Analysis of model equations for stress-enhanced diffusion in coal layers

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Intro

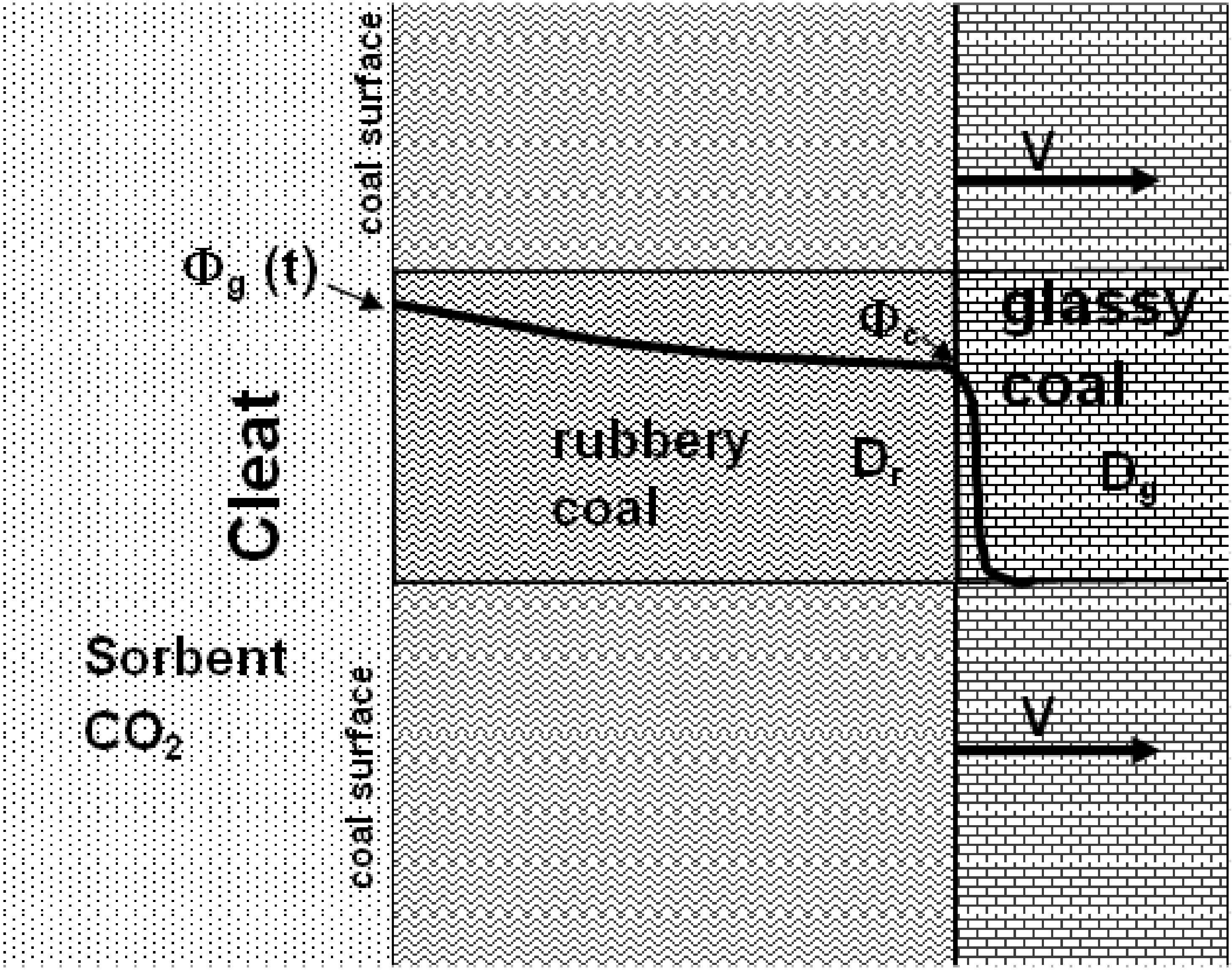
One of the promising methods to reduce the discharge of the ”greenhouse gas” carbon dioxide (CO_2) into the atmosphere is its sequestration in unminable coal seams. A typical procedure is the injection of carbon dioxide via deviated wells drilled inside the coal seams. Carbon dioxide displaces the methane adsorbed on the internal surface of the coal. A production well gathers the methane as free gas. This process, known as carbon dioxide-enhanced coal bed methane production (CO_2-ECBM), is a producer of energy and at the same time reduces greenhouse concentrations as about two carbon dioxide molecules displace one molecule of methane. World-wide application of ECBM can reduce greenhouse gas emissions by a few percent.

