A unilateral L^2 gradient flow and its quasi-static limit in phase-field fracture by an alternate minimizing movement

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Abstract. We consider an evolution in phase field fracture which combines, in a system of PDEs, an irreversible gradient-flow for the phase-field variable with the equilibrium equation for the displacement field. We introduce a discretization in time and define a discrete solution by means of a 1-step alternate minimization scheme, with a quadratic L^2-penalty in the phase-field variable (i.e. an alternate minimizing movement). First, we prove that discrete solutions converge to a solution of our system of PDEs. Then, we show that the vanishing viscosity limit is a quasi-static (parametrized) BV-evolution. All these solutions are described both in terms of energy balance and, equivalently, by PDEs within the natural framework of $W^{1,2}(0,T;L^2)$.

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1 Introduction

Phase field approaches are widely used to simulate crack propagation in academical and industrial applications: even within the linear-elastic setting there are plenty of phase field models, based on different choices of potentials and evolution laws, see e.g [2, 6, 10, 11, 19, 23, 25, 26, 34, 35] or the recent review [3]. In the present work, we will study in detail a couple of evolutions generated by a phase-field energy of the form

$$\mathcal{F}(t,u,v) = \int_\Omega (v^2+\eta) W(D\tilde{u}(t))\,dx + \tfrac{1}{2}\,G_c \int_\Omega (v-1)^2 + |\nabla v|^2\,dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, $\tilde{u}(t) = u + g(t)$ is the displacement fields with $u \in H^1_0(\Omega)$ (so that $\tilde{u}(t) = g(t)$ on $\partial\Omega$), $W(D\tilde{u})$ is a linear elastic energy density, $v \in H^1(\Omega;[0,1])$ is the phase field variable, $G_c > 0$ is toughness while $\eta > 0$ is a regularization parameter.

Functionals like $\mathcal{F}$ provide an elliptic regularization of free discontinuity functional: for instance, neglecting boundary conditions, for $\varepsilon \to 0^+$ and $0 < \eta_\varepsilon = o(\varepsilon)$ the Γ-limit of the functionals

$$\mathcal{F}_\varepsilon(u,v) = \int_\Omega (v^2+\eta_\varepsilon) W(Du)\,dx + G_c \int_\Omega (4\varepsilon)^{-1}(v-1)^2 + \varepsilon|\nabla v|^2\,dx$$

is of the form

$$\mathcal{F}_0(u) = \int_\Omega W(u)\,dx + G_c \mathcal{H}^1(J(u)),$$

where $J(u)$ is the set of discontinuity points of the displacement field u and $\mathcal{H}^1$ denotes Hausdorff measure. Roughly speaking, the set $J(u)$ represents the crack. Convergence has been rigorously proved in the framework of SBD^2 and $GSBD^2$ spaces respectively in [12] and [16] while in the scalar framework of the space $GSBV^2$ it was proved in the well known [5]. In our work, we will study an evolution in time, rather then a limit for $\varepsilon \to 0^+$, hence we will work with the functional $\mathcal{F}$, omitting, for simplicity of notation, the dependence on the "internal length" ε.

Our starting point is a time-discrete evolution generated by an alternate unilateral minimizing movement. For $\tau > 0$ let $t_n = n\tau \in [0,T]$ for $n \in \mathbb{N}$ be the time discretization. Then, the incremental problem is the following: given (u_{n-1}, v_{n-1}) (at time t_{n-1}) the configuration (u_n, v_n) at time t_n is obtained by solving

$$\begin{cases} v_n \in \operatorname{argmin}\{\mathcal{F}(t_n, u_n, v) + \frac{1}{2\tau}\|v - v_{n-1}\|^2 : v \le v_{n-1},\, v \in H^1\} \\ u_n \in \operatorname{argmin}\{\mathcal{F}(t_n, u, v_{n-1}) : u \in H^1_0\}. \end{cases} \tag{1}$$

Similar discrete schemes have been used in applications, e.g. by [23, 35]. Note that the updated configuration is determined by a single iteration in each variable, that each minimization problem is well posed (thank to $\eta > 0$) and that the constraint $v \le v_{n-1}$ models irreversibility. This scheme takes, of course, full advantage of the separate quadratic structure of $\mathcal{F}(t,\cdot,\cdot)$. Our first result proves that (as $\tau \to 0$) the time-discrete evolutions converge to a time-continuous evolution $t \mapsto (u(t), v(t))$, where $v(\cdot)$ is monotone non-increasing and satisfies for every $t \in [0,T]$ the energy balance

$$\begin{aligned} \mathcal{F}(t, u(t), v(t)) = {}& \mathcal{F}(0, u_0, v_0) - \tfrac{1}{2}\int_0^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r, u(r), v(r))|^2_{L^2}\, dr\, + \\ & + \int_0^t \partial_t \mathcal{F}(r, u(r), v(r))\, dr, \end{aligned} \tag{2}$$

where $|\partial_v^- \mathcal{F}(t,u,v)|_{L^2} = \sup\{-\partial_v \mathcal{F}(t,u,v)[\xi] : \xi \le 0,\, \|\xi\|_{L^2} \le 1\}$ is the unilateral slope. Note that, by irreversibility, only negative variations are allowed; for this reason a minus sign is added to the notation $|\partial_v \mathcal{F}|_{L^2}$ of the (unconstrained) slope. Equation (2) can be considered as De Giorgi's integral characterization of gradient flows; indeed, for a.e. $t \in [0,T]$ the time continuous limit solves also the following system of PDEs:

$$\begin{cases} \dot v(t) = -\big[v(t) W(D\tilde u(t)) + G_c(v(t) - 1) - G_c \Delta v(t)\big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{v(t)}(\tilde u(t))\big) = 0, \end{cases} \tag{3}$$

where $\boldsymbol{\sigma}_v(\tilde u) = (v^2 + \eta)\, \boldsymbol{\sigma}(\tilde u)$ denotes the phase-field stress. Technically, we will see that $v \in W^{1,2}(0,T; L^2(\Omega))$ and that $v(t)W(D\tilde u(t)) + G_c(v(t)-1) - G_c\Delta v(t)$ is a finite Radon measure with positive part $[\,\cdot\,]^+$ in $L^2(\Omega)$. To better understand the variational structure behind this evolution problem, note that formally the partial derivatives of $\mathcal{F}$ read

$$\begin{aligned} \partial_v \mathcal{F}(t,u,v)[\xi] &= \int_\Omega \big(v\, W(D\tilde u(t)) + G_c(v-1) - G_c \Delta v\big)\xi\, dx, \\ \partial_u \mathcal{F}(t,u,v)[\phi] &= -\int_\Omega \operatorname{div}(\boldsymbol{\sigma}_v(\tilde u(t)))\phi\, dx. \end{aligned}$$

The positive part $[\,\cdot\,]^+$ appearing in (3) comes from the irreversibility constraint; more precisely the term $-\big[v(t)W(D\tilde u(t)) + G_c(v(t)-1) - G_c\Delta v(t)\big]^+$ is (in a suitable sense) the "projection" of $-\partial_v \mathcal{F}(t, u(t), v(t))$ on the set of negative variations ξ, therefore the parabolic equation can be interpreted as a unilateral gradient flow, constrained by irreversibility.

In a larger perspective, an alternate minimizing scheme has been employed in different ways also in a dynamic visco-elastic setting [24], in another gradient flow setting [7] and in a quasi-static setting [21]. In general, alternate scheme are very useful in numerical simulation since they require only the minimization of convex (in our case, quadratic) functionals. On the other hand, different approaches can provide existence of solutions for (3) or similar problems. For instance: [22] obtains existence introducing a unilateral minimizing movement for a "reduced" (non-convex) energy, which in our setting would be of the form $\mathcal{F}(t, u_{t,v}, u)$ for $u_{t,v} \in \operatorname{argmin}\{\mathcal{F}(t,u,v) : u \in H^1_0\}$, while [9] proves existence, for a similar problem, by a fixed point argument. In the applications, "Ginzburg-Landau" models are used in phase-field fracture for instance [1, 19, 23].

In the second part of the paper we consider the vanishing viscosity limit of (3). More precisely, we start with the system

$$\begin{cases} \varepsilon \dot v^\varepsilon(t) = -\big[v^\varepsilon(t) W(D\tilde u^\varepsilon(t)) + G_c (v^\varepsilon(t) - 1) - G_c \Delta v^\varepsilon(t) \big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{v^\varepsilon(t)}(\tilde u^\varepsilon(t)) \big) = 0, \end{cases} \tag{4}$$

where $\varepsilon > 0$ is a "mobility parameter" or "viscosity". Our goal is the characterization of the quasi-static limit, obtained as $\varepsilon \to 0$. Since in the limit we expect discontinuous evolutions and since the limit is rate independent, we first parametrize the evolutions by an arc-length parameter in L^2. In this way, we get a Lipschitz map $s \mapsto (t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))$ where $w_\varepsilon(s) = u^\varepsilon \circ t_\varepsilon(s)$ and $z_\varepsilon(s) = v^\varepsilon \circ t_\varepsilon(s)$. Passing to the limit, as $\varepsilon \to 0$, we get a map $s \mapsto (t(s), w(s), z(s))$ which satisfies $w(s) \in \operatorname{argmin} \{ \mathcal{F}(t(s), w, z(s)) : w = g \circ t(s) \text{ on } \partial\Omega \}$ together with the energy balance

$$\mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2} \, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r)) \, t'(r) \, dr. \tag{5}$$

It is noteworthy that this balance together with the Lipschitz continuity of parametrized solutions imply the main properties of the quasi-static parametrized solution $s \mapsto (t(s), w(s), z(s))$. Indeed, if $t'(s) > 0$ (i.e. in continuity points) we have equilibrium, i.e.

$$\begin{cases} \big[z(s) W(D\tilde w(s)) + G_c(z(s) - 1) - G_c \Delta z(s) \big]^+ = 0 \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde w(s)) \big) = 0, \end{cases} \tag{6}$$

if $t(s)$ is constant in $(s^\flat, s^\sharp)$ (i.e. in discontinuity points) we have the following re-parametrization of (5)

$$\begin{cases} \lambda(s) z'(s) = -\big[z(s) W(D\tilde w(s)) + G_c(z(s) - 1) - G_c \Delta z(s) \big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde w(s)) \big) = 0, \end{cases} \tag{7}$$

where $\lambda(s) = \| [z(s) W(D\tilde w(s)) + G_c(z(s) - 1) - G_c \Delta z(s)]^+ \|_{L^2}$. Hence the vanishing viscosity limit is labelled "parametrized BV-evolution" [28, 32].

It is interesting to compare, at least qualitatively, the quasi-static limit obtained here with the one obtained in [21]. The latter is based on the alternate minimization scheme of [11], which reads, in the one iteration version,

$$\begin{cases} v_n \in \operatorname{argmin} \{ \mathcal{F}(t_n, u_n, v) : v \le v_{n-1}, \, v \in H^1 \} \\ u_n \in \operatorname{argmin} \{ \mathcal{F}(t_n, u, v_{n-1}) : u \in H^1_0 \}. \end{cases}$$

Note that in this scheme there is no additional viscosity. As a matter of fact, using the separate quadratic structure of the energy $\mathcal{F}(t, \cdot, \cdot)$, the above minimization problems, in u and v, are recast in [21] as the minimization of linear energies with suitable (state depending) dissipation-norms. Such dissipation-norms are called "intrinsic", as opposed to "artificial" dissipations appearing in the vanishing viscocity approach, like the L^2 norm in (1). Roughly speaking, the quasi-static limit of [21] is a parametrized BV-evolution for the energy $\mathcal{F}$ with respect to these intrinsic dissipation-norms; for the detailed characterization together with several fine properties we refer to [21].

Last, but not least, let us provide some technical considerations about our results. First of all, the proofs of (2) and (3) rely essentially on the separate convexity of the energy $\mathcal{F}(t, \cdot, \cdot)$, the lower semi-continuity of the unilateral slope and a sort of upper gradient inequality, based on a measure theoretic argument employed in [15]. All these ingredients, put together, allow us to work with evolutions of class $W^{1,2}(0, T; L^2)$, which seems to be the natural weak setting for (2) and, in perspective, for more complex systems, e.g. [1], and higher dimensional problems. We remark that our proof of existence for the unilateral gradient flow does not rely on the chain rule in $W^{1,2}(0,T;H^1)$ (see Lemma A.7). However, in order to study the quasi-static limit, and prove (5), it is necessary to have a uniform bound on the length of the curves $s \mapsto (t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))$. This is a delicate technical point, which is obtained by means of a discrete Gronwall argument, cf. [22, 33], and which gives, as a by-product, a uniform bound in $W^{1,2}(0,T;H^1)$. This bound is used, together with the chain rule, in the proof (7). Finally, we remark that our analysis works for domains Ω contained in $\mathbb{R}^2$ since it relies on Sobolev embeddings, see for instance Lemma A.6. For similar technical reasons, in the N-dimensional setting gradient flows of this type have been studied, e.g. [7] and [22], by modifying the energy $\mathcal{F}$ with some dimension depending variants; for instance $\int_\Omega |v-1|^2+|\nabla v|^2\,dx = \|v-1\|^2_{H^1}$ is replaced by $\|v-1\|^p_{W^{1,p}}$ (for $p>N$) or by $\|v-1\|^2_{L^2}+|\nabla v|^2_{H^{1/2}}$ (for $N=3$).

Contents

2 Setting, energy and its derivatives

First of all we collect the set of assumptions used through the work.

Assumptions. *We assume that Ω is an open, bounded, connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Deformations are assumed to be of the form $\tilde u = u + g(t)$ for $u \in \mathcal{U} = H^1_0(\Omega,\mathbb{R}^2)$ and $g \in C^1([0,T]; W^{1,\bar p}(\Omega,\mathbb{R}^2))$ for $\bar p > 2$. The phase-field "space" $\mathcal{V}$ is $H^1(\Omega,[0,1])$.*

The potential energy $\mathcal{F} : [0,T]\times\mathcal{U}\times\mathcal{V} \to [0,+\infty)$ is given by the following [5, 11] phase field energy for brittle fracture

$$\mathcal{F}(t,u,v) = \tfrac12 \int_\Omega (v^2+\eta) W(D\tilde u(t))\,dx + \tfrac12 G_c \int_\Omega (v-1)^2 + |\nabla v|^2\,dx, \tag{8}$$

where $\tilde u(t) = u + g(t)$ and $W(D\tilde u) = D\tilde u : \boldsymbol{C} D\tilde u = \boldsymbol{\varepsilon}(\tilde u) : \boldsymbol{\sigma}(\tilde u)$ is the linear elastic energy density, $G_c > 0$ is the toughess while $\eta > 0$ is a (small) regularization parameter. For convenience of notation, let

$$\mathcal{E}(t,u,v) = \tfrac12 \int_\Omega (v^2+\eta) W(D\tilde u(t))\,dx, \qquad \mathcal{D}(v) = \tfrac12 G_c \int_\Omega (v-1)^2 + |\nabla v|^2\,dx$$

denote respectively the elastic and the fracture (dissipated) phase-field energy.

