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Materials with Memory: free energies & Solution Exponential Decay

- General Framework of the Problem
- Constitutive and Functional Assumptions
- Evolution Problems
- Semigroups Theory: Exponential Decay
- Bäcklund Transformations
- Traveling Wave Solutions

General Framework of the Problem

the model

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- thermally passive environment $\Longrightarrow$ environment's thermal status is not affected by the conductor's one.

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...and many more !!

Quantities of Interest

$\theta = \theta\,(\mathbf{x}, t)$	*temperature*
θ_0	*fixed reference temperature*
$u := \theta\,(\mathbf{x}, t) - \theta_0$	*relative temperature*
$\mathbf{g} := \nabla u\,(\mathbf{x}, t)$	*temperature gradient*
$\mathbf{q} = \mathbf{q}\,(\mathbf{x}, t)$	*heat flux*
$e = e\,(\mathbf{x}, t)$	*internal energy*
$\alpha_0 = const$	*energy relaxation coefficient*
$k = k\,(\mathbf{x}, t)$	*heat flux relaxation function*
$k_0(\mathbf{x}) := k\,(\mathbf{x}, 0)$	*initial heat flux relaxation function*

furthemore . . . . . . constitutive assumptions

Constitutive Assumptions

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$$
\begin{aligned}
e(\mathbf{x},t) &= \alpha_0 \; u(\mathbf{x},t) \quad , \quad \alpha_0 \; const \\
\mathbf{q}(\mathbf{x},t) &= -\int_0^\infty k(\mathbf{x},\tau)\; \nabla u(\mathbf{x},t-\tau)\; d\tau \\
k(\mathbf{x},t) &= k_0(\mathbf{x}) + \int_0^t \dot{k}(\mathbf{x},s)\; ds
\end{aligned}
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\end{aligned}
$$

the heat flux can be equivalently written in the form:

$$
\boxed{\mathbf{q}\,(\mathbf{x},t) = \int_0^\infty \dot{k}(\mathbf{x},\tau)\,\bar{\mathbf{g}}^t\,(\mathbf{x},\tau)\;d\tau} \, . \tag{-1}
$$

where $\quad \mathbf{g} := \nabla u, \quad$ and $\quad \bar{\mathbf{g}}^t(\mathbf{x},\tau) := \int_{t-\tau}^t \nabla u(\mathbf{x},s)\;ds$

$\bar{\mathbf{g}}^t(\mathbf{x},\tau)$ **. . . integrated history of the temperature gradient**

Functional Assumptions

- $\dot{k} \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \qquad \forall\ \mathbf{x} \in \Omega$
- $k \in L^1(\mathbb{R}^+) \qquad \forall\ \mathbf{x} \in \Omega$
- $k(\mathbf{x}, t) = k_0(\mathbf{x}) + \int_0^t \dot{k}(\mathbf{x}, s)\ ds$
- $k(\mathbf{x}, \infty) := \lim_{t \to \infty} k(\mathbf{x}, t) = 0$

fading memory property $\forall\ \mathbf{x} \in \Omega$

$$\forall \varepsilon > 0\ \exists\ \tilde{a} = a\,(\varepsilon, \bar{\mathbf{g}}^t) \in \mathbb{R}^+ s.t.\ \forall a > \tilde{a} \Rightarrow \left| \int_0^\infty \dot{k}(s + a)\bar{\mathbf{g}}^t(s)\ ds \right| < \varepsilon$$

…i.e., no heat flux when, at infinity, equilibrium is reached!

The evolution problem

$$\begin{cases} \alpha_0 u_t = -\nabla \cdot \mathbf{q} + r(\mathbf{x}, t)\,, \\ \mathbf{q}(\mathbf{x}, t) = \displaystyle\int_0^\infty \dot{k}(\mathbf{x}, \tau) \overline{\nabla u}^{\,t}(\mathbf{x}, \tau)\, d\tau \end{cases}$$

$r(\mathbf{x}, t)$ heat supply

in $Q_T = \Omega \times [0, T)$

initial and boundary conditions Problem 1

no heat flux through the body boundary (Neumann cond.s)

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initial and boundary conditions — Problem 1

no heat flux through the body boundary (Neumann cond.s)

$$\begin{aligned} u(\mathbf{x},0) &= u_0(\mathbf{x}) && \forall\, \mathbf{x} \in \Omega \\ \mathbf{q}(\mathbf{x},t)\cdot \mathbf{n}(\mathbf{x})|_{\partial\Omega} &= 0, && \forall\, t \in [0,T) \end{aligned}$$

where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the (smooth) boundary $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$ and, in addition, say $r(\mathbf{x},t) = 0$,i.e no heat supply.