Intro 2

Coal has an extensive fracturing system called the cleat system.

Matrix blocks consist of polymeric structure (dehydrated cellulose), which provides the adsorption sites for the gases. At low temperatures or low sorption concentration the coal structure behaves like a rigid glassy polymer, in which movement is difficult. At high temperatures or high sorption concentrations the glassy structure is converted to the less rigid and open rubber like (swollen) structure. As coal is less dense in the rubber like state a conversion from the glassy state to the rubber like state exhibits **swelling**. Therefore modelling of diffusion is not only relevant for modelling transport into the matrix blocks, but also for the modelling of swelling, which affects the permeability of the coal seam.

Intro 3



coal surface
Φ_g (t)
Φ_c
glassy coal
Cleat
rubbery coal
D_r
D_g
V
Sorbent
CO_2
coal surface
V

Intro 4

Caption: A coal face exposed to a sorbent (CO_2) . Far to the right the virgin coal, which behaves as a glassy polymer. As the sorbent penetrates in the coal a reorientation of the polymeric coal structure occurs and the coal becomes rubber like. The diffusion coefficient in the rubber like structure is much higher $(> 1000\ \times)$ than in the glassy structure. The rubber like structure has also a lower density leading to swelling.

Thomas and Windle (1982):
the diffusion transport was enhanced by stress gradients that resulted from the accommodation of large molecules in the small cavities providing the adsorption sites.
(superdiffusion or case II diffusion).

Model 1

- volume fraction ϕ i.e., $\phi = c/\Omega$, where c is the molecular concentration and Ω is the molecular volume.
- the molar (diffusive) flux J is not only driven by the volume fraction (ϕ) (concentration) gradient , but also by the stress (P_{xx}) gradient, i.e.

$$J = -D\left(\frac{\partial \phi}{\partial x} + \frac{\Omega \phi}{kT}\frac{\partial P_{xx}}{\partial x}\right), \tag{1}$$

where k is the Boltzmann constant.

- stress (P_{xx}) is related to the volumetric flux gradient as

$$P_{xx} = -\eta_l \frac{\partial J}{\partial x} = \eta_l \frac{\partial \phi}{\partial t}, \tag{2}$$

Model 2

With η_l we denote the elongational viscosity, i.e. the resistance of movement due to a velocity gradient $\dfrac{\partial J}{\partial x}$ in the direction of flow.

The elongational viscosity η_l is supposed to depend on the volume fraction of the penetrant as

$$\eta_l = \eta_o \exp\left(-m\phi\right), \qquad (3)$$

where m is a material constant and η_0 is the volumetric viscosity of the unswollen coal sample.

(1) – (2) implies that in $Q_T = (0, L) \times (0, T)$ we have

$$\partial_t \phi = \partial_x \left\{ D\left(\phi\right) \partial_x \phi + \frac{D\left(\phi\right)\phi}{B} \partial_x \left(e^{-m\phi} \partial_t \phi \right) \right\}, \qquad (4)$$

Model 3

As initial condition we have that the concentration is

$$\phi\left(x,t=0\right)=0 \quad \text{on} \ \left(0,T\right). \tag{5}$$

The boundary condition at $x=0$ must be derived from thermodynamic arguments and it reads $\phi(0,x)=\phi_0$ with

$$t=-\phi_o\frac{\eta_0\Omega}{k_BT}\int\limits_0^{\phi/\phi_o}\frac{\exp(-m\phi_o y)}{\ln y}dy, \tag{6}$$

$B=k_BT/\left(\eta_o\Omega\right)$. Next at $x=L$

$$D\left(\phi\right)\left(\partial_x\phi+\frac{1}{B}\phi\partial_x\left(\exp\left(-m\phi\right)\partial_t\phi\right)\right)_{x=L}=0. \tag{7}$$

Model 3a

Nonlinear diffusion equations with a pseudoparabolic regularizing term being the Laplacean of the time derivative are considered in by Novick-Cohen and Pego in *Transactions of the American Mathematical Society*, 1991 and by Padron in *Comm. Partial Differential Equations*, 1998.