For sake of simplicity we will assume that the initial configuration u_0, v_0 (at time $t = 0$) is in equilibrium; since the energy $\mathcal{F}(t,\cdot,\cdot)$ is separately quadratic, equilibrium is equivalent to separate minimality, i.e.

$$u_0 \in \operatorname{argmin}\{\mathcal{E}(0, v_0, \cdot) : u \in \mathcal{U}\}, \qquad v_0 \in \operatorname{argmin}\{\mathcal{F}(0, \cdot, u_0) : v \leq v_0\,, v \in \mathcal{V}\}.$$

Next, we provide the properties of energy and derivatives which will be used in the sequel.

Lemma 2.1 *If $t_n \to t$, $u_n \rightharpoonup u$ in $\mathcal{U}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$\mathcal{F}(t, u, v) \leq \liminf_{n\to+\infty} \mathcal{F}(t_n, u_n, v_n).$$

Proof. Since $v_n \rightharpoonup v$ in $\mathcal{V}$ it is clear that $\mathcal{D}(v) \leq \liminf_{n\to+\infty} \mathcal{D}(v_n)$. Thus, it is enough to show that

$$\mathcal{E}(t, u, v) \leq \liminf_{n\to+\infty} \mathcal{E}(t_n, u_n, v_n).$$

First, extract a subsequence (not relabeled) such that $\liminf_n \mathcal{E}(t_n, u_n, v_n) = \lim_n \mathcal{E}(t_n, u_n, v_n)$. Since v_n is bounded, we can extract a further subsequence (again not relabeled) such that $v_n \to v$ a.e. in Ω. By Egorov's Theorem, for every $\varepsilon \ll 1$ there exists $\Omega_\varepsilon \subset \Omega$ with $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ such that $v_n \to v$ uniformly in Ω_ε. Hence for $\delta \ll 1$ and $n \gg 1$ in Ω_ε it holds $0 \leq (v^2 + \eta) - \delta \leq (v_n^2 + \eta)$. Then

$$\tfrac{1}{2} \int_\Omega (v_n^2 + \eta) W(D\tilde{u}_n(t_n))\, dx \geq \tfrac{1}{2} \int_\Omega (v^2 + \eta - \delta) W(D\tilde{u}_n(t_n)) \chi_{\Omega_\varepsilon}\, dx.$$

Defining the density $0 \leq W_\varepsilon(x, \xi) = (v^2(x) + \eta - \delta) W(\xi) \chi_{\Omega_\varepsilon}(x)$ the weak lower semi-continuity of the right hand side (see e.g. [14, Theorem 3.4]) yields

$$\liminf_{n\to+\infty} \mathcal{E}(t_n, u_n, v_n) \geq \tfrac{1}{2} \int_\Omega (v^2 + \eta - \delta) W(D\tilde{u}(t)) \chi_{\Omega_\varepsilon}\, dx.$$

To conclude, it is sufficient to take first the supremum for $\delta \searrow 0$ and then the supremum for $\varepsilon \searrow 0$. ■

If the displacement field u is sufficiently regular (and this is the case for our evolutions) variations of energy take a simple form; more precisely, if $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ then, by Lemma A.6, the energy $\mathcal{F}(t, u, \cdot)$ is Gateaux differentiable with

$$\partial_v \mathcal{F}(t, u, v)[\xi] = \int_\Omega v\xi\, W(D\tilde{u}(t))\, dx + G_c \int_\Omega (v - 1)\xi + \nabla v \cdot \nabla \xi\, dx \quad \forall\, \xi \in H^1(\Omega). \tag{9}$$

In the evolution, irreversibility is modeled by monotonicity of v. For this reason, both the gradient flow and the BV-evolution will be defined in terms of the following unilateral L^2-slope of $\mathcal{F}(t, u, \cdot)$: if $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ let

$$|\partial_v^- \mathcal{F}(t, u, v)|_{L^2} = |\inf\{\partial_v \mathcal{F}(t, u, v)[\xi] : \xi \in H^1(\Omega),\, \xi \leq 0,\, \|\xi\|_{L^2} \leq 1\}|. \tag{10}$$

For future convenience denote $\Xi = \{\xi \in H^1(\Omega),\, \xi \leq 0,\, \|\xi\|_{L^2} \leq 1\}$, so that the unilateral slope is equivalently given by

$$|\partial_v^- \mathcal{F}(t, u, v)|_{L^2} = \sup\{-\partial_v \mathcal{F}(t, u, v)[\xi] : \xi \in \Xi\}. \tag{11}$$

Lemma 2.2 *If $t_n \to t$, $u_n \to u$ in $W^{1,p}(\Omega, \mathbb{R}^2)$ for $p > 2$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$|\partial_v^- \mathcal{F}(t, u, v)|_{L^2} \leq \liminf_{n\to+\infty} |\partial_v^- \mathcal{F}(t_n, u_n, v_n)|_{L^2}. \tag{12}$$

Proof. First, we show that for every $\xi \in \Xi$ we have

$$\lim_{n\to+\infty} \partial_v \mathcal{F}(t_n,u_n,v_n)[\xi] = \partial_v \mathcal{F}(t,u,v)[\xi]. \tag{13}$$

By weak convergence in $H^1(\Omega)$

$$\lim_{n\to+\infty} \int_\Omega (v_n-1)\xi + \nabla v_n \cdot \nabla \xi \, dx = \int_\Omega (v-1)\xi + \nabla v \cdot \nabla \xi \, dx,$$

while

$$\lim_{n\to+\infty} \int_\Omega v_n \xi \, W(D\tilde{u}_n(t_n))\, dx = \int_\Omega v\xi \, W(D\tilde{u}(t))\, dx$$

because $v_n \to v$ in $L^q(\Omega)$ for every $q<\infty$ (by compact embedding) while

$$D\tilde{u}(t_n) = Du_n + Dg(t_n) \to D\tilde{u}(t) = Du + Dg(t) \text{ in } L^r(\Omega,\mathbb{R}^{2\times 2}) \text{ for } r = p \wedge \bar{p}$$

and thus $W(D\tilde{u}_n(t_n)) \to W(D\tilde{u}(t))$ in $L^{r/2}$ for $r/2>1$.

By (11) for every $\xi \in \Xi$

$$|\partial_v^- \mathcal{F}(t_n,u_n,v_n)|_{L^2} \ge -\partial_v \mathcal{F}(t_n,u_n,v_n)[\xi]$$

and hence by (13) for every $\xi \in \Xi$ we get

$$\liminf_{n\to+\infty} |\partial_v^- \mathcal{F}(t_n,u_n,v_n)|_{L^2} \ge -\lim_{n\to+\infty} \partial_v \mathcal{F}(t_n,u_n,v_n)[\xi] = -\partial_v \mathcal{F}(t,u,v)[\xi].$$

Taking the supremum with respect to $\xi \in \Xi$ concludes the proof. ■

By the regularity in time of g the partial time derivative takes the form

$$\partial_t \mathcal{F}(t,u,v) = \int_\Omega (v^2+\eta)\boldsymbol{\varepsilon}(\tilde{u}(t)) : \boldsymbol{\sigma}(\dot{g}(t))\, dx. \tag{14}$$

In particular, for $v \in \mathcal{V}$ by continuity and coercivity of the elastic energy we have

$$|\partial_t \mathcal{F}(t,u,v)| \le C\|\boldsymbol{\varepsilon}(\tilde{u}(t))\|_{L^2} \le C'\left(\|\boldsymbol{\varepsilon}(\tilde{u}(t))\|^2_{L^2}+1\right) \le C''(\mathcal{F}(t,u,v)+1). \tag{15}$$

Lemma 2.3 *If $t_n \to t$, $u_n \rightharpoonup u$ in $\mathcal{U}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$\lim_{n\to+\infty} \partial_t \mathcal{F}(t_n,u_n,v_n) = \partial_t \mathcal{F}(t,u,v). \tag{16}$$

Proof. It is sufficient to pass to the limit in (14) using the fact that $D\dot{g}(t_n)(v_n^2+\eta) \to D\dot{g}(t)(v^2+\eta)$ in $L^2(\Omega,\mathbb{R}^{2\times 2})$. ■

Finally, the energy $\mathcal{F}(t,\cdot,v)$ is Fréchet differentiable in $H^1_0(\Omega,\mathbb{R}^2)$, with respect to the natural norm of $H^1_0(\Omega,\mathbb{R}^2)$, and for every $\zeta \in H^1_0(\Omega;\mathbb{R}^2)$ we have

$$\partial_u \mathcal{F}(t,u,v)[\zeta] = \int_\Omega (v^2+\eta)\boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(\zeta)\, dx = \langle -\mathrm{div}\big((v^2+\eta)\boldsymbol{\sigma}(u)\big), \zeta\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^1_0(\Omega;\mathbb{R}^2)$ and $H^{-1}(\Omega;\mathbb{R}^2)$.

3 Gradient flows

3.1 Incremental problems

Our gradient flow will be defined as an "alternate minimizing movement", i.e. as the limit of an alternate implicit Euler discretization.

Fix $\tau_m = T/m > 0$ (for some $m \in \mathbb{N}$ with $m > 0$) and for $k = 0, ..., m$ consider the discrete times $t_{m,k} = k\tau_m \in [0, T]$. Given $u_{m,k-1} = u(t_{m,k-1})$ and $v_{m,k-1} = v(t_{m,k-1})$ the irreversible alternate minimizing movement is defined by

$$\begin{cases} v_{m,k} \in \operatorname{argmin} \{\mathcal{F}(t_{m,k}, u_{m,k-1}, v) + \frac{1}{2\tau_m} \|v - v_{m,k-1}\|^2_{L^2} : v \le v_{m,k-1}\,, v \in \mathcal{V}\} \\ u_{m,k} \in \operatorname{argmin} \{\mathcal{F}(t_{m,k}, u, v_{m,k}) : u \in \mathcal{U}\} = \operatorname{argmin} \{\mathcal{E}(t_{m,k}, u, v_{m,k}) : u \in \mathcal{U}\}. \end{cases} \tag{17}$$

By Lemma A.5 we get the regularity and the continuous dependence of $u_{m,k}$ stated in the next Lemma.

Lemma 3.1 *There exists $2 < \tilde{p} < \bar{p}$ and $p \in (2, \bar{p})$, independent of τ_m and k, such that $u_{m,k} \in W^{1,p}(\Omega, \mathbb{R}^2)$. Moreover, there exists $C > 0$, independent of τ_m and k, such that*

$$\|u_{m,k} - u_{m,k-1}\|_{W^{1,p}} \le C\big(|t_{m,k} - t_{m,k-1}| + \|v_{m,k} - v_{m,k-1}\|_{L^q}\big) \tag{18}$$

for $1/q = 1/p - 1/\tilde{p}$. Finally, $u_{m,k}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^2)$ uniformly with respect to τ_m and k.

The next two lemmas provide the main ingredients in the proof of the convergence Theorem 3.6.

Lemma 3.2 *For every $k \ge 1$ let $\dot{v}_{m,k} = (v_{m,k} - v_{m,k-1})/(t_{m,k} - t_{m,k-1})$, then*

$$\langle \dot{v}_{m,k}, \xi \rangle_{L^2} + \partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[\xi] \ge 0 \quad \text{for every } \xi \in \Xi, \tag{19}$$

$$\|\dot{v}_{m,k}\|^2_{L^2} = |\partial^-_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2} \, \|\dot{v}_{m,k}\|_{L^2} = -\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[\dot{v}_{m,k}]. \tag{20}$$

Proof. First of all, by a standard truncation argument we know that we can replace $\mathcal{V}$ with the whole $H^1(\Omega)$ in (17), hence

$$v_{m,k} \in \operatorname{argmin} \{\mathcal{F}(t_{m,k}, u_{m,k-1}, v) + \tfrac{1}{2\tau_m} \|v - v_{m,k-1}\|^2_{L^2} : v \le v_{m,k-1}\,, v \in H^1(\Omega)\}.$$

By minimality, $v_{m,k}$ solves the variational inequality

$$\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[w - v_{m,k}] + \langle \dot{v}_{m,k}, w - v_{m,k} \rangle_{L^2} \ge 0 \tag{21}$$

for every $w \in H^1(\Omega)$ with $w \le v_{m,k-1}$. Inequality (19) follows. Choosing $w - v_{m,k} = \pm\tau_m \dot{v}_{m,k}$ (corresponding to $w = v_{m,k-1}$ and $w = 2v_{m,k} - v_{m,k-1}$) provides

$$\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[\dot{v}_{m,k}] + \|\dot{v}_{m,k}\|^2_{L^2} = 0. \tag{22}$$

Next, by (11) and (21) with $w - v_{m,k} = \xi \in \Xi$ we get

$$\begin{aligned} |\partial^-_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2} &= \sup\{-\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[\xi] : \xi \in \Xi\} \\ &\le \sup\{\langle \dot{v}_{m,k}, \xi \rangle_{L^2} : \xi \in \Xi\} = \|\dot{v}_{m,k}\|_{L^2}. \end{aligned} \tag{23}$$

If $\dot{v}_{m,k} = 0$ there is nothing else to prove. Otherwise, $\xi = \dot{v}_{m,k}/\|\dot{v}_{m,k}\|_{L^2}$ is an admissible variation and by (22) the inequality in (23) becomes an equality, thus $|\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2} = \|\dot{v}_{m,k}\|_{L^2}$. ■

Lemma 3.3 *For every $k \geq 1$ it holds the following energy estimate*

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + \int_{t_{m,k-1}}^{t_{m,k}} \partial_t \mathcal{F}(t, u_{m,k-1}, v_{m,k-1})\, dt + \\ - \tfrac{1}{2} \int_{t_{m,k-1}}^{t_{m,k}} \|\dot v_{m,k}\|_{L^2}^2 + |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2}^2 \, dt. \tag{24}$$

Proof. By minimality of $u_{m,k}$ and convexity of $\mathcal{F}(t_{m,k}, u_{m,k-1}, \cdot)$ we get

$$\begin{aligned} \mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) &\leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k}) \\ &\leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k-1}) - \partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[v_{m,k-1} - v_{m,k}]. \end{aligned} \tag{25}$$

By Lemma 3.2

$$\partial_v \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})[v_{m,k-1} - v_{m,k}] = \tau_m \|\dot v_{m,k}\|_{L^2}^2 = \tau_m \tfrac{1}{2} \big(\|\dot v_{m,k}\|_{L^2}^2 + |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2}^2 \big)$$

and hence by (25)