The evolution problem

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$r(\mathbf{x}, t)$ heat supply

in $Q_T = \Omega \times [0, T)$

initial and boundary conditions

Problem 2

fixed temperature at the body boundary (Dirichlet cond.s)

$$\begin{aligned} u(\mathbf{x}, 0) &= u_0(\mathbf{x}) && \forall\, \mathbf{x} \in \Omega \\ u(\mathbf{x}, t)|_{\partial\Omega} &= 0, && \forall\, t \in [0, T) \end{aligned}$$

at the boundary of Ω , namely $\forall \mathbf{x} \in \partial\Omega$, and, in addition, say $r(\mathbf{x}, t) = 0$,i.e no heat supply.

Semigroups Theory

Semigroups Theory

- write evolution problem as a dynamical system

$$\begin{cases} \dot{u}(x,t) = \nabla \cdot \displaystyle\int_0^{+\infty} \dot{k}(\tau)\bar{\mathbf{g}}^t(\tau)\, d\tau \\ \dot{\bar{\mathbf{g}}}^t(\tau) = \nabla u(t) - \nabla u(t-\tau) \end{cases}$$

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- energy inner product ...

$$\boxed{\langle \chi_1(t), \chi_2(t) \rangle = \frac{1}{2}\int_\Omega \alpha(\mathbf{x}) u_1(\mathbf{x},t) u_2(\mathbf{x},t) d\mathbf{x} - \frac{1}{2}\int_\Omega \int_0^\infty \dot{k}(\tau)\bar{\mathbf{g}}_1^t(\tau)\cdot\bar{\mathbf{g}}_2^t(\tau) d\tau d\mathbf{x}}$$

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- ... and induced norm

$$\boxed{\|\chi(t)\|^2 = \frac{1}{2}\int_\Omega \alpha(\mathbf{x}) u^2(\mathbf{x},t) d\mathbf{x} - \frac{1}{2}\int_\Omega \int_0^\infty \dot{k}(\tau)\left[\bar{\mathbf{g}}^t(\tau)\right]^2 d\tau d\mathbf{x}}$$

Exponential Decay

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- recall Graffi's free energy

$$\zeta_G\left(u, \bar{\mathbf{g}}^t\right) := \frac{\alpha_0}{2}[u(\mathbf{x}, t)]^2 - \frac{1}{2}\int_0^\infty k'(\tau)\left[\bar{\mathbf{g}}^t(\mathbf{x}, \tau)\right]^2 d\tau$$

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▷ Graffi's free energy satisfies

$$\boxed{\frac{\partial \zeta_G}{\partial t}(\mathbf{x},t) - \frac{\partial e}{\partial t}(\mathbf{x},t)u(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t)\cdot\mathbf{g}(\mathbf{x},t) = -\frac{1}{2}\int_0^\infty k''(\tau)\left[\bar{\mathbf{g}}^t(\mathbf{x},\tau)\right]^2 d\tau}$$

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▷ introduce the energy

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \alpha_0\,[u(\mathbf{x},t)]^2\,d\mathbf{x} - \frac{1}{2}\int_\Omega\int_0^\infty k'(\tau)\left[\bar{\mathbf{g}}^t(\mathbf{x},\tau)\right]^2 d\tau d\mathbf{x}$$

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▷ under the assumption

$$0 < -\dot{k}(\tau) \leq \delta \ddot{k}(\tau) \quad , \quad \dot{k}(0) \;\; bounded$$

since

$$\boxed{\int_\Omega \dot{\zeta}_G\left(u, \bar{\mathbf{g}}^t\right) d\mathbf{x} = -\frac{1}{2}\int_\Omega \int_0^\infty \ddot{k}(\mathbf{x}, \tau)\left[\bar{\mathbf{g}}^t(\tau)\right]^2 d\tau d\mathbf{x} \leq 0}$$