Global existence of a strong solution is proved by writing the problem as a linear elliptic operator, acting on the time derivative, equal to the nonlinear diffusion term. Then the linear elliptic operator, acting on the time derivative, was inverted and the standard geometric theory of nonlinear parabolic equations is applicable.

Model 4

Equations like equation (4) can occur in many transport problems in which the flux is calculated using classical irreversible thermodynamics (CIT) or extended irreversible thermodynamics (EIT). A well known example for CIT in porous media flow is that the deviation of the capillary pressure P_c from its equilibrium value at a given oil saturation S_o, i.e., $P_c^o = P_c^o(S_o)$ is a driving force leading to a rate of change of the saturation (scalar flux). This leads, as introduced by M. Hassanizadeh, to $\partial_t S_o = L\left(P_c - P_c^o\right)$, and to the transport equation for counter current imbibition

$$\begin{aligned}\varphi \partial_t S_o &= \partial_x\left(\Lambda\left(S_o\right)\partial_x P_c\right) = \\ &= \partial_x\left(\Lambda\left(S_o\right)\partial_x P_c^o\left(S_o\right)\right) + \partial_x\left(\Lambda\left(S_o\right)\partial_x \frac{1}{L\left(S_o\right)}\partial_t S_o\right).\end{aligned}$$

Model 4a

See e.g. Hassanizadeh, Gray: *Water Resources Research* 1993 and Beliaev, Hassanizadeh, *Transport in Porous Media* 2001.

This application to multiphase and unsaturated flows through porous media motivated a number of recent papers. In paper Hulshof, King, *SIAM J. Appl. Math.*, 1998, one finds a detailed study of possible travelling wave solutions and in particular of the behavior of such travelling waves near fronts where the concentration is zero. Further studies of the travelling waves are in the papers Cuesta, van Duijn, Hulshof, *European J. Appl. Math.*, 2000, and Cuesta, Hulshof *Nonlinear Analysis-Theory Methods & Applications*, 2003. The small- and waiting time behavior of the equations was studied in King, Cuesta, *SIAM J. Appl. Math.*, 2006.

Model 4b

Study of the viscosity limit for the linear relaxation model of the dynamic term is in van Duijn, Peletier, Pop, *SIAM J. Math. Anal.*, 2007.

Nevertheless, the study of existence of a solution to the nonlinear model from Hassanizadeh, Gray: *Water Resources Research* 1993 was undertaken only in papers Beliaev. *European J. Appl. Math.*, 2003, and Beliaev, Hassanizadeh, *Transport in Porous Media* 2001, where the non-degeneracy was supposed and existence is local in time.

Another existence result, also local in time, is in the paper Düll. *Comm. Partial Differential Equations*, 2006, where a related pseudoparabolic equation modeling solvent uptake in polymeric solids was studied.

Model 4c

Düll proved the short time existence of a solution with for the problem in $\mathbb{R}$, supposing non-negative compactly supported initial datum. Contrary to our approach, the problem was written as a system containing a linear elliptic equation and an evolution equation. With such approach, we did not manage to get as good estimates as with the entropy approach undertaken in this paper.

Model 5

EIT differs from CIT as it not only characterizes a system by its local thermodynamic variables (pressure, temperature, concentration) but also by its gradients. In isothermal systems and in the absence of other applied fields, e.g. electric fields, the volumetric flux J is, according to EIT, given by the following system of equations

$$\tau_1 \partial_t J + J = -D \left(\frac{\partial \phi}{\partial x} + \frac{\Omega \phi}{kT} \frac{\partial P_{xx}}{\partial x} \right), \tag{8}$$

$$\tau_2 \partial_t P_{xx} + P_{xx} = -\eta_l \frac{\partial J}{\partial x}. \tag{9}$$

Here τ_1, τ_2 are time constants, which are small with respect to L^2/D. Hence EIT or CIT can lead to transport equations of the form of equation (4).

Entropy 1

Engineering community tries to solve the problem (4), (5), (6) and (7) using direct discretization.