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k-1}) - \tfrac{1}{2} \int_{t_{m,k-1}}^{t_{m,k}} \|\dot v_{m,k}\|_{L^2}^2 + |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|_{L^2}^2 \, dt.$$

Finally,

$$\mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k-1}) = \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + \int_{t_{m,k-1}}^{t_{m,k}} \partial_t \mathcal{F}(t, u_{m,k-1}, v_{m,k-1})\, dt$$

and the proof is concluded. ■

Lemma 3.4 *There exists $C > 0$, independent of τ_m and k, such that*

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq C(\mathcal{F}(t_0, u_0, v_0) + 1).$$

Proof. By minimality of $u_{m,k}$ and $v_{m,k}$

$$\begin{aligned} \mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) &\leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k}) \\ &\leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k}) + \tfrac{1}{2\tau_m} \|v_{m,k} - v_{m,k-1}\|_{L^2}^2 \leq \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k-1}). \end{aligned}$$

Further,

$$\mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k-1}) = \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + \int_{t_{m,k-1}}^{t_{m,k}} \partial_t \mathcal{F}(t, u_{m,k-1}, v_{m,k-1})\, dt$$

and

$$\partial_t \mathcal{F}(t, u_{m,k-1}, v_{m,k-1}) = \int_\Omega (v_{m,k-1}^2 + \eta) \boldsymbol{\varepsilon}(u_{m,k-1} + g(t)) : \boldsymbol{\sigma}(\dot g(t))\, dx.$$

Clearly $\|v_{m,k-1}^2 + 1\|_{L^\infty} \leq (1 + \eta)$ and $\|\boldsymbol{\sigma}(\dot g)\|_{L^2} \leq C$. Moreover, by (15) and by the Lipschitz continuity of $g(\cdot)$ we have

$$\begin{aligned} \|\boldsymbol{\varepsilon}(u_{m,k-1} + g(t))\|_{L^2} &\leq \|\boldsymbol{\varepsilon}(u_{m,k-1} + g_{m,k-1})\|_{L^2} + \|\boldsymbol{\varepsilon}(g(t) - g_{m,k-1})\|_{L^2} \\ &\leq C(\mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + 1). \end{aligned}$$

In summary, we can write

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + C\tau_m(\mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + 1)$$

and then

$$(\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) + 1) \leq (1 + C\tau_m)(\mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) + 1).$$

It follows that $\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq (1 + C\tau_m)^k (\mathcal{F}(t_0, u_0, v_0) + 1)$ and then, since $\tau_m \leq T/k$,

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \leq (1 + CT/k)^k (\mathcal{F}(t_0, u_0, v_0) + 1),$$

since $(1 + CT/k)^k \to e^{CT}$ the required estimate follows. ■

3.2 Compactness and convergence

Let us denote by $u_m : [0,T] \to \mathcal{U}$ and $v_m : [0,T] \to \mathcal{V}$ the evolutions obtained by piecewise affine interpolation of $u_{m,k} = u_m(t_{m,k})$ and $v_{m,k} = v_m(t_{m,k})$.

Lemma 3.5 *The sequence v_m is bounded in $L^\infty(0,T;H^1(\Omega))$ and in $H^1(0,T;L^2(\Omega))$ and thus, upon extracting a (non-relabeled) subsequence, $v_m \overset{*}{\rightharpoonup} v$ in $L^\infty(0,T;H^1(\Omega))$ and $v_m \rightharpoonup v$ in $H^1(0,T;L^2(\Omega))$.*

As a consequence, if $t_m \to t$ then $v_m(t_m) \rightharpoonup v(t)$ in $H^1(\Omega)$ and $u_m(t_m) \to u(t)$ in $W^{1,p}(\Omega,\mathbb{R}^2)$, for some $p > 2$, where $u(t) \in \operatorname{argmin} \{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$.

Proof. From Lemma 3.4 we known that $\mathcal{F}(t_{m,k}, u_{m,k}, v_{m.k})$ is uniformly bounded and thus v_m is bounded in $L^\infty(0,T;H^1(\Omega))$ while u_m is bounded in $L^\infty(0,T;H^1_0(\Omega,\mathbb{R}^2))$ by Korn's inequality. Then from the energy balance (24), together with identity (20), and from the uniform bound on the time derivative (15) we get

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \le \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) - \int_{t_{m,k-1}}^{t_{m,k}} \|\dot v_{m,k}\|^2_{L^2}\, dt + C\tau_m.$$

By induction we get

$$\int_0^T \|\dot v_{m,k}\|^2_{L^2}\, dt \le \mathcal{F}(0,u_0,v_0) + CT$$

and thus v_m is bounded in $H^1(0,T;L^2(\Omega))$. As a consequence, (up to subsequences) $v_m \rightharpoonup v$ in $H^1(0,T;L^2(\Omega))$ and $v_m(t_m) \rightharpoonup v(t)$ in $L^2(\Omega)$ for $t_m \to t$; since $v_m(t_m)$ is bounded in $H^1(\Omega)$ it turns out that $v_m(t_m) \rightharpoonup v(t)$ in $H^1(\Omega)$.

Let k_m such that $t_{k_m} \le t < t_{k_m+1}$. Being $u_m(t_{m,k_m}) = u_{m,k_m} \in \operatorname{argmin} \{\mathcal{E}(t_{m,k_m}, u, v_{m,k_m}) : u \in \mathcal{U}\}$ we have

$$\int_\Omega (v^2_{m,k_m} + 1) D\tilde u_{m,k_n} : \mathbf{C} D\phi \, dx = 0 \qquad \forall \phi \in \mathcal{U}.$$

Since u_{m,k_m} is bounded in $H^1_0(\Omega,\mathbb{R}^2)$ there exists a subsequence (non-relabeled) weakly converging to some $u_\infty \in H^1_0(\Omega,\mathbb{R}^2)$. Clearly

$$D\tilde u_{m,k_n} = Du_{m,k_m} + Dg(t_{m,k_m}) \rightharpoonup Du_\infty + Dg(t) = D\tilde u(t) \quad \text{and} \quad (v^2_{m,k_m}+1)D\phi \to (v^2(t)+1)D\phi$$

in $L^2(\Omega,\mathbb{R}^{2\times 2})$, thus

$$\int_\Omega (v^2(t)+1) D\tilde u_\infty(t) : \mathbf{C} D\phi \, dx = 0 \qquad \forall \phi \in \mathcal{U}$$

and $u_\infty = u(t) \in \operatorname{argmin} \{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$. Since the limit is uniquely determined the whole sequence converges. We can argue exactly in the same way for $u_m(t_{m,k_m+1})$.

By compact embedding $v_{m,k_m} \to v(t)$ in $L^q(\Omega)$ for every $q < +\infty$. Then, invoking Lemma A.5 we have

$$\|u(t) - u_{m,k_m}\|_{W^{1,p}} \le C\|g(t) - g(t_{m,k_m})\|_{L^q} + C\|v(t) - v_{m,k_m}\|_{L^q}$$

and similarly for u_{m,k_m+1}. As $u_m(t_m)$ is the affine interpolation of u_{m,k_m} and u_{m,k_m+1}, the strong convergence of displacements in $W^{1,p}(\Omega,\mathbb{R}^2)$ follows. ■

Theorem 3.6 *Let v be a limit of v_m (as in Lemma 3.5) and let u be the corresponding limit of u_m; then $v \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;\mathcal{V})$ and for every $t \in [0,T]$ it holds $u(t) \in \operatorname{argmin} = \{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$ and*

$$\mathcal{F}(t,u(t),v(t)) = \mathcal{F}(0,u_0,v_0) - \tfrac{1}{2}\int_0^t \|\dot v(r)\|^2_{L^2} + |\partial^-_v \mathcal{F}(r,u(r),v(r))|^2_{L^2}\, dr + \\ + \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\, dr. \tag{26}$$

Moreover for almost every $t \in [0,T]$ we have

$$\|\dot v(t)\|_{L^2} = |\partial^-_v \mathcal{F}(t,u(t),v(t))|_{L^2}. \tag{27}$$

For sake clarity the proof of the previous Theorem will be split into a couple of Propositions.

Proposition 3.7 *Under the hypotheses of Theorem 3.6 for every $t \in [0,T]$ it holds*

$$\mathcal{F}(t,u(t),v(t)) \le \mathcal{F}(0,u_0,v_0) - \tfrac12 \int_0^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\, dr + \\ + \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\, dr. \tag{28}$$

Proof. Given $t \in [0,T]$ let $1 \le k_m \le m$ such that $t_{m,k_m} \to t$. Then, by induction (24) provides

$$\mathcal{F}(t_{m,k_m}, u(t_{m,k_m}), v(t_{m,k_m})) + \tfrac12 \int_0^{t_{m,k_m}} \|\dot v_m(r)\|^2_{L^2}\, dr + \\ + \tfrac12 \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|^2_{L^2}\, dr \le \\ \le \mathcal{F}(0,u_0,v_0) + \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} \partial_t \mathcal{F}(r, u_{m,k-1}, v_{m,k-1})\, dr. \tag{29}$$

By assumption $t_{m,k_m} \to t$, then by Lemma 3.5 $v_m(t_{m,k_m}) \rightharpoonup v(t)$ in $H^1(\Omega)$ and $u_m(t_{m,k_m}) \to u(t)$ in $W^{1,p}(\Omega,\mathbb{R}^2)$. Then by Lemma 2.1 we get

$$\mathcal{F}(t,u(t),v(t)) \le \liminf_{m\to\infty} \mathcal{F}(t_{m,k_m}, u(t_{m,k_m}), v(t_{m,k_m})).$$

Since $v_m \rightharpoonup v$ in $H^1(0,T;L^2(\Omega))$ we have

$$\int_0^t \|\dot v(r)\|^2_{L^2}\, dr \le \liminf_{m\to\infty} \int_0^{t_{m,k_m}} \|\dot v_m(r)\|^2_{L^2}\, dr.$$

Next, given $r \in (0,t)$ let $r \in [t_{m,k'_m}, t_{m,k'_m+1})$ for $k'_m \le k_m$. Clearly, both $t_{m,k'_m} \to r$ and $t_{m,k'_m-1} \to r$. By Lemma 3.5 we know that $u_m(t_{m,k'_m-1}) \to u(r)$ strongly in $W^{1,p}(\Omega,\mathbb{R}^2)$ (for some $p > 2$) while $v_m(t_{m,k'_m}) \rightharpoonup v(r)$ in $H^1(\Omega)$ and then by Lemma 2.2 we get the pointwise estimate

$$|\partial_v^- \mathcal{F}(r,u(r),v(r))|_{L^2} \le \liminf_{m\to+\infty} |\partial_v^- \mathcal{F}(t_{m,k'_m}, u_m(t_{m,k'_m-1}), v_m(t_{m,k'_m-1}))|_{L^2}.$$

We remark that $|\partial_v^- \mathcal{F}(\cdot,u(\cdot),v(\cdot))|_{L^2}$ is measurable in $(0,T)$. Indeed, given $\xi \in \Xi$, the functional $\partial_v \mathcal{F}(t,u,v)[\xi]$ is continuous in $[0,T] \times W^{1,p}(\Omega,\mathbb{R}^2) \times H^1(\Omega)$ and thus $t \mapsto \partial_v \mathcal{F}(t,u(t),v(t))[\xi]$ is measurable. To conclude, it is enough to employ (11) and write $|\partial_v^- \mathcal{F}(\cdot,u(\cdot),v(\cdot))|_{L^2}$ as the (pointwise) supremum of $-\partial_v \mathcal{F}(\cdot,u(\cdot),v(\cdot))[\xi_j]$ for ξ_j in a dense, countable subset of Ξ. So, by Fatou's Lemma we conclude that

$$\int_0^t |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\, dr \le \liminf_{m\to+\infty} \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|^2_{L^2}\, dr.$$

By Lemma 2.3 and (15) we get, by dominated convergence,

$$\limsup_{m\to+\infty} \sum_{k=0}^{k_m-1} \int_{t_{m,k-1}}^{t_{m,k}} \partial_t \mathcal{F}(r, u_{m,k-1}, v_{m,k-1})\, dr \le \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\, dr.$$

Taking respectively the liminf on the left hand side and the limsup on the right hand side of (29) we get the energy inequality

$$\mathcal{F}(t,u(t),v(t)) + \tfrac12 \int_0^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\, dr \le \\ \le \mathcal{F}(0,u_0,v_0) + \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\, dr,$$

which conclude the proof. ■

There are different ways to prove the "upper gradient inequality"

$$\begin{aligned}\mathcal{F}(0,u_0,v_0)-\mathcal{F}(t,u(t),v(t)) &\le \int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|_{L^2}\ \|\dot v(r)\|_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr \\ &\le \tfrac12\int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|^2_{L^2}+\|\dot v(r)\|^2_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr.\end{aligned}$$

For instance, the estimate will follow from the chain rule Lemma A.7 once we will know (from Theorem 5.1) that the limit evolution v belongs to $H^1(0,T;H^1(\Omega))$. Actually, Proposition 3.8 below provides the required inequality for v in $H^1(0,T;L^2(\Omega))$; its proof, based only on measure theory and separate convexity, is inspired by [15, Theorem 4.12]. In some sense (30) corresponds to [4, Corollary 2.4.10].

Proposition 3.8 *Let $v\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;\mathcal{V})$ and $u(t)\in \operatorname{argmin}\{\mathcal{F}(t,v(t),u): u\in\mathcal{U}\}$ such that $t\mapsto|\partial_v^-\mathcal{F}(t,u(t),v(t))|$ belongs to $L^2(0,T)$. Then for every $t\in(0,T)$*

$$\begin{aligned}\mathcal{F}(0,u_0,v_0)-\mathcal{F}(t,u(t),v(t)) &\le \int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|_{L^2}\ \|\dot v(r)\|_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr. \end{aligned}\tag{30}$$

Proof. Step I. Since the slope belongs to $L^2(0,t)$ there exists a sequence of finite subdivisions $t_{j,i}$ (for $j\in\mathbb{N}$ and $i=0,...,I_j$) of the time interval $[0,t]$ with $0=t_{j,0}<...<t_{j,i}<t_{j,i+1}<...<t_{j,I_j}=t$, with $\lim_{j\to+\infty}\max_i\{t_{j,i+1}-t_{j,i}\}=0$ and such that the piecewise constant functions

$$F_j(r)=\sum_{i=0}^{I_j-1}\chi_{(t_{j,i},\,t_{j,i+1})}(r)\,|\partial_v^-\mathcal{F}(t_{j,i},u(t_{j,i}),v(t_{j,i}))|_{L^2}$$

converge to $|\partial_v^-\mathcal{F}(\cdot,u(\cdot),v(\cdot))|_{L^2}$ strongly in $L^2(0,t)$ (cf. Theorem 4.12 in [15] or [18]).