Exponential Decay

- recall that is $\mathcal{E}(t) = \int_\Omega \zeta_G\left(u, \bar{\mathbf{g}}^t\right) d\mathbf{x}$

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- exponential decay follows

$$\boxed{\frac{d}{dt}\mathcal{E}(t) \leq 0 \Longrightarrow 0 \leq \mathcal{E}(t) \leq \mathcal{E}(0)}$$

Bäcklund Transformations

Bäcklund Transformations

- given two Evolution Equations, K and G are C^∞-vector field

$$
\begin{aligned}
u_t &= K(u), \quad K : M_1 \to TM_1, \quad u : (x,t) \in \mathbb{R}^n \times \mathbb{R} \to u(x,t) \in \mathbb{R}^r \\
v_t &= G(v), \quad G : M_2 \to TM_2, \quad v : (x,t) \in \mathbb{R}^n \times \mathbb{R} \to v(x,t) \in \mathbb{R}^n
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- Bäcklund Transformations [Fuchssteiner, Nonlin. An. TMA, 3, 849,80]. let, in turn, $u(x,t)$ and $v(x,t)$ be solutions to the two evolution equations

if $\boxed{B(u,v)|_{t=0} = 0 \quad \Rightarrow \quad B(u,v) = 0 \quad \forall t}$

then $\boxed{B(u,v) = 0}$ is a Bäcklund Transformations

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- if $u_t = K(u)$ admits a Recursion Operator Φ it can be written

$$\boxed{u_t = \Phi(u)[u_x]}$$

Bäcklund Chart

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- Bäcklund Chart: net of links connecting evolution equations

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- Example 1 [Cole(1950),Hopf(1950)].

$$\boxed{u_t \;=\; u_{xx} + 2uu_x} \overset{BT}{\longleftrightarrow} \boxed{v_t \;=\; v_{xx}}$$

$$B(u,v) = 0 \; reads \; \boxed{v_x - uv = 0}$$

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- Example 2

$$\boxed{u_t \;=\; u_{xxx} + 6uu_x} \overset{BT}{\longleftrightarrow} \boxed{v_t \;=\; v_{xxx} - 6v^2 v_x}$$

$$B(u,v) = 0 \; reads \; \boxed{u + v_x + v^2 = 0}$$

...Latest Bäcklund Chart

[S. C., 2008]

...Latest Bäcklund Chart

[S. C., 2008]

Heat Condution with memory:

...Latest Bäcklund Chart

[S. C., 2008]

Heat Condution with memory:

- Consider a 1-dimesional heat conductor with memory

$$\alpha_0 \, v_t = - \left[\int_0^t \dot{k}(\tau) \overline{\nabla v}^t(\mathbf{x}, \tau) \, d\tau \right]_x$$

which ... after a Cole-Hopf transformations is related to

$$\boxed{u_{tt} = k_0 \left[u_{xx} + 2 u u_x \right] \quad , \quad k_0 = k(0)}$$

... look for solutions of the latter ...

Traveling Wave Solutions

Traveling Wave Solutions

Traveling Wave Solutions

- Inspection shows [S. C., Vietri 2008]

$$u(x,t) = \frac{k_0 - c^2}{k_0(x \pm ct + C)}, \ \forall C \in \mathbb{R}, \ \forall c \in \mathbb{R}^+.$$

$c^2 = k_0 \implies$ trivial zero solution.

Traveling Wave Solutions

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$c^2 = k_0 \quad \Longrightarrow$ trivial zero solution.

- further solutions are $\forall c \in \mathbb{R}^+ \backslash \{\sqrt{k_0}\}, \forall c_1, c_2 \in \mathbb{R}$,

$$u(x \pm ct) = -\frac{c_1 e^{\frac{k_0}{(c^2 - k_0)}(x \pm ct)} - c_2 e^{-\frac{k_0}{(c^2 - k_0)}(x \pm ct)}}{c_1 e^{\frac{k_0}{(c^2 - k_0)}(x \pm ct)} + c_2 e^{-\frac{k_0}{(c^2 - k_0)}(x \pm ct)}} \ ,$$

which also represent *traveling wave* solutions of the nonlinear wave equation

$$\boxed{u_{tt} = k_0 \left[u_{xx} + 2 u u_x \right] \quad , \quad k_0 = k(0)}$$

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