We will propose the **ENTROPY** approach. Motivation:

- Here we deal with a nonlinear degenerate pseudo-parabolic equation.
- Straightforward discretization leads to numerical difficulties (oscillations, blow-up . . .)
- What is the source of the difficulties? Using the solution (if there is a solution!), as a test function, does not give an energy estimate, because of the third order derivative term. For discretized problem this means that we do not control well the discretized energy. Furthermore, the problem is not explicit in time derivative and it could happen that the time stepping does not work.

Entropy 2

In the case of the linear heat equation, Kullback's entropy functional from statistical physics plays a special role. It is given by $\mathcal{E}(\varphi) = \varphi \log \varphi$, $\varphi > 0$. Our PDE allows a natural generalization of the classic Kullback entropy:

$$\mathcal{E}(\varphi) = \int_0^{\varphi} \frac{\varphi - \xi}{\xi} \Big(e^{-m\xi} \frac{1}{D(\xi)} - \frac{1}{D(0)} \Big) \, d\xi + \frac{1}{D(0)} (\varphi \log \varphi - \varphi). \tag{10}$$

Following ideas from the work of A. Mikelić and R. Robert (SIAM J. Math Anal. 1998) on the equation of Robert and Sommeria from statistical hydrodynamics, we will use $\mathcal{E}'(\varphi)$ as a test function, with the hope to obtain a convenient a priori estimate. For more information about the entropy methods, see the book to appear by L.C. Evans and the survey of entropy methods for PDEs, by the same author, in *Bulletin of the American Mathematical Society*, 2004.

Entropy 3

Formal calculation gives the equality

$$\partial_t \int_0^L \left\{ \mathcal{E}(\phi) - \varphi \mathcal{E}'(\varphi_g) + \frac{1}{2B}(e^{-m\phi}\partial_x \phi)^2 \right\} dx + \int_0^L \left(\frac{1}{\phi} e^{-m\phi} (\partial_x \phi)^2 + \phi \partial_t \mathcal{E}'(\varphi_g) \right) dx = 0. \quad (11)$$

Presence of the initial and the boundary conditions lead to unbounded non-integrable $\mathcal{E}'$. The equality (11) can not be directly used and we have to do careful regularizations.

We will use regularized $\mathcal{E}'(\varphi)$ as the unknown. Existence is proven by showing that the 'energy' of the system remains bounded during the time evolution of the system.

Existence 1

We introduce Φ_δ by

$$\Phi_\delta' := \frac{e^{-m \min\{|\xi|, 1/\delta\}}}{(|\xi| + \delta) D(\xi)}, \quad \delta > 0,\ \xi \in \mathbb{R}, \tag{12}$$

and

$$\Phi_\delta(\phi) := \begin{cases} \displaystyle\int_0^\phi \frac{e^{-m \min\{\xi, 1/\delta\}}}{(\xi+\delta) D(\xi)} d\xi, & \phi > 0 \\ \displaystyle -\int_\phi^0 \frac{e^{-m \min\{-\xi, 1/\delta\}}}{(-\xi+\delta) D(-\xi)} d\xi, & \phi < 0. \end{cases} \tag{13}$$

we study the following regularized problem in $Q_T = (0, L) \times (0, T)$:

Existence 2

$$\partial_t \phi = \partial_x \left\{ D(\phi)\, \partial_x \phi + \frac{D(\phi)\,(|\phi| + \delta)}{B} \partial_x \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi \right) \right\} \tag{14}$$

with boundary condition at $x = L$

$$D(\phi)\, \partial_x \phi + \frac{D(\phi)\,(|\phi| + \delta)}{B} \partial_x \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi \right) \Big|_{x=L} = 0 \tag{15}$$

and boundary and initial conditions (5) and (6).
We start by introducing a variational solution for the problem (14), (15), (5) and (6).

Existence 3

Definition: Let

$$\mathcal{V} := \left\{ z \in C^{\infty}[0,L],\ z|_{x=0} = 0 \right\} \text{ and } \mathcal{H} := \left\{ C^{\infty}[0,T], h(T) = 0 \right\} \quad (16)$$

Then the variational formulation corresponding to the problem (5), (6), (14) and (15) is

$$-\int_0^T \int_0^L \phi g(x)\, \partial_t h(t)\, dxdt + \int_0^T \int_0^L D(\phi)\, \partial_x \phi\ \partial_x g(x) h(t)\ dxdt$$

$$+ \int_0^T \int_0^L \frac{D(\phi)\left(|\phi| + \delta\right)}{B} \partial_x g(x)\ h(t)\ \partial_x \left(e^{-m \min\{|\phi|, 1/\delta\}} \partial_t \phi \right) dxdt = 0$$

$$\forall g \in \mathcal{V} \text{ and } \forall h \in \mathcal{H} \quad (17)$$

Existence 4

and at the boundary $x = 0$, we have

$$\phi - \phi_g = 0. \tag{18}$$

Our first goal is to prove existence for (17)-(18).

STRATEGY

Let $z := \Phi_\delta(\phi)$, $\phi = \Phi_\delta^{-1}(z)$, $z|_{x=0} = \Phi_\delta(\phi_g(t))$. We reformulate the problem (5), (6), (14) and (15) in terms of z:

$$\frac{1}{\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)}\partial_t z = \partial_x \left\{ \frac{D\left(\Phi_\delta^{-1}(z)\right)}{\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)}\partial_x z \right.$$

$$\left. + \frac{D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right)}{B}\partial_x\left(D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right)\partial_t z\right)\right\}$$

$$\text{in } Q_T \tag{19}$$

Existence 5

Moreover we can express the boundary and initial conditions in z as

$$z(0,t) = \Phi_\delta(\phi_g(t)) \text{ on } (0,T); \quad z(x,t=0) = \Phi_\delta(0) = 0 \text{ on } (0,L) \tag{20}$$

$$\frac{1}{\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)}\partial_x z + \frac{\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right)}{B}\partial_x\left(D(\Phi_\delta^{-1}(z))\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right)\partial_t z\right) = 0 \quad \text{on } x = L. \tag{21}$$

$$d_1(z) := \frac{1}{\left(\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)\right)}, d_2(z) := \frac{D\left(\Phi_\delta^{-1}(z)\right)}{\left(\Phi'_\delta\left(\Phi_\delta^{-1}(z)\right)\right)}$$
$$d(z) := D\left(\Phi_\delta^{-1}(z)\right)\left(\left|\Phi_\delta^{-1}(z)\right| + \delta\right). \tag{22}$$

Existence 6

- we introduce the discretization of the problem (19)-(21): Find

$$z_N = \sum_{j=1}^{N} c_j\,(t)\,\alpha_j\,(x) + \Phi_\delta\,(\phi_g\,(t)) \in W^{1,q}([0,T];V_N),$$

$q \in (2, +\infty)$, such that

$$\int_0^L \partial_t z_N d_1\,(z_N)\,\alpha_k\;dx + \int_0^L d_2\,(z_N)\,\partial_x z_N \partial_x \alpha_k\;dx +$$

$$\int_0^L \frac{1}{B} d\,(z_N)\,\partial_x\,(d\,(z_N)\,\partial_t z_N)\,\partial_x \alpha_k\;dx = 0, \qquad (23)$$

for $k = 1, ..., N$ and $z_N|_{t=0} == 0, \qquad (24)$

Existence 7

- Let the vector valued function $\mathbf{F}$ be given by $F_\kappa\left(t,\mathbf{c},\partial_t\mathbf{c}\right) =$ left part of equation (23) and $\mathbf{c}$ is the column vector consisting of elements $\left(c_1\left(t\right)...c_N\left(t\right)\right)$, then equations (23), (24) are equivalent to the following Cauchy Problem in $\mathbb{R}^N$:

$$\begin{cases} \mathbf{F}\left(t,\mathbf{c},\partial_t\mathbf{c}\right) = 0 \\ \mathbf{c}\big|_{t=0} = 0. \end{cases} \tag{25}$$

- we see that the problem (23)–(24) is equivalent to the Cauchy problem

Find $\mathbf{c} \in W^{1,q}(0,T)^N$ such that

Existence 7a

$$\mathcal{A}(\mathbf{c})\frac{d\mathbf{c}}{dt} = -\mathcal{B}(\mathbf{c})\mathbf{c} - \mathbf{f}(\mathbf{c}) \text{ a.e. in } (0,T); \quad \mathbf{c}|_{t=0} = 0. \qquad (26)$$

We note that $\mathcal{A}$ could loose its positive definiteness:
For $b_\alpha(x) = \mathbf{b}\cdot\alpha(x) = \sum_{j=1}^N b_j\alpha_j(x)$ we have

$$(\mathcal{A}\mathbf{b})\cdot\mathbf{b} \geq \int_0^L \left\{d_1(z_N) - \frac{1}{4B}(d'(z_N))^2(\partial_x z_N)^2\right\}(b_\alpha)^2\,dx. \quad (27)$$

Since we start from "good" initial state, we are able to prove:

Existence 8

- **Proposition 1:** There is a $T_N > 0$ such that the problem (23)–(24) has a unique solution $z_N \in W^{1,q}(0, T_N; V_N)$, for all $q < +\infty$.
- **Proposition 2:** There is a constant C, independent of N, such that

$$\|\partial_x z_N\|_{L^\infty(0,T_N;L^2(0,L))} \leq C. \tag{28}$$

  Consequently, the vector valued function $\mathbf{c}$ remains bounded at $t = T_N$.
- Nevertheless, since the matrix $\mathcal{A}$ could degenerate, some components of $\dfrac{\partial \mathbf{c}}{\partial t}$ could blow up at $t = T_N$. In other to exclude this possibility and to prove that the maximal solution for (25) exists on $[0, T]$, we need

an estimate for the time derivatives. Furthermore, if we want to pass to the limit $N \to +\infty$ in equation (23), estimate (28) is not sufficient. Our strategy is to obtain an estimate, uniform with respect to N, for $\partial_{xt} z_N$ in $L^2(Q_T)$.

Theorem 1: There exists $T_0 > 0$, independent of N, such that

$$\|\partial_x z_N\|_{L^\infty(0,T_0;L^2(0,L))} + \|\partial_t z_N\|_{L^2(0,T_0;L^2(0,L))} \leq C \qquad (29)$$

$$\|\partial_{xt} z_N\|_{L^2(0,T_0;L^2(0,L))} \leq C \qquad (30)$$

$$\left\| \partial_{xt} \int_0^{z_N} d(\xi)\, d\xi \right\|_{L^2((0,T_0)\times(0,L))} \leq C, \qquad (31)$$

with constants independent of N. Consequently, the maximal solution for (25) exists on $[0, T_0]$.

Existence 10

The estimate (31) allow us to pass to the limit $N \to +\infty$. Using classical compactness and weak compactness arguments and due to the a priori estimate (31) we can extract a subsequence of z_N, denoted by the same subscripts, which converges to an element $z \in H^1((0,T_0)\times(0,L))$, $\partial_{xt} z \in L^2((0,T_0)\times(0,L))$, in the following sense

$$z_N \to z \text{ strongly in } L^2((0,T_0)\times(0,L)) \text{ and a.e. on } (0,T_0)\times(0,L) \quad (32)$$

$$\partial_x z_N \rightharpoonup \partial_x z \text{ weakly in } L^2((0,T_0)\times(0,L)) \quad (33)$$

$$\partial_t z_N \rightharpoonup \partial_t z \text{ weakly in } L^2((0,T_0)\times(0,L)) \quad (34)$$

$$\partial_{xt} z_N \rightharpoonup \partial_{xt} z \text{ weakly in } L^2((0,T_0)\times(0,L)) \quad (35)$$

Existence 11

$$\partial_{xt}\int_0^{z_N} d(\xi)\,d\xi \rightharpoonup \partial_{xt}\int_0^{z} d(\xi)\,d\xi \text{ weakly in } L^2((0,T_0)\times(0,L)). \quad (36)$$

Now passing to the limit $N \to \infty$ in equation (23) does not pose problems and we conclude that z satisfies (19)-(21). $\Rightarrow$

Theorem 2: Let $\phi_g \in H^1(0,T)$. Then there exists $T_0 > 0$ such that problem (19)-(21) has at least one variational solution $z \in H^1((0,T_0)\times(0,L))$, $\partial_{xt} z \in L^2((0,T_0)\times(0,L))$.

Corollary 1: Let $\phi_g \in H^1(0,T)$. Then there exists $T_0 > 0$ such that the variational formulation (17)-(18) has at least one solution $\phi = \Phi_\delta^{-1}(z) \in H^1((0,T_0)\times(0,L))$, $\partial_{xt}\phi \in L^2((0,T_0)\times(0,L))$.

Global 1

Now we test (17) by $\Phi_\delta(\phi) - \Phi_\delta(\phi_g(t))$ and then by

$$e^{-m\,\min\{|\phi|,1/\delta\}}\partial_t\phi - e^{-m\,\min\{|\phi_g|,1/\delta\}}\partial_t\phi_g$$

and obtain

Theorem 3: Let $\phi_g \in H^1(0,T)$.Then for all $T>0$ there exists a weak solution $\phi \in H^1((0,T)\times(0,L))$, $\partial_{xt}\phi \in L^2((0,T)\times(0,L))$ for the variational formulation (17)-(18) of the problem (5), (6), (14) and (15).

We conclude this section by establishing uniform L^∞-bounds for ϕ. we have

Proposition 3: Let $\phi_g \in H^1(0,T)$ and $\phi_g \geq 0$. Then any weak solution ϕ of the problem (5), (6), (14) and (15), obtained in Theorem 3, satisfies $\phi(x,t) \geq 0$, a.e. on Q_T.

Global 2

Proposition 4: Let $\phi_g \in H^1(0,T)$, $\phi_g \geq 0$ and $\partial_t \phi_g \geq 0$ a.e. on $(0,T)$. Then any weak solution ϕ of the problem (5), (6), (14) and (15), obtained in Theorem 3, satisfies
$\phi_g(t) \geq \phi(x,t)$, a.e. on Q_T.

Proposition 5: Let $\phi_g \in H^1(0,T)$ and let us suppose in addition that there are constants $A_0 > 0$, $\alpha > 0$ and $C_0 > 0$ such that

$$A_0 \geq \phi_g(t) \geq C_0 t^\alpha, \quad \forall t \in [0,T]. \tag{37}$$

Then any weak solution ϕ of the problem (5), (6), (14)and (15), obtained in Theorem 3, satisfies $A_0 \geq \phi(x,t) \geq C_0 t^\alpha$, a.e. on Q_T.

Theorem 4: Let $\phi_g \in H^1(0,T)$, $A_0 = \max_{0 \le t \le T} \phi_g(t)$, $A_0 \ge \phi_g \ge C_0 t^\alpha$ and $\alpha > 1$.Then there exists a weak solution ϕ, $C_0 t^\alpha \le \phi(x,t) \le A_0$, $\partial_{xt}\phi \in L^2((0,T)\times(0,L))$, $\phi \in H^1((0,T)\times(0,L))$, for the problem (5), (6), (14), (15).

Remark 1:

- By choosing $\delta < 1/A_0$, we can replace $e^{-m \min\{|\phi|, 1/\delta\}}$ by $e^{-m\phi}$ and $|\phi| + \delta$ by $\phi + \delta$.
- In addition to the assumptions of Theorem 4 let us suppose that $\partial_t \phi_g \ge 0$. Then there exists a weak solution ϕ, $C_0 t^\alpha \le \phi(x,t) \le \phi_g(t)$, $\partial_{xt}\phi \in L^2((0,T)\times(0,L))$, $\phi \in H^1((0,T)\times(0,L))$, for the problem (5), (6), (14) and (15).

It remains to pass to the limit $\delta \to 0$. This limit will give us the solvability of the starting problem.

Theorem 5: Let $\alpha > 0$, C_0 and A_0 be positive constants and

$$\phi_g \in H^1(0,T), \quad C_0 t^\alpha \leq \phi_g \leq A_0 \text{ and } \log \phi_g \in L^2(0,T). \tag{38}$$

Then problem (4)-(7) has at least one weak solution $\phi \in H^1((0,T)\times(0,L))$, such that $\sqrt{\phi}\,\partial_x\left(e^{-m\phi}\partial_t\phi\right) \in L^2((0,T)\times(0,L))$ and $C_0 t^\alpha \leq \phi \leq A_0$.

Conclusion

Open questions and further challenges:

- Uniqueness. Long time behavior. Convergence of the fully discretized numerical schemes.
- Similar model 1: Unsaturated flows with dynamic capillary pressure.
- Similar model 2: Multiphase flows with dynamic capillary pressure.