Denote for simplicity $u_{j,i}=u(t_{j,i})$ and $\chi_{j,i}=\chi_{(t_{j,i},t_{j,i+1})}$ etc. For each $j\in\mathbb{N}$ and $i=0,...,I_j$ write

$$\begin{aligned}\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i+1},u_{j,i+1},v_{j,i+1}) =&\ \mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})\ + \\ &+\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})\ + \\ &+\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i+1},v_{j,i+1}).\end{aligned}$$

We will consider the three lines above separately, starting with the first. By convexity of $\mathcal{F}(t_{j,i},u_{j,i},\cdot)$ we get

$$\begin{aligned}\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1}) &\le -\partial_v\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})[v_{j,i+1}-v_{j,i}] \\ &\le |\partial_v^-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \|v_{j,i+1}-v_{j,i}\|_{L^2} \\ &\le \int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \ \|\dot v_{j,i+1}\|_{L^2}\ dr\end{aligned}$$

where $\dot v_{j,i}=(v_{j,i+1}-v_{j,i})/(t_{j,i+1}-t_{j,i})$ denotes the "discrete" velocity. For the second term we will just write

$$\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})=-\int_{t_{j,i}}^{t_{j,i+1}}\partial_t\mathcal{F}(r,u_{j,i},v_{j,i+1})\,dr.$$

For the third term, remember that by minimality

$$\int_\Omega (v_{j,i+1}^2 + \eta)\, \boldsymbol{\sigma}(u_{j,i+1} + g_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i} - u_{j,i+1})\, dx = 0$$

and that $\mathcal{F}(t_{j,i+1}, \cdot, v_{j,i+1})$ is quadratic, then

$$\begin{aligned}
\mathcal{F}(t_{j,i+1}, u_{j,i}, v_{j,i+1}) - \mathcal{F}(t_{j,i+1}, u_{j,i+1}, v_{j,i+1}) = & \\
&= \tfrac{1}{2} \int_\Omega (v_{j,i+1}^2 + \eta)\big(W(Du_{j,i} + Dg_{j,i+1}) - W(Du_{j,i+1} + Dg_{j,i+1})\big)\, dx \\
&= \tfrac{1}{2} \int_\Omega (v_{j,i+1}^2 + \eta)\, \boldsymbol{\sigma}(u_{j,i} + u_{j,i+1} + 2g_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i} - u_{j,i+1})\, dx \\
&= \tfrac{1}{2} \int_\Omega (v_{j,i+1}^2 + \eta)\, \boldsymbol{\sigma}(u_{j,i} - u_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i} - u_{j,i+1})\, dx \\
&\le C \|u_{j,i+1} - u_{j,i}\|_{H^1}^2 = C \fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1} - u_{j,i}\|_{H^1}^2\, dr.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\mathcal{F}(t_{j,i}, u_{j,i}, v_{j,i}) - \mathcal{F}(t_{j,i+1},\, u_{j,i+1}, v_{j,i+1}) \le & \\
\le \int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^- \mathcal{F}(t_{j,i}, u_{j,i}, v_{j,i})|_{L^2}\ \|\dot v_{j,i+1}\|_{L^2}\ dr\ &- \int_{t_{j,i}}^{t_{j,i+1}} \partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1})\, dr + \\
+ C \fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1} - u_{j,i}\|_{H^1}^2\, dr. &
\end{aligned}$$

Taking the sum for $i = 0, ..., I_j$ yields

$$\begin{aligned}
\mathcal{F}(0, u_0, v_0) - \mathcal{F}(t, u(t), v(t)) \le \sum_{i=0}^{I_j - 1} \int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^- \mathcal{F}(t_{j,i}, u_{j,i}, v_{j,i})|_{L^2}\ \|\dot v_{j,i+1}\|_{L^2}\ dr + & \\
- \sum_{i=0}^{I_j-1} \int_{t_{j,i}}^{t_{j,i+1}} \partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1})\, dr\ + \sum_{i=0}^{I_j-1} \fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1} - u_{j,i}\|_{H^1}^2\, dr. & \qquad (31)
\end{aligned}$$

Step II. Let us re-write (31) as

$$\mathcal{F}(0, u_0, v_0) - \mathcal{F}(t, u(t), v(t)) \le \int_0^t F_j(r) V_j(r) - P_j(r) + E_j(r)\, dr$$

in terms of the piecewise constant functions F_j (defined above) and

$$V_j(r) = \sum_{i=0}^{I_j-1} \chi_{j,i}(r)\ \|\dot v_{j,i+1}\|_{L^2}, \qquad P_j(r) = \sum_{i=0}^{I_j-1} \chi_{j,i}(r)\, \partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1}),$$

$$E_j(r) = \sum_{i=0}^{I_j-1} \chi_{j,i}(r)\, |t_{j,i+1} - t_{j,i}|^{-1} \|u_{j,i+1} - u_{j,i}\|_{H^1}^2.$$

Since the above estimate holds for every subdivision $t_{j,i}$ it must hold also

$$\mathcal{F}(0, u_0, v_0) - \mathcal{F}(t, u(t), v(t)) \le \lim_{j \to +\infty} \int_0^t F_j(r) V_j(r) - P_j(r) + E_j(r)\, dr.$$

We will show that

$$\lim_j \int_0^t F_j(r)V_j(r)dr = \int_0^t |\partial_v^- \mathcal{F}(r,u(r),v(r))|_{L^2} \, \|\dot v(r)\|_{L^2}\,dr, \tag{32}$$

$$\lim_j \int_0^t P_j(r)\,dr = \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\,dr, \tag{33}$$

$$\lim_j \int_0^t E_j(r)\,dr = 0, \tag{34}$$

which will prove (30).

Since F_j converge strongly in $L^2(0,t)$ (by construction) to prove (32) it is enough to see that

$$V_j = \sum_{i=0}^{I_j-1} \chi_{j,i}\, \|\dot v_{j,i+1}\|_{L^2} \rightharpoonup \|\dot v\|_{L^2} \quad \text{weakly in } L^2(0,t).$$

Note that $V_j \to \|\dot v\|$ a.e. in $[0,t]$ since $v \in W^{1,2}(0,t;L^2)$. Thus, it is enough to check that V_j is bounded in $L^2(0,t)$. Write,

$$\|\dot v_{j,i+1}\|_{L^2}^2 = \left\| \frac{v_{j,i+1}-v_{j,i}}{t_{j,i+1}-t_{j,i}} \right\|_{L^2}^2 = \left\| \fint_{t_{j,i}}^{t_{j,i+1}} \dot v(r)\,dr \right\|_{L^2}^2 \le \fint_{t_{j,i}}^{t_{j,i+1}} \|\dot v(r)\|_{L^2}^2\,dr \tag{35}$$

so that

$$\int_0^t V_j^2(r)\,dr \le \sum_{i=0}^{I_j-1} (t_{j,i+1}-t_{j,i}) \fint_{t_{j,i}}^{t_{j,i+1}} \|\dot v(r)\|_{L^2}^2\,dr = \int_0^t \|\dot v(r)\|_{L^2}^2\,dr.$$

Let us prove (33). Fix $r \in (0,t)$ and let $t_{j,i} \le r < t_{j,i+1}$ (with i depending on j). For a.e. $r \in (0,t)$ we have

$$\partial_t \mathcal{F}(r,u_{j,i},v_{j,i+1}) = \int_\Omega (v_{j,i+1}^2+\eta)\boldsymbol{\sigma}(u_{j,i}+g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\,dr.$$

Remember that $v \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. Using the arguments of Lemma 3.5 we get $v_{j,i+1} = v(t_{j,i+1}) \to v(r)$ and $v_{j,i} = v(t_{j,i}) \to v(r)$ in $L^q(\Omega)$ for every $q < \infty$. Thus $u_{j,i} = u(t_{j,i}) \to u(r)$ in $W^{1,p}(\Omega,\mathbb{R}^2)$ for $p>2$ (by Lemma A.5). As a consequence

$$\int_\Omega (v_{j,i+1}^2+\eta)\boldsymbol{\sigma}(u_{j,i}+g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\,dr \;\to\; \int_\Omega (v^2(r)+\eta)\;\boldsymbol{\sigma}(u(r)+g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\,dr.$$

Therefore $\partial_t \mathcal{F}(r,u_{j,i},v_{j,i+1}) \to \partial_t \mathcal{F}(r,u(r),v(r))$ a.e. in $(0,t)$. Since $v \in L^\infty(0,t;H^1)$ by (15) we get that $|\partial_t \mathcal{F}(r,u_{j,i},v_{j,i+1})|$ is uniformly bounded and thus (33) follows by dominated convergence.

Finally, let us prove (34). Since $u_{j,i} \in \operatorname{argmin}\{\mathcal{E}(t_{j,i},\cdot,v_{j,i})\}$ by Lemma 3.1 we known that

$$\|u_{j,i+1}-u_{j,i}\|_{H^1}^2 \le C|t_{j,i+1}-t_{j,i}|^2 + C\|v_{j,i+1}-v_{j,i}\|_{L^q}^2,$$

for some q sufficiently large. Since $v_{j,i} \in L^p(\Omega)$ for every $p<\infty$ (by Sobolev embedding) we can apply the interpolation inequality

$$\|v_{j,i+1}-v_{j,i}\|_{L^q} \le \|v_{j,i+1}-v_{j,i}\|_{L^2}^\alpha \; \|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}}^{1-\alpha}$$

with $1/q = \alpha/2 + (1-\alpha)/\bar q$ (for a suitable $\bar q$ depending on α). Hence, for $\alpha = 1/2$ we get

$$|t_{j,i+1}-t_{j,i}|^{-1}\|v_{j,i+1}-v_{j,i}\|_{L^q}^2 \le \|\dot v_{j,i+1}\|_{L^2}\, \|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}}.$$

Then,

$$\begin{aligned}\int_0^t E_j(r)\,dr &\le \sum_{i=0}^{I_j-1}\int_{t_{j,i}}^{t_{j,i+1}} |t_{j,i+1}-t_{j,i}|^{-1}\|u_{j,i+1}-u_{j,i}\|^2_{H^1}\\ &\le \sum_{i=0}^{I_j-1}\int_{t_{j,i}}^{t_{j,i+1}} C|t_{j,i+1}-t_{j,i}| + C\,\|\dot v_{j,i+1}\|_{L^2}\,\|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}}\,dr\\ &\le C\int_0^t |t_{j,i+1}-t_{j,i}| + V_j(r)D_j(r)\,dr, \end{aligned} \tag{36}$$

where V_j has been defined before while $D_j(r)=\sum_i \chi_{j,i}\|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}}$. We have already seen that $V_j \rightharpoonup \|\dot v\|_{L^2}$ weakly in $L^2(0,t)$ and that $\|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}} \to 0$ a.e. in $(0,t)$. Since $v_{j,i}$ is uniformly bounded in $H^1(\Omega)$, and thus in $L^{\bar q}(\Omega)$, by dominated convergence $\|v_{j,i+1}-v_{j,i}\|_{L^{\bar q}} \to 0$ in $L^2(0,t)$. As $E_j \ge 0$ from (36) follows (34). ■

Theorem 3.9 *Let v,u be the limits obtained by Lemma 3.5, then $v\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;\mathcal{V})$ and for a.e. $t\in[0,T]$ it holds*

$$\begin{cases}\dot v(t) = -\big[v(t)W(D\tilde u(t)) + G_c(v(t)-1) - G_c\Delta v(t)\big]^+\\ \operatorname{div}\big(\boldsymbol{\sigma}_{v(t)}(\tilde u(t))\big)=0\,,\end{cases} \tag{37}$$

Note that $v(t)W(D\tilde u(t)) + G_c(v(t)-1) - G_c\Delta v(t)$ is a finite Radon measure with positive part in $L^2(\Omega)$ while $\boldsymbol{\sigma}_v(\tilde u) = (v^2+\eta)\,\mathbf{C}D\tilde u$ is the phase-field stress (and $\tilde u(t)$ denotes the displacement $u(t)+g(t)$). In particular, the first equation holds in $L^2(\Omega)$ and v is monotone non-increasing while the second holds in $H^{-1}(\Omega,\mathbb{R}^2)$.

Proof. By Lemma 3.2 we known that $\langle \dot v_{m,k},\xi\rangle + \partial_v\mathcal{F}(t_{m,k},u_{m,k-1},v_{m,k})[\xi]\ge 0$ for every $\xi\in\Xi$ and for every choice of the indices m and k. Besides the piecewise affine function v_m we introduce the (auxiliary) piecewise constant functions t^m, u^m and v^m given by

$$t^m(t)=t_{m,k},\quad u^m(t)=u_{m,k-1},\quad v^m(t)=v_{m,k}\qquad \text{for } t\in[t_{m,k-1},t_{m,k}).$$

Then, for $0\le t_a<t_b\le T$ we can write

$$\int_{t_a}^{t_b}\langle \dot v_m,\xi\rangle_{L^2}\,dt \ge \int_{t_a}^{t_b} -\partial_v\mathcal{F}(t^m,u^m,v^m)[\xi]\,dt.$$

Since $\dot v_m \rightharpoonup \dot v$ in $L^2(0,T;L^2(\Omega))$ we can easily pass to the limit in the first term. By Lemma 3.5 we know that $v^m\rightharpoonup v$ in $H^1(\Omega)$ and $u^m\to u$ strongly in $W^{1,p}(\Omega,\mathbb{R}^2)$ (for some $p>2$) pointwise in $(0,T)$. Thus by (13)

$$\int_{t_a}^{t_b}\langle \dot v(t),\xi\rangle_{L^2}\,dt \ge \int_{t_a}^{t_b} -\partial_v\mathcal{F}(t,u(t),v(t))[\xi]\,dt,$$

which holds for every $0\le t_a<t_b\le T$. As a consequence $\langle\dot v(t),\xi\rangle \ge -\partial_v\mathcal{F}(t,u(t),v(t))[\xi]$ holds a.e. in time for every $\xi\in\Xi$.

From Theorem 3.6 we know that $\|\dot v(t)\|_{L^2} = |\partial_v^-\mathcal{F}(t,u(t),v(t))|_{L^2}$ holds a.e. in time, then by Lemma A.3 it follows that $\partial_v\mathcal{F}(t,u(t),v(t))$ is a finite Radon measure μ with positive part $\mu^+\in L^2(\Omega)$ and that

$$\|\dot v(t)\|_{L^2} = |\partial_v^-\mathcal{F}(t,u(t),v(t))|_{L^2} = \|\mu^+\|_{L^2}.$$

Moreover for every $\xi\in\Xi\cap C_0^\infty(\Omega)$

$$\langle -\dot v(t),-\xi\rangle_{L^2} \ge \partial_v\mathcal{F}(t,u(t),v(t))[-\xi] = \langle \mu^+-\mu^-,-\xi\rangle = \int_\Omega \xi\,d\mu^- - \int_\Omega \xi\mu^+dx,$$

where the duality is in the sense of distributions. Let $\Omega^\pm$ be the disjoint supports of $\mu^\pm$. Let $\varphi \in L^2(\Omega)$ a non-negative function supported in Ω^+. Arguing as in the proof of Lemma A.3 we can find a sequence $\varphi_n \in C_0^\infty(\Omega)$ such that $\langle \mu^-, \varphi_n \rangle \to 0$ and $\langle \mu^+, \varphi_n \rangle \to \langle \mu^+, \varphi \rangle$. Hence, for every non-negative $\varphi \in L^2(\Omega)$ supported in Ω^+ we get

$$\langle -\dot v(t), \varphi \rangle_{L^2} \geq \langle \mu^+, \varphi \rangle_{L^2}.$$

It follows that $-\dot v(t) \geq \mu^+ \geq 0$ in Ω^+. The same inequality hold (obviously) in Ω^-, where $\mu^+ = 0$. Since $\| - \dot v(t) \|_{L^2} = \| \mu^+ \|_{L^2}$ we get $-\dot v(t) = \mu^+$. To conclude, it is enough to represent the functional $\partial_v \mathcal{F}(t, u(t), v(t))[\phi]$. For $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \partial_v \mathcal{F}(t, u(t), v(t))[\phi] &= \int_\Omega v \phi W(D\tilde u(t))\, dx + G_c \int_\Omega (v(t) - 1)\phi + \nabla v \cdot \nabla \phi \, dx \\ &= \langle v W(D\tilde u(t)) + G_c(v(t) - 1) - G_c \Delta v(t), \phi \rangle, \end{aligned} \tag{38}$$

where the last duality is in the sense of distributions. Hence

$$\mu^+ = [v W(D\tilde u(t)) + G_c(v(t) - 1) - G_c \Delta v(t)]^+$$

and the proof is concluded. ■

4 Time rescaling

In order to find the quasi-static limit we first rescale the time variable, in order to get a "slow" evolution in the rescaled physical time interval $[0, T/\varepsilon]$. For $\varepsilon > 0$ let us consider the boundary condition $g_\varepsilon(t) = g(\varepsilon t)$ defined in $[0, T_\varepsilon]$, for $T_\varepsilon = T/\varepsilon$. Clearly g_ε is Lipschitz continuous in $W^{1,\bar p}(\Omega, \mathbb{R}^2)$ with

$$\| g_\varepsilon(t_2) - g_\varepsilon(t_1) \|_{W^{1,\bar p}} \leq \varepsilon C |t_2 - t_1|.$$

Next, we define $\mathcal{F}_\varepsilon : [0, T_\varepsilon] \times \mathcal{U} \times \mathcal{V} \to [0, +\infty)$ by

$$\mathcal{F}_\varepsilon(t, u, v) = \tfrac{1}{2} \int_\Omega (v^2 + \eta) W(Du + Dg_\varepsilon(t))\, dx + \tfrac{1}{2}\, G_c \int_\Omega (v - 1)^2 + |\nabla v|^2 \, dx.$$

As in §3 fix $\tau_m = T_\varepsilon / m > 0$ and let $t_{m,k} = k \tau_m$ for $k = 0, ..., m$. Given $u_{\varepsilon,m,k-1}$ and $v_{\varepsilon,m,k-1}$ define by induction

$$\begin{cases} v_{\varepsilon,m,k} \in \operatorname{argmin} \left\{ \mathcal{F}_\varepsilon(t_{m,k}, u_{\varepsilon,m,k-1}, v) + \frac{1}{2\tau_m} \| v - v_{\varepsilon,m,k-1} \|^2_{L^2} : v \leq v_{\varepsilon,m,k-1}\,,\ v \in \mathcal{V} \right\} \\ u_{\varepsilon,m,k} \in \operatorname{argmin} \left\{ \mathcal{F}_\varepsilon(t_{m,k}, u, v_{\varepsilon,m,k}) : u \in \mathcal{U} \right\}. \end{cases} \tag{39}$$

Denote by $u_{\varepsilon,m}$ and $v_{\varepsilon,m}$ the corresponding piecewise affine interpolate. By Lemma 3.5, Theorem 3.6 and Theorem 3.9 we easily get the following result.

Theorem 4.1 *There exists a subsequence (not relabelled) of $v_{\varepsilon,m}$ such that $v_{\varepsilon,m} \rightharpoonup v_\varepsilon$ in $H^1(0, T_\varepsilon; L^2(\Omega))$. Let u_ε be the corresponding limit of $u_{\varepsilon,m}$. Then, for every $t \in [0, T_\varepsilon]$ we have $u_\varepsilon(t) \in \operatorname{argmin} \{ \mathcal{E}(t, v_\varepsilon(t), u) : u \in \mathcal{U} \}$ and*

$$\begin{aligned} \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)) = \mathcal{F}_\varepsilon(0, u_0, v_0) - \tfrac{1}{2} \int_0^t \| \dot v_\varepsilon(r) \|^2_{L^2} + | \partial_v^- \mathcal{F}_\varepsilon(r, u_\varepsilon(r), v_\varepsilon(r)) |^2_{L^2} \, dr \; + \\ + \int_{t_0}^t \partial_t \mathcal{F}_\varepsilon(r, u_\varepsilon(r), v_\varepsilon(r))\, dr. \end{aligned} \tag{40}$$

Moreover for almost every $t \in [0, T_\varepsilon]$ we have

$$\| \dot v_\varepsilon(t) \|_{L^2} = | \partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)) |_{L^2} \tag{41}$$

$$\begin{cases} \dot v_\varepsilon(t) = - \big[v_\varepsilon(t) W(D\tilde u_\varepsilon(t)) + G_c(v_\varepsilon(t) - 1) - G_c \Delta v_\varepsilon(t) \big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{v_\varepsilon(t)}(\tilde u_\varepsilon(t)) \big) = 0\,. \end{cases} \tag{42}$$

Corollary 4.2 *For every $\lambda \in [0,1]$ it holds*

$$\mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)) = \mathcal{F}_\varepsilon(0, u_0, v_0) + \int_0^t \partial_t \mathcal{F}_\varepsilon(r, u_\varepsilon(r), v_\varepsilon(r))\, dr + \\ - \int_0^t \lambda \, \|\dot v_\varepsilon(r)\|^2_{L^2} + (1-\lambda) |\partial_v \mathcal{F}_\varepsilon(r, u_\varepsilon(r), v_\varepsilon(r))|^2_{L^2}\, dr. \tag{43}$$

Proof. It is sufficient to re-write (40) taking into account (41). ■

Remark 4.3 *We remark that in general* (43) *does not provide a characterization of the gradient flow, unless it holds for $\lambda = 1/2$. Moreover, introducing the rescaled variable* $\mathrm{t} = (t/\varepsilon) \in [0, T]$*, the functions* $\mathrm{v}^\varepsilon(\mathrm{t}) = v_\varepsilon(\mathrm{t}/\varepsilon)$ *and* $\mathrm{u}^\varepsilon(\mathrm{t}) = u_\varepsilon(\mathrm{t}/\varepsilon)$ *solve the system*

$$\begin{cases} \varepsilon \dot{\mathrm{v}}^\varepsilon(\mathrm{t}) = -\big[\mathrm{v}^\varepsilon(\mathrm{t}) W(D\tilde{\mathrm{u}}^\varepsilon(\mathrm{t})) + G_c(\mathrm{v}^\varepsilon(\mathrm{t}) - 1) - G_c \Delta \mathrm{v}^\varepsilon(\mathrm{t})\big]^+ \\ \mathrm{div}\big(\boldsymbol{\sigma}_{\mathrm{v}^\varepsilon(\mathrm{t})}(\tilde{\mathrm{u}}^\varepsilon(\mathrm{t}))\big) = 0\,. \end{cases}$$

5 Quasi-static parametrized limit

In this section we will apply the change of variable

$$t \mapsto s^\varepsilon(t) = \varepsilon t + \int_0^t \|\dot v_\varepsilon(r)\|_{L^2}\, dr$$

in order to obtain a parametrization of v_ε (originally defined for $t \in [0, T/\varepsilon]$) in terms of an arc-length parameter $s \in [0, S_\varepsilon]$. This is a convenient way of writing the quasi-static (rate-independent) evolution and in particular to characterize its behavior in the discontinuity points. Note that here the parametrization is in L^2 and not in H^1 as in [22].

First of all, let us see that s^ε maps the physical time interval $[0, T/\varepsilon]$ onto a reference parametrization interval $[0, S_\varepsilon]$ with $S_\varepsilon = s^\varepsilon(T_\varepsilon)$ uniformly bounded with respect to $\varepsilon > 0$.

5.1 Finite length and boundedness

Theorem 5.1 *The length of the discrete curves $v_{\varepsilon,m}$ is uniformly bounded in L^2, i.e., there exists $C > 0$ (independent of ε and m) such that for τ sufficiently small it holds*

$$\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}(t)\|_{L^2}\, dt \le C(T + |\Omega|).$$

Moreover,

$$\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}(t)\|^2_{H^1}\, dt \le C(\varepsilon T + |\Omega|).$$

Proof. The proof is quite technical and is based on a discrete Gronwall argument, provided in Lemma A.1, see also [33, 22, 31, 29]. We will prove this property employing again the time discretization scheme. However, for notational convenience, we will write v_k instead of $v_{\varepsilon,m,k}$ etc.

Step I. By Lemma 3.2 for $k \ge 0$ we have

$$\partial_v \mathcal{F}_\varepsilon(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] + \|\dot v_{k+1}\|^2_{L^2} = 0. \tag{44}$$

while for $k \ge 1$

$$\partial_v \mathcal{F}_\varepsilon(t_k, u_{k-1}, v_k)[w - v_k] + \langle \dot v_k, w - v_k \rangle_{L^2} \ge 0 \quad \text{for every } w \in H^1(\Omega) \text{ with } w \le v_{k-1}.$$

Choosing $w = v_k + \dot v_{k+1}$ provides

$$\partial_v \mathcal{F}_\varepsilon(t_k, u_{k-1}, v_k)[\dot v_{k+1}] + \langle \dot v_k, \dot v_{k+1} \rangle_{L^2} \ge 0. \tag{45}$$

For $k = 0$ we have, by equilibrium,

$$\partial_v \mathcal{F}_\varepsilon(0, u_0, v_0)[\dot v_1] \ge 0. \tag{46}$$

Hence, from (44) and (45) we obtain, for $k \ge 1$,

$$\begin{aligned} &\partial_v \mathcal{F}_\varepsilon(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{F}_\varepsilon(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] \ge \\ &\qquad \ge \|\dot v_{k+1}\|^2_{L^2} - \langle \dot v_k, \dot v_{k+1}\rangle_{L^2} \ge \tfrac12 \|\dot v_{k+1}\|^2_{L^2} - \tfrac12 \|\dot v_k\|^2_{L^2}. \end{aligned} \tag{47}$$

For $k = 0$ from (44) and (46) we get

$$\partial_v \mathcal{F}_\varepsilon(0, u_0, v_0)[\dot v_1] - \partial_v \mathcal{F}_\varepsilon(t_1, u_0, v_1)[\dot v_1] \ge \|\dot v_1\|^2_{L^2}.$$

Setting $\dot v_0 = 0$ we can also write

$$\partial_v \mathcal{F}_\varepsilon(0, u_0, v_0)[\dot v_1] - \partial_v \mathcal{F}_\varepsilon(t_1, u_0, v_1)[\dot v_1] \ge \tfrac12 \|\dot v_1\|^2_{L^2} - \tfrac12 \|\dot v_0\|^2_{L^2} \tag{48}$$

and thus (47) actually holds for every $k \ge 0$.

For the left hand side of (47) we proceed as follows.

$$\begin{aligned} &\partial_v \mathcal{F}_\varepsilon(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{F}_\varepsilon(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] = \\ &\qquad = \partial_v \mathcal{E}(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{E}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] + \\ &\qquad + \partial_v \mathcal{D}(v_k)[\dot v_{k+1}] - \partial_v \mathcal{D}(v_{k+1})[\dot v_{k+1}]. \end{aligned}$$

For $\partial_v \mathcal{D}(v_k)[\dot v_{k+1}] - \partial_v \mathcal{D}(v_{k+1})[\dot v_{k+1}]$ we write

$$\begin{aligned} &\int_\Omega (v_k - 1)\, \dot v_{k+1} + \nabla v_k \cdot \nabla \dot v_{k+1}\, dx - \int_\Omega (v_{k+1} - 1)\, \dot v_{k+1} + \nabla v_{k+1} \cdot \nabla \dot v_{k+1}\, dx = \\ &\qquad = \int_\Omega (v_k - v_{k+1})\, \dot v_{k+1} + \nabla(v_k - v_{k+1}) \cdot \nabla \dot v_{k+1}\, dx = -\tau \|\dot v_{k+1}\|^2_{H^1}. \end{aligned} \tag{49}$$

We can estimate $\partial_v \mathcal{E}(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{E}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}]$ by

$$\begin{aligned} &\int_\Omega v_k \dot v_{k+1} W(Du_{k-1} + Dg_{\varepsilon,k})\, dx - \int_\Omega v_{k+1}\, \dot v_{k+1}\, W(Du_k + Dg_{\varepsilon,k+1})\, dx \le \\ &\quad \le \int_\Omega v_k \dot v_{k+1} W(Du_{k-1} + Dg_{\varepsilon,k})\, dx - \int_\Omega v_{k+1} \dot v_{k+1} W(Du_{k-1} + Dg_{\varepsilon,k})\, dx + \\ &\qquad + \int_\Omega v_{k+1} \dot v_{k+1} W(Du_{k-1} + Dg_{\varepsilon,k})\, dx - \int_\Omega v_{k+1} \dot v_{k+1} W(Du_k + Dg_{\varepsilon,k+1})\, dx \\ &\quad \le \int_\Omega (v_k - v_{k+1}) \dot v_{k+1} W(Du_{k-1} + Dg_{\varepsilon,k})\, dx + \\ &\qquad + \int_\Omega v_{k+1} \dot v_{k+1} \big(W(Du_{k-1} + Dg_{\varepsilon,k}) - W(Du_k + Dg_{\varepsilon,k+1}) \big)\, dx \\ &\quad \le \int_\Omega v_{k+1} \dot v_{k+1} \big(W(Du_{k-1} + Dg_{\varepsilon,k}) - W(Du_k + Dg_{\varepsilon,k+1}) \big)\, dx, \end{aligned}$$

where last inequality follows from $(v_k - v_{k+1})\, \dot v_{k+1}\, W(Du_k + Dg_{\varepsilon,k}) \le 0$. For $1/s + 2/p = 1$ we get by Hölder inequality

$$\begin{aligned} &\int_\Omega \dot v_{k+1} v_{k+1} \big(W(Du_{k-1} + Dg_{\varepsilon,k}) - W(Du_k + Dg_{\varepsilon,k+1}) \big)\, dx \le \\ &\qquad \le \|\dot v_{k+1}\|_{L^s}\, \|W(Du_{k-1} + Dg_{\varepsilon,k}) - W(Du_k + Dg_{\varepsilon,k+1})\|_{L^{p/2}}. \end{aligned}$$

Since u_k is uniformly bounded in $W^{1,p}(\Omega,\mathbb{R}^2)$ for $2 < p < \bar p$ (by Lemma A.5) and since $g \in W^{1,\infty}(0,T;W^{1,\bar p}(\Omega,\mathbb{R}^2))$ we get by Lemma A.5

$$\begin{aligned}\sup_k \|(Du_{k-1}+Dg_{\varepsilon,k})+(Du_k+Dg_{\varepsilon,k+1})\|_{L^p} < +\infty,\\ \|(Du_{k-1}+Dg_{\varepsilon,k})-(Du_k+Dg_{\varepsilon,k+1})\|_{L^p} \le \\ \le C(\varepsilon|t_k-t_{k-1}|+\|v_k-v_{k-1}\|_{L^r}+\varepsilon|t_{k+1}-t_k|)\\ \le C\tau(\varepsilon+\|\dot v_k\|_{L^r}),\end{aligned}$$

where $1/r = 1/p - 1/\bar p$. Thus for $q = s \vee r$ we have

$$\int_\Omega \dot v_{k+1} v_{k+1}\big(W(Du_{k-1}+Dg_{\varepsilon,k}) - W(Du_k+Dg_{\varepsilon,k+1})\big)\,dx \le C\tau\|\dot v_{k+1}\|_{L^q}(\varepsilon+\|\dot v_k\|_{L^q}).$$

In conclusion, for $k \ge 0$ we have

$$\tfrac12\|\dot v_{k+1}\|^2_{L^2} - \tfrac12\|\dot v_k\|^2_{L^2} \le -\tau G_c\|\dot v_{k+1}\|^2_{H^1} + C\tau\|\dot v_{k+1}\|_{L^q}(\varepsilon+\|\dot v_k\|_{L^q}). \tag{50}$$

Step II. By Young's inequality, for $0<\mu\ll 1$ and $C_\mu > 1$

$$\begin{aligned}\|\dot v_{k+1}\|_{L^q}(\varepsilon+\|\dot v_k\|_{L^q}) &\le \varepsilon\|\dot v_{k+1}\|_{L^q} + C_\mu\|\dot v_{k+1}\|^2_{L^q} + \mu\|\dot v_k\|^2_{L^q} \le C'_\mu(\varepsilon^2+\|\dot v_{k+1}\|^2_{L^q}) + \mu\|\dot v_k\|^2_{L^q}\\ &\le C'_\mu(\varepsilon^2+\|\dot v_{k+1}\|^2_{L^q}) + C\mu\|\dot v_k\|^2_{H^1}.\end{aligned} \tag{51}$$

Write $1/q = \alpha + (1-\alpha)/\bar q$ for $\alpha\in(0,1)$ and $q<\bar q<+\infty$. Then, by interpolation and Young's inequality, with $p = 1/\alpha$, for $0<\lambda\ll 1$ we have

$$\begin{aligned}\|\dot v_{k+1}\|^2_{L^q} &\le \|\dot v_{k+1}\|^{2\alpha}_{L^1}\,\|\dot v_{k+1}\|^{2(1-\alpha)}_{L^{\bar q}} \le C_\lambda\|\dot v_{k+1}\|^2_{L^1} + \lambda\|\dot v_{k+1}\|^2_{L^{\bar q}}\\ &\le C'_\lambda\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + C\lambda\|\dot v_{k+1}\|^2_{H^1}.\end{aligned} \tag{52}$$

Hence, upon choosing μ and λ sufficiently small, respectively in (51) and in (52), we infer that for every $0<\delta\ll 1$ there exists C_δ such that

$$C\|\dot v_{k+1}\|_{L^q}(\varepsilon+\|\dot v_k\|_{L^q}) \le C_\delta(\varepsilon^2+\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2}) + \delta\|\dot v_{k+1}\|^2_{H^1} + \delta\|\dot v_k\|^2_{H^1}. \tag{53}$$

Joining (50) and (53) yields the estimate

$$\begin{aligned}\tfrac12\|\dot v_{k+1}\|^2_{L^2} - \tfrac12\|\dot v_k\|^2_{L^2} \le &-\tau G_c\|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_k\|^2_{H^1} +\\ &+\tau C_\delta\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + \tau C_\delta\varepsilon^2.\end{aligned} \tag{54}$$

Step III. In order to apply the discrete Gronwall Lemma A.1 we need to re-write (54). First,

$$\begin{aligned}\tfrac12\|\dot v_{k+1}\|^2_{L^2} - \tfrac12\|\dot v_k\|^2_{L^2} \le &-\tau(G_c-2\delta)\|\dot v_{k+1}\|^2_{H^1} - \delta\tau\|\dot v_{k+1}\|^2_{H^1} + \delta\tau\|\dot v_k\|^2_{H^1} +\\ &+\tau C_\delta\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + \tau C_\delta\varepsilon^2.\end{aligned}$$

Hence,

$$\begin{aligned}\big(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\big) - \big(\tfrac12\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\big) \le\\ -\tau(G_c-2\delta)\|\dot v_{k+1}\|^2_{H^1} + \tau C_\delta\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + \tau C_\delta\varepsilon^2.\end{aligned}$$

For $\tau,\delta\ll 1$ and $\gamma>0$ we can write

$$\gamma\left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right) \le (G_c-2\delta)\|\dot v_{k+1}\|^2_{H^1}.$$

Therefore, we get

$$\begin{aligned}\big(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\big) - \big(\tfrac12\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\big) \le\\ \le -\gamma\tau\left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right) +\\ + C'_\delta\tau\|\dot v_{k+1}\|_{L^1}\left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right)^{1/2} + C_\delta\tau\varepsilon^2.\end{aligned} \tag{55}$$

Define

$$a_k = \left(\tfrac{1}{2}\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\right)^{1/2}, \qquad b_k = C'_\delta\|\dot v_k\|_{L^1}, \qquad c_k^2 = C_\delta\varepsilon^2.$$

Hence (55) reads: for every $0 \le k \le m-1$

$$a_{k+1}^2 - a_k^2 \le -\tau\gamma a_{k+1}^2 + \tau a_{k+1}b_{k+1} + \tau c_{k+1}^2.$$

Then, for $0 < \beta < \gamma/2$ by Lemma A.1 we get

$$a_k \le \left(\sum_{i=0}^{k}\tau e^{-2\beta(t_k-t_i)}c_i^2\right)^{1/2} + \sum_{i=0}^{k}\tau e^{-\beta(t_k-t_i)}b_i.$$

Remembering the definition of a_k, b_k and c_k, the previous estimate gives

$$\begin{aligned}
\tfrac{1}{2}\|\dot v_k\|_{L^2} &\le \left(\tfrac{1}{2}\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\right)^{1/2}\\
&\le C\left(\sum_{i=0}^{k} e^{-2\beta(t_k-t_i)}\tau\varepsilon^2\right)^{1/2} + C\sum_{i=0}^{k}\tau e^{-\beta(t_k-t_i)}\|\dot v_i\|_{L^1}\\
&\le C\varepsilon\left(\int_0^{t_k} e^{-2\beta(t_k-r)}\,dr\right)^{1/2} + C\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr\\
&\le C\varepsilon/2\beta + C\int_0^{t_k}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr \le C'(\varepsilon + |\Omega|),
\end{aligned} \tag{56}$$

where the last inequality follows from monotonicity (in time) and boundedness of $v_{\varepsilon,m}$. Hence $v_{\varepsilon,m}$ is bounded in $W^{1,\infty}(0,T_\varepsilon;L^2)$. Moreover,

$$\int_0^{T_\varepsilon}\|\dot v_{\varepsilon,m}(t)\|\,dt \le \sum_{k=0}^{m}\tau\|\dot v_k\|_{L^2} \le C\varepsilon T_\varepsilon + C\sum_{k=0}^{m}\tau\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr.$$

Then, for $t \in [t_k, t_{k+1}]$ we can write

$$\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr \le \int_0^{t} e^{-\beta(t-\tau-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr$$

and thus

$$\begin{aligned}
\sum_{k=0}^{K}\tau\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr &\le \int_0^{T/\varepsilon}\int_0^{t} e^{-\beta(t-\tau-r)}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\,dr\,dt\\
&\le e^{\beta\tau}\int_0^{T/\varepsilon}\|\dot v_{\varepsilon,m}(r)\|_{L^1}\int_r^{T/\varepsilon} e^{-\beta(t-r)}\,dt\,dr\\
&\le C\int_0^{T/\varepsilon}\|\dot v_{\varepsilon,m}(r)\|_{L^1}dr = C|\Omega|.
\end{aligned}$$

Step IV. Let us go back to (54), i.e.

$$\begin{aligned}
&\tfrac{1}{2}\|\dot v_{k+1}\|^2_{L^2} - \tfrac{1}{2}\|\dot v_k\|^2_{L^2} \le\\
&\qquad \le -\tau(G_c-\delta)\|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_k\|^2_{H^1} + \tau C_\delta\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + \tau C_\delta\varepsilon^2.
\end{aligned}$$

Let $0 < C = G_c - \delta$ for $0 < \delta \ll 1$; being $\|\dot v_k\|_{L^2}$ uniformly bounded (by the previous step) the above estimate can be written as

$$\tau C\|\dot v_{k+1}\|^2_{H^1} \le \left(\tfrac{1}{2}\|\dot v_k\|^2_{L^2} - \tfrac{1}{2}\|\dot v_{k+1}\|^2_{L^2}\right) + \tau\delta\|\dot v_k\|^2_{H^1} + \tau C'_\delta(\varepsilon^2 t + \|\dot v_{k+1}\|_{L^1}).$$

Taking the sum for $k = 0, ..., m-1$ and remembering that $v_{\varepsilon,m}$ is piecewise affine we get

$$C\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}\|^2_{H^1}\,dt = C\sum_{k=0}^{m-1} \tau\|\dot v_{k+1}\|^2_{H^1} \le \sum_{k=0}^{m-1}\left(\tfrac12\|\dot v_k\|^2_{L^2} - \tfrac12\|\dot v_{k+1}\|^2_{L^2}\right) +$$
$$+\delta\sum_{k=0}^{m-1}\tau\|\dot v_k\|^2_{H^1} + C'_\delta\sum_{k=0}^{m-1}\tau(\varepsilon^2 + \|\dot v_{k+1}\|_{L^1})$$
$$\le \tfrac12\|\dot v_0\|^2_{L^2} + \delta\int_0^{T_\varepsilon}\|\dot v_{\varepsilon,m}\|^2_{H^1}\,dt + C'_\delta\int_0^{T_\varepsilon}\varepsilon^2 + \|\dot v_{\varepsilon,m}\|_{L^1}\,dt$$
$$\le \delta\int_0^{T_\varepsilon}\|\dot v_{\varepsilon,m}\|^2_{H^1}\,dt + C_\delta(\varepsilon T_\varepsilon + |\Omega|). \tag{57}$$

Choosing $0 < \delta < C$ it follows that $v_{\varepsilon,m}$ is bounded in $H^1(0, T_\varepsilon; H^1)$. ■

Passing to the limit for $\tau_m \to 0$ in the previous Theorem we get the following result.

Corollary 5.2 *The limit evolution v_ε (provided by Theorem 4.1) satisfies*

$$\int_0^{T_\varepsilon}\|\dot v_\varepsilon(t)\|_{L^2} + \|\dot v_\varepsilon(t)\|^2_{H^1}\,dt \le C(T + |\Omega|).$$

In particular $S_\varepsilon = s^\varepsilon(T/\varepsilon)$ is uniformly bounded.

5.2 Rescaled parametrized gradient flows

Let us go back to our parametrization

$$t \mapsto s^\varepsilon(t) = \varepsilon t + \int_0^t \|\dot v_\varepsilon(r)\|_{L^2}\,dr \tag{58}$$

from $[0, T_\varepsilon]$ onto $[0, S_\varepsilon]$. The map $t \mapsto s^\varepsilon(t)$ is absolutely continuous and strictly monotone. We denote by $t^\varepsilon(s)$ be its inverse; moreover we denote

$$t_\varepsilon(s) = \varepsilon t^\varepsilon(s), \qquad z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s), \qquad w_\varepsilon(s) = u_\varepsilon \circ t^\varepsilon(s). \tag{59}$$

Accordingly, let $w_0 = u_0$ and $z_0 = v_0$.

Lemma 5.3 *The functions $s \mapsto t_\varepsilon(s)$ and $s \mapsto z_\varepsilon(s)$ are Lipschitz continuous in $[0, S_\varepsilon]$, more precisely for a.e. $s \in [0, S_\varepsilon]$ it holds*

$$t'_\varepsilon(s) + \|z'_\varepsilon(s)\|_{L^2} = 1, \qquad t'_\varepsilon(s) = \frac{\varepsilon}{\varepsilon + |\partial_v^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}}.$$

Note that t_ε is onto $[0, T]$.

Proof. As $t \mapsto s^\varepsilon(t)$ is absolutely continuous with $\dot s^\varepsilon(t) \ge \varepsilon$ a.e. in $[0, T_\varepsilon]$ the inverse function $s \mapsto t^\varepsilon(s)$ turns out to be Lipschitz continuous with $(t^\varepsilon)'(s) = 1/\dot s^\varepsilon(t^\varepsilon(s))$ a.e. in $[0, S_\varepsilon]$. Hence, by (58) and (59)

$$1 = \dot s^\varepsilon(t^\varepsilon(s))\,(t^\varepsilon)'(s) = (\varepsilon + \|\dot v_\varepsilon(t^\varepsilon(s))\|_{L^2})\,(t^\varepsilon)'(s) = t'_\varepsilon(s) + \|z'_\varepsilon(s)\|_{L^2}.$$

Moreover, by (27) for a.e. $t \in [0, T_\varepsilon]$ we have

$$\|\dot v_\varepsilon(t)\|_{L^2} = |\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2} = |\partial_v^- \mathcal{F}(\varepsilon t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2}.$$

Thus

$$\|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2} = |\partial_v^- \mathcal{F}(\varepsilon t^\varepsilon(s), u_\varepsilon \circ t^\varepsilon(s), v_\varepsilon \circ t^\varepsilon(s))|_{L^2} = |\partial_z^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}.$$

Since sets of measure zero are mapped to sets of measure zero, both by $s \mapsto t^\varepsilon(s)$ and by $t \mapsto s^\varepsilon(t)$, for a.e. $s \in [0, S_\varepsilon]$ we have

$$t_\varepsilon'(s) = \frac{\varepsilon}{\dot{s}^\varepsilon(t^\varepsilon(s))} = \frac{\varepsilon}{\varepsilon + \|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2}} = \frac{\varepsilon}{\varepsilon + |\partial_v \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}}. \tag{60}$$

Since t_ε is the rescaled inverse of s^ε it is surjective, taking values in $[0, T]$. ■

Lemma 5.4 *For $\varepsilon > 0$, the rescaled parametrized evolutions $(t_\varepsilon, z_\varepsilon)$ are (uniformly) bounded in $W^{1,\infty}(0, S_\varepsilon; [0,T] \times L^2)$ and in $L^\infty(0, S_\varepsilon; [0,T] \times \mathcal{V})$ with $t_\varepsilon' \geq 0$, $z_\varepsilon' \leq 0$ and $t_\varepsilon' + \|z_\varepsilon'\|_{L^2} \leq 1$. Further, for every $s \in [0, S_\varepsilon]$ and every $\lambda \in [0,1]$ the following energy balance holds:*

$$\begin{aligned}\mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s)) = {} & \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t_\varepsilon(r), w_\varepsilon(r), z_\varepsilon(r))\, t_\varepsilon'(r)\, dr + \\ & - \int_0^s \lambda \Psi_\varepsilon(\|z_\varepsilon'(r)\|_{L^2}) + (1-\lambda)\Phi_\varepsilon(|\partial_z^- \mathcal{F}(t_\varepsilon(r), w_\varepsilon(r), z_\varepsilon(r))|_{L^2})\, dr,\end{aligned} \tag{61}$$

where

$$\Psi_\varepsilon(\xi) = \begin{cases} \varepsilon\xi^2/(1-\xi) & 0 \leq \xi < 1 \\ +\infty & \xi \geq 1, \end{cases} \qquad \Phi_\varepsilon(\xi) = \xi^2/(\varepsilon + \xi).$$

We consider both Ψ_ε and Φ_ε to be defined in $[0, +\infty)$. Clearly, $w_\varepsilon(s) \in \operatorname{argmin}\{\mathcal{E}(t_\varepsilon(s), w, z_\varepsilon(s)) : w \in \mathcal{U}\}$

Proof. By Corollary 4.2 we known that for every $\bar{t} \in [0, T_\varepsilon]$ it holds

$$\begin{aligned}\mathcal{F}_\varepsilon(\bar{t}, u_\varepsilon(\bar{t}), v_\varepsilon(\bar{t})) = \mathcal{F}_\varepsilon(0, u_0, v_0) + \int_0^{\bar{t}} \partial_t \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))\, dt \, + \\ - \int_0^{\bar{t}} \lambda\, \|\dot{v}_\varepsilon(t)\|_{L^2}^2 + (1-\lambda)|\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2}^2\, dt.\end{aligned}$$

Remember that $\mathcal{F}_\varepsilon(t, u, v) = \mathcal{F}(\varepsilon t, u, v)$ and thus

$$\partial_t \mathcal{F}_\varepsilon(t, u, v) = \varepsilon\, \partial_t \mathcal{F}(\varepsilon t, u, v), \qquad \partial_v \mathcal{F}_\varepsilon(t, u, v) = \partial_v \mathcal{F}(\varepsilon t, u, v).$$

Hence, by the change of variable $t = t^\varepsilon(s) = t_\varepsilon(s)/\varepsilon$, the energy balance in parametrized form reads: for a.e. $\bar{s} \in [0, S_\varepsilon]$ it holds

$$\begin{aligned}\mathcal{F}(t_\varepsilon(\bar{s}), w_\varepsilon(\bar{s}), z_\varepsilon(\bar{s})) = \mathcal{F}(0, w_0, z_0) + \int_0^{\bar{s}} \partial_t \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))\, t_\varepsilon'(s)\, ds \, + \\ - \int_0^s \big[\lambda\, \|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2}^2 + (1-\lambda)|\partial_v^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}^2\big] (t^\varepsilon)'(s)\, ds.\end{aligned}$$

Since $\|\dot{v}_\varepsilon(t^\varepsilon(r))\|_{L^2}\, (t^\varepsilon)'(r) = \|z_\varepsilon'(r)\|_{L^2}$ and $(t^\varepsilon)'(r) = t_\varepsilon'(r)/\varepsilon$ it follows by Lemma 5.3 that

$$\|\dot{v}_\varepsilon(t_\varepsilon(r))\|_{L^2}^2\, (t^\varepsilon)'(r) = \varepsilon \|z_\varepsilon'(r)\|_{L^2}^2 / t_\varepsilon'(r) = \varepsilon \|z_\varepsilon'(r)\|_{L^2}^2 / (1 - \|z_\varepsilon'(r)\|_{L^2}) = \Psi_\varepsilon(\|z_\varepsilon'(r)\|_{L^2}).$$

Again by Lemma 5.3, $(t^\varepsilon)'(r) = 1/(\varepsilon + |\partial_z^- \mathcal{F}(t_\varepsilon(r), w_\varepsilon(r), z_\varepsilon(r))|_{L^2})$. ■

Since S_ε is uniformly bounded, by Corollary 5.2, we have $S = \liminf_\varepsilon S_\varepsilon < +\infty$. For compactness, it will be convenient to consider parametrized evolutions t_ε and z_ε to be defined in $[0, S]$ with a constant extension in $(S_\varepsilon, S]$ (clearly only in the case $S_\varepsilon < S$). In this way all the (possibly extended) evolutions enjoy the compactness properties of the previous Lemma in the parametrization interval $[0, S]$. Note however that, with this simple extension, the energy balance is not true, in general, for $s \in (S_\varepsilon, S]$. Note that, independently of $\varepsilon > 0$, we have $t_\varepsilon(0) = 0$ and $t_\varepsilon(S) = T$. Using Lemma 5.3 and Lemma A.5 it is now immediate to prove the following compactness property.

Corollary 5.5 *For $\varepsilon_n \to 0$ there exists a subsequence (not relabeled) such that*

$$(t_{\varepsilon_n}, z_{\varepsilon_n}) \stackrel{*}{\rightharpoonup} (t, z) \text{ in } W^{1,\infty}(0, S; [0,T] \times L^2).$$

For a.e. $s \in [0,S]$ we have $z_{\varepsilon_n}(s) \rightharpoonup z(s)$ in H^1 and thus $w_{\varepsilon_n}(s) \to w(s)$ in $W^{1,p}$ (for $p > 2$) where $w(s) \in \operatorname{argmin}\{\mathcal{E}(t(s), w, z(s)) : w \in \mathcal{U}\}$. Finally, $t(0) = 0$ and $t(S) = T$ and thus t maps $[0,S]$ onto $[0,T]$.

5.3 Quasi-static parametrized BV-limit

Theorem 5.6 *Every limit evolution obtained by Corollary 5.5 satisfies $z' \le 0$, $t' \ge 0$ and $t' + \|z'\|_{L^2} \le 1$, $t(0) = 0$ and $t(S) = T$. Moreover, for every $s \in [0,S)$ we have $w(s) \in \operatorname{argmin}\{\mathcal{E}(t(s), w, z(s)) : w \in \mathcal{U}\}$ and the following energy balance*

$$\mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr + \\ - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr. \tag{62}$$

Any such limit will be called a parametrized BV-evolution.

Proof. Part I. The proof follows closely that of [32, Theorem 4.4]. If $s < S$ then $s \in [0, S_{\varepsilon_n})$ for $\varepsilon_n \ll 1$; thus (61), with $\lambda = 0$, provides

$$\mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)) = \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(t))\, t'_{\varepsilon_n}(r)\, dr + \\ - \int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{63}$$

By Corollary 5.5 we known that $t_{\varepsilon_n}(s) \to t(s)$, $z_{\varepsilon_n}(s) \rightharpoonup z(s)$ in H^1 and $w_{\varepsilon_n}(s) \to w(s)$ in $W^{1,p}$ (for $p > 2$). As a consequence, by Lemma 2.2

$$\mathcal{F}(t(s), w(s), z(s)) \le \liminf_{\varepsilon_n \to 0} \mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)). \tag{64}$$

Next, taking the $\limsup_{\varepsilon_n \to 0}$ in (63) we get

$$\limsup_{\varepsilon_n \to 0} \mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)) \le \mathcal{F}(0, w_0, z_0) + \\ + \limsup_{\varepsilon_n \to 0} \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))\, t'_{\varepsilon_n}(r)\, dr \\ - \liminf_{\varepsilon_n \to 0} \int_0^s \Phi_{\varepsilon_n}(|\partial_z \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{65}$$

First, let us see that

$$\lim_{\varepsilon_n \to 0} \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))\, t'_{\varepsilon_n}(r)\, dr = \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \tag{66}$$

By Lemma 2.3 we know that $\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(t)) \to \partial_t \mathcal{F}(t(r), w(r), z(r))$ for a.e. $r \in [0,s]$. Moreover $\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))$ is uniformly bounded since

$$|\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))| \le C(\mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r)) + 1) < \bar{C}.$$

Hence $\partial_t \mathcal{F}(t_{\varepsilon_n}(\cdot), w_{\varepsilon_n}(\cdot), z_{\varepsilon_n}(\cdot))$ converge to $\partial_t \mathcal{F}(t(\cdot), w(\cdot), z(\cdot))$ strongly in $L^1(0,s)$ (by dominated convergence). Since $t_{\varepsilon_n} \stackrel{*}{\rightharpoonup} t$ in $L^\infty(0,s)$ we get (66).

Finally, let us show that

$$\int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr \le \liminf_{\varepsilon_n \to 0} \int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{67}$$

It is not difficult to check that $\Phi_\varepsilon(\xi) \ge \xi - \varepsilon$ for every $\xi \in [0, +\infty)$. Thus we can write

$$\int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr \ge \int_0^s |\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2}\, dr - \varepsilon_n s.$$

By Lemma 2.2

$$|\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2} \le \liminf_{\varepsilon_n \to 0} |\partial_v^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2}$$

and thus (67) follows from Fatou's Lemma. Joining (64)-(67) yields

$$\begin{aligned} \mathcal{F}(t(s), u(s), v(s)) \le \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \end{aligned}$$

Part II. To prove the opposite inequality we employ the "upper gradient inequality" as in Proposition 3.8. In this setting, $t \in W^{1,\infty}(0,s)$, $z \in W^{1,\infty}(0,s; L^2(\Omega)) \cap L^\infty(0,s;\mathcal{V})$, $w(r) \in \operatorname{argmin}\{\mathcal{F}(t(r), w, z(r)) : w \in \mathcal{U}\}$ and $r \mapsto |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}$ belongs to $L^1(0,s)$. Then, following step by step the proof of Proposition 3.8 it is not difficult (but lengthy) to check that

$$\begin{aligned} \mathcal{F}(0, w_0, z_0) - \mathcal{F}(t(s), w(s), z(s)) \le \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, \|z'(r)\|_{L^2}\, dr + \\ - \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \end{aligned} \tag{68}$$

Since $\|z'(r)\|_{L^2} \le 1$ we get

$$\begin{aligned} \mathcal{F}(0, w_0, z_0) - \mathcal{F}(t(s), w(s), z(s)) \le \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ - \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr, \end{aligned} \tag{69}$$

which concludes the proof. ■

Corollary 5.7 *If $s_{\varepsilon_n} \to s$ then $\mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \to \mathcal{F}(t(s), w(s), z(s))$ and $z_{\varepsilon_n}(s_{\varepsilon_n}) \to z(s)$ in $H^1(\Omega)$. Moreover $s \mapsto z(s)$ is continuous from $(0,S)$ to $H^1(\Omega)$.*

Proof. Following the proof of Theorem 5.6 it is easy to check, using (65)-(69), that

$$\begin{aligned} \mathcal{F}(t(s), w(s), z(s)) &\le \liminf_{n \to +\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \\ &\le \limsup_{n \to +\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \le \mathcal{F}(t(s), w(s), z(s)). \end{aligned}$$

Thus $\lim_{n \to +\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) = \mathcal{F}(t(s), w(s), z(s))$.

If $s_\varepsilon \to s$ then by compactness (cf. Corollary 5.5) $z_{\varepsilon_n}(s_{\varepsilon_n})$ converge to $z(s)$ weakly in H^1 and thus, by compact embedding, strongly in L^q for every $q < +\infty$. Since $t_{\varepsilon_n} \stackrel{*}{\rightharpoonup} t$ we get $t_{\varepsilon_n}(s_{\varepsilon_n}) \to t(s)$. Then by Lemma A.5 we have $w_\varepsilon(s_\varepsilon) \to w(s)$ in $W^{1,p}$ for some $p > 2$. Hence

$$\int_\Omega (z_{\varepsilon_n}^2(s_{\varepsilon_n}) + \eta) W(D\tilde{u}_{\varepsilon_n}(s_{\varepsilon_n}))\, dx \to \int_\Omega (z^2(s) + \eta) W(D\tilde{u}(s))\, dx,$$

$$\int_\Omega (z_{\varepsilon_n}(s_{\varepsilon_n}) - 1)^2\, dx \to \int_\Omega (z(s) - 1)^2\, dx.$$

By convergence of the energy it follows that

$$\int_\Omega |\nabla z_{\varepsilon_n}(s_{\varepsilon_n})|^2\,dx \;\to\; \int_\Omega |\nabla z(s)|^2\,dx,$$

from which follows the strong convergence in H^1. Since $z_{\varepsilon_n}(s_{\varepsilon_n}) \to z(s)$ in H^1 for every sequence $s_{\varepsilon_n} \to s$ we get that $z_{\varepsilon_n} \to z$ (strongly in H^1) locally uniformly in $(0, S)$.

Remember that, by Corollary 5.2, the evolution v_ε is bounded in $W^{1,\infty}(0, T_\varepsilon; L^2) \cap H^1(0, T_\varepsilon; H^1)$ and thus it is continuous in H^1. As a consequence $s \mapsto z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s)$ is continuous from $[0, S_\varepsilon]$ to H^1. Since z_ε converge to z locally uniformly, its limit z is continuous as well. ∎

Remark 5.8 *Using the Legendre transform it is possible to write* (62) *"in gradient flow fashion". Let*

$$\tilde\Psi(z) = \begin{cases} 0 & z \le 1 \\ +\infty & z > 1, \end{cases} \qquad \tilde\Phi(z) = \begin{cases} +\infty & z < 0 \\ z & z \ge 0\,. \end{cases}$$

Note that $\tilde\Phi(z) = \tilde\Psi^*(z)$. *With this notation* (62) *reads*

$$\begin{aligned} \mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\,dr \;+ \\ - \int_0^s \tilde\Psi(\|z'(r)\|_{L^2}) + \tilde\Psi^*(|\partial_v^- \mathcal{F}(t(r), w(r), z(r))|_{L^2})\,dr\,. \end{aligned} \tag{70}$$

5.4 From energy balance to PDEs

In this last subsection we provide some properties, in terms of PDEs, of the parametrized evolution characterized by Theorem 5.6. Intuitively such an evolution is an "arc-length" parametrization of a BV-evolution [27, 30].

Remember that quasi-static evolutions for non-convex energies may have discontinuity in time and that characterization of these points makes the difference between different notion of quasi-static evolution, e.g. energetic, BV or local [27, 30]. Remember also that discontinuity points t_d (in time) correspond in the parametric picture to intervals $(s^\flat, s^\sharp)$ with $t(s) = t_d$, $z(s^\flat) = z^-(t_d)$ and $z(s^\sharp) = z^+(t_d)$. "Vice versa" if $t'(s_c) > 0$ then $t_c = t(s_c)$ is a continuity point in time.

Most of the informations are provided by the relationship between the derivative $t'(s)$ and the slope $|\partial_v^- \mathcal{F}(t(s), w(s), z(s))|_{L^2}$, which is the subject of Proposition 5.9 ; its PDEs form is provided in Corollary 5.10. First, in order to employ the chain rule, we prove the following lemma.

Proposition 5.9 *Let* (t, w, z) *be a parametrized evolution (provided by Theorem 5.6) then for a.e.* $s \in [0, S]$ *it holds*

- $\partial_w \mathcal{F}(t(s), w(s), z(s)) = \partial_w \mathcal{E}(t(s), w(s)) = 0,$
- *if* $t'(s) > 0$ *then* $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0,$
- *if* $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \neq 0$ *then* $t'(s) = 0$ *and*

$$z'(s) \in \operatorname{argmin}\{\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi,\ \|\xi\|_{L^2} \le 1\},$$

  in particular $\|z'(s)\|_{L^2} = 1.$

Proof. Equilibrium for the displacement field follows from the minimality of $w(s)$.

Since $\|z'(s)\|_{L^2} \le 1$ for a.e. $s \in [0,S]$ by (62) and (68) we can write

$$\begin{aligned} \mathcal{F}(t(s), w(s), z(s)) &= \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ &\quad + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr \\ &\le \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_v^- \mathcal{F}(t(r), w(r), z(r))|_{L^2} \|z'(r)\|_{L^2}\, dr + \\ &\quad + \int_0^s \partial_t \mathcal{F}(t(r), u(r), v(r))\, t'(r)\, dr \le \mathcal{F}(t(s), u(s), v(s)). \end{aligned}$$

Hence all inequalities becomes equalities and hold in every subinterval of $(0,S)$. In particular, for a.e. $s \in (0,S)$ we have

$$|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} (1 - \|z'(s)\|_{L^2}) = 0. \tag{71}$$

Hence, if $t'(s) > 0$ then $\|z'(s)\|_{L^2} < 1$ (simply because $t'(s) + \|z'(s)\|_{L^2} \le 1$) and thus

$$|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0.$$

On the contrary, if $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \neq 0$ then $\|z'(s)\|_{L^2} = 1$ and $t'(s) = 0$.

Let $\bar s \in [0,S]$ such that $|\partial_v^- \mathcal{F}(t(\bar s), w(\bar s), z(\bar s))|_{L^2} \neq 0$, let us show that

$$z'(\bar s) \in \operatorname{argmin} \{ \partial_v \mathcal{F}(t(\bar s), w(\bar s), z(\bar s))[\xi] : \xi \in \Xi,\ \|\xi\|_{L^2} \le 1 \}.$$

In order to apply the chain rule (80) we will show first that $z \in W^{1,2}(s_1, s_2; H^1)$ for $\bar s \in (s_1, s_2)$. By the lower semi-continuity of the slope (cf. Lemma 2.2) for $\delta \ll 1$ it holds $|\partial_v^- \mathcal{F}(t, w, z)|_{L^2} \ge C > 0$ for

$$(t, z) \in I_\delta \times B_\delta = \{ |t - t(\bar s)| \le \delta \} \times \{ \|z - z(\bar s)\|_{H^1} \le \delta \}$$

and $w \in \operatorname{argmin} \{ \mathcal{F}(t, \cdot, z) \}$. Since $s \mapsto (t(s), z(s))$ is continuous in $[0,T] \times H^1$ and since t_ε and z_ε converge locally uniformly (cf. Corollaries 5.5 and 5.7) there exists $s_1 < s_2$ such that both $(t(s), z(s)) \in I_\delta \times B_\delta$ and $(t_\varepsilon(s), z_\varepsilon(s)) \in I_\delta \times B_\delta$ for $s \in [s_1, s_2]$. Thus,

$$|\partial_v^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2} \ge C > 0 \qquad \text{for } s \in [s_1, s_2].$$

In other terms, let $t_1^\varepsilon = t^\varepsilon(s_1)$ and $t_2^\varepsilon = t^\varepsilon(s_2)$. Then we have $s^\varepsilon(t) \in [s_1, s_2]$ for $t \in [t_1^\varepsilon, t_2^\varepsilon]$ and

$$|\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2} \ge C > 0 \qquad \text{for } t \in [t_1^\varepsilon, t_2^\varepsilon].$$

Remember that $z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s)$ and that $(t^\varepsilon)'(s) = 1/\dot s^\varepsilon(t^\varepsilon(s))$ (being t^ε the inverse of s^ε); then, by the change of variable $s = s^\varepsilon(t)$ we get

$$\begin{aligned} \int_{s_1}^{s_2} \|z_\varepsilon'(s)\|_{H^1}^2\, ds &= \int_{s_1}^{s_2} \|\dot v_\varepsilon(t^\varepsilon(s))\|_{H^1}^2 |(t^\varepsilon)'(s)|^2\, ds \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|_{H^1}^2}{\dot s^\varepsilon(t)}\, dt \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|_{H^1}^2}{\varepsilon + \|\dot v_\varepsilon(t)\|_{L^2}}\, dt \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|_{H^1}^2}{\varepsilon + |\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)|_{L^2}}\, dr \\ &\le \frac{1}{\varepsilon + C} \int_{t_1^\varepsilon}^{t_2^\varepsilon} \|\dot v_\varepsilon(t)\|_{H^1}^2 \le +\infty, \end{aligned}$$

where the last bound follow from Corollary 5.2. Thus, z_ε and its limit z belong to $W^{1,2}(s_1, s_2; H^1)$.

Hence, by the chain rule

$$\mathcal{F}'(t(s), w(s), z(s)) = \partial_z \mathcal{F}(t(s), w(s), z(s))[z'(s)] + \partial_t \mathcal{F}(t(s), w(s), z(s))\, t'(s) \tag{72}$$

for a.e. in $s \in (s_1, s_2)$. On the other hand, by Theorem 5.6 for a.e. $s \in (s_1, s_2)$ it holds

$$\mathcal{F}'(t(s), w(s), z(s)) = -|\partial_v \mathcal{F}(t(s), w(s), z(s)|_{L^2} + \partial_t \mathcal{F}(t(s), w(s), z(s))\, t'(s).$$

Hence,

$$\partial_z \mathcal{F}(t(s), w(s), z(s))[z'(s)] = -|\partial_v \mathcal{F}(t(s), w(s), z(s))|_{L^2}.$$

Therefore $z'(s) \in \operatorname{argmin} \{\partial_v \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi,\ \|\xi\|_{L^2} \le 1\}$. ■

Corollary 5.10 *Let (t, w, z) be a parametrized evolution (provided by Theorem 5.6) then for a.e. $s \in [0, S]$ we have*

- *if $t'(s) > 0$ then*

$$\begin{cases} \big[z(s)W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)\big]^+ = 0 \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde{w}(s))\big) = 0, \end{cases} \tag{73}$$

- *if $t(s) = t_d$ in $(s^\flat, s^\sharp)$ then*

$$\begin{cases} \lambda(s) z'(s) = -\big[z(s)W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)\big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde{w}(s))\big) = 0, \end{cases} \tag{74}$$

  where $\lambda(s) = \|[z(s)W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)]^+\|_{L^2}$.

Remember that the first case corresponds to a continuity point in time, the second describes instead the "instantaneous evolution" in the discontinuity point t_d. As in (37) the first equation in both the previous systems holds in $L^2(\Omega)$ while the second holds in $H^{-1}(\Omega, \mathbb{R}^2)$.

Proof. If $t'(s) > 0$ then by Proposition 5.9 $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0$, i.e.

$$\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] \ge 0 \quad \text{for every } \xi \in H^1 \text{ with } \xi \le 0.$$

In other terms, $\partial_z \mathcal{F}(t(s), w(s), z(s))$ is a negative Radon measure or, equivalently, a Radon measure μ with $\mu^+ = 0$. As in (38), writing $\partial_v \mathcal{F}(t(s), w(s), z(s))$ in the sense of distributions yields (73).

By Proposition 5.9 we know that $z'(s) \in \operatorname{argmin} \{\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi,\ \|\xi\|_{L^2} \le 1\}$ and thus by Lemma A.4 we get (74). ■

A Some Lemmas

A.1 Discrete Gronwall

First of all let us provide the Gronwall estimate to be used in the proof of Theorem 5.1. It's proof originates from [33] and [22].

Lemma A.1 *Let $\gamma > 0$, $a_k, b_k, c_k \ge 0$ and $a_0 = 0$ such that*

$$a_{k+1}^2 - a_k^2 \le -\tau\gamma a_{k+1}^2 + \tau a_{k+1} b_{k+1} + \tau c_{k+1}^2 \quad \text{for } k \in \mathbb{N}. \tag{75}$$

Denote $t_k = k\tau$ for $k \in \mathbb{N}$. Then for $0 < \beta < \gamma/2$ and $\tau \ll 1$ it holds

$$a_k \le \left(\sum_{i=0}^{k} \tau e^{-2\beta(t_k - t_i)} c_i^2 \right)^{1/2} + \sum_{i=0}^{k} \tau e^{-\beta(t_k - t_i)} b_i \quad \text{for } k \in \mathbb{N}.$$

Proof. for $\lambda = (1+\tau\gamma)^{1/2}$ and for $\tau < 1$ let us re-write (75) as $\lambda^2 a_{k+1}^2 - a_k^2 - a_{k+1}b_{k+1} \le c_{k+1}^2$. Denote

$$A_k = \lambda^{-k}(C_k + B_k)\,, \qquad C_k = \left(\sum_{i=0}^{k} \lambda^{2i} c_i^2\right)^{1/2}, \qquad B_k = \sum_{i=0}^{k} \lambda^i b_i\,.$$

Let us show that A_k satisfies

$$\lambda^2 A_{k+1}^2 - A_k^2 - A_{k+1}b_{k+1} \ge c_{k+1}^2 \tag{76}$$

In terms of C_k and B_k, the left hand side reads

$$\lambda^{-2(k+1)+2}\big(C_{k+1}^2 + B_{k+1}^2 + 2C_{k+1}B_{k+1}\big) - \lambda^{-2k}\big(C_k^2 + B_k^2 + 2C_kB_k\big) - \lambda^{-(k+1)}\big(C_{k+1} + B_{k+1}\big)b_{k+1}$$

Let us see that (76) holds. First, since $\lambda > 1$

$$\lambda^{-2k}C_{k+1}^2 - \lambda^{-2k}C_K^2 = \lambda^{-2k}(C_{k+1}^2 - C_k^2) \ge \lambda^{-2k+2(k+1)}c_{k+1}^2 \ge c_{k+1}^2.$$

Next,

$$\begin{aligned}\lambda^{-2k}B_{k+1}^2 \;-\; &\lambda^{-2k}B_k^2 - \lambda^{-(k+1)}B_{k+1}b_{k+1} = \\ &= \lambda^{-2k}(B_k + \lambda^{k+1}b_{k+1})^2 - \lambda^{-2k}B_k^2 - \lambda^{-(k+1)}(B_k + \lambda^{k+1}b_{k+1})b_{k+1} \\ &= \big(\lambda^{-2k+2(k+1)} - 1\big)b_{k+1}^2 + \big(2\lambda^{-2k+(k+1)} - \lambda^{-(k+1)}\big)B_kb_{k+1} \ge 0,\end{aligned}$$

where the last inequality follows again from $\lambda > 1$. Finally,

$$\begin{aligned}2\lambda^{-2k}C_{k+1}\big(B_k + \lambda^{k+1}b_{k+1}\big) \;-\; 2\lambda^{-2k}C_kB_k - \lambda^{-(k+1)}C_{k+1}b_{k+1} = \\ = 2\lambda^{-2k}\big(C_{k+1} - C_k\big)B_k + \big(2\lambda^{-2k+(k+1)} - \lambda^{-(k+1)}\big)C_{k+1}b_{k+1} \ge 0,\end{aligned}$$

again because $\lambda > 1$.

Since $\lambda^2 a_{k+1}^2 - a_k^2 - a_{k+1}b_{k+1} \le c_{k+1}^2$ and $a_{k+1} \ge 0$ we get

$$a_{k+1} \le \frac{1}{2\lambda^2}\left(b_{k+1} + \sqrt{b_{k+1}^2 + 4\lambda^2(a_k^2 + c_{k+1}^2)}\right).$$

In the same way

$$A_{k+1} \ge \frac{1}{2\lambda^2}\left(b_{k+1} + \sqrt{b_{k+1}^2 + 4\lambda^2(A_k^2 + c_{k+1}^2)}\right).$$

Hence by induction $a_k \le A_k$ for every $k \in \mathbb{N}$, i.e.

$$a_k \le \left(\sum_{i=0}^{k} \tau\lambda^{2(i-k)}c_i^2\right)^{1/2} + \sum_{i=0}^{k} \tau\lambda^{i-k}b_i\,.$$

Finally, it is not hard to check that for $0 < \beta < \gamma/2$ and $0 < \tau \ll 1$ it holds

$$\lambda^{-1} = (1+\tau\gamma)^{-1/2} \le 1 - \beta\tau.$$

Hence, for $t_k = k\tau$ we have

$$\lambda^{(i-k)} = \lambda^{-(k-i)} \le (1-\beta\tau)^{(k-i)} = e^{(k-i)\ln(1-\beta\tau)} \le e^{-\beta(k-i)\tau} = e^{-\beta(t_k - t_i)}.$$

Then

$