\cdot,z)$ uniformly monotone.

Regularization $\mathcal{E}_\varepsilon$ of $\mathcal{E}$ by a term $\varepsilon|e(u)|^6$: then
$(e,z)\mapsto z\mathbb{C}e{:}e+\varepsilon|e|^6+\ell|e|^2$ is strictly convex for ℓ large, as need above.

Joint recovery sequence:
A.Mielke & T.R. (for $p>d$), A.Mielke & M.Thomas (also for $p\leq d$).

After having the energetic solution of the regularized problem, passage $\varepsilon\to0$ possible because $\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}$.

Delamination:

$\Omega \subset \mathbb{R}^d$ a bounded domain,
Γ a $d-1$ dimensional manifold inside Ω,
$u =$ displacement,
$z =$ a scalar delamination parameter,
$\mathcal{U} = W^{1,2}(\Omega \setminus \Gamma; \mathbb{R}^d)$,
$\mathcal{Z} = L^\infty(\Gamma)$,

$$\mathcal{E}_\varepsilon(t,u,z) = \begin{cases} \displaystyle\int_\Omega \frac{\mathbb{C}e(u){:}e(u)}{2} - f\cdot u \mathrm{d}x + \int_\Gamma \frac{z}{\varepsilon}[u]_\Gamma^2 \mathrm{d}S & \text{if } [u]\cdot\nu \geq 0 \text{ on } \Gamma, \\ & 0 \leq z \leq 1 \text{ on } \Gamma, \\ +\infty & \text{elsewhere.} \end{cases}$$

with ν the normal to Γ,

$$\mathcal{R}_1(\dot z) = \int_\Gamma \delta_{(-\infty,0]}(\dot z) - \kappa \dot z \, \mathrm{d}S, \quad \text{with}$$

$\kappa > 0$ the energy per $d-1$-dimensional surface dissipated by delamination,

$$\mathcal{R}_2(\dot u) = \int_\Omega \frac{1}{2}\mathbb{D}e(\dot u) : e(\dot u)\,\mathrm{d}x,$$

$$\mathcal{T}(\dot u) = \int_\Omega \frac{\varrho}{2}|\dot u|^2\,\mathrm{d}x.$$

Main features:

$\mathcal{R}_1$ discontinous and $\mathcal{E}(t,\cdot,\cdot)$ nonconvex
(⇒ a regularization $\int_\Gamma \varepsilon\big|[u]\big|^6 \mathrm{d}S$ helps),

but we benefit compactness of trace operator on Γ
(⇒ no gradient of z nedeed),

$\partial_u\mathcal{E}(t,\cdot,z)$ uniformly monotone.

Γ-limit of $\mathcal{E}_\varepsilon$ for $\varepsilon \to 0$: a brittle delamination:

$$\mathcal{E}_\infty(t,u,z) = \begin{cases} \displaystyle\int_\Omega \frac{\mathbb{C}e(u){:}e(u)}{2} - f\cdot u\mathrm{d}x & \text{if } [u]\cdot\nu \geq 0 \text{ on } \Gamma,\ 0 \leq z \leq 1 \text{ on } \Gamma, \text{ and} \\ & [u(x)]_\Gamma = 0 \text{ for a.a. } x{\in}\Gamma \text{ such that } z(x) > 0, \\ +\infty & \text{elsewhere.} \end{cases}$$

Joint recovery sequence: T.R. & L.Scardia & C.Zanini (in preparation)

Applications in geophysics of short-time range:
spontaneous rupture of faults in lithosperic plates with kinetic-energy emission via sesmic waves (attenuated by viscosity) that may

1) activate another rupture on another distant fault
2) manifest as an earthsquake on the earth surface.

Modifications:
z the slip along Γ and $\mathcal{R}_1(z,\dot{z})$ with monotonic $\mathcal{R}_1(\cdot,\dot{z})$
(so-called "slip weakening concept used in geophysics")
Combination of delamination and slip weakening possible too.

Some references:

- A.Mielke, T.Roubíček, U.Stefanelli: Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. P.D.E.* **31** (2008), 387-416.
- A.Mielke, T.Roubíček, J.Zeman: Complete damage in elastic and viscoelastic media and its energetics. *Comp. Methods Appl. Mech. Engr.*, submitted.
- T.Roubíček: Rate independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.*, printed electronically.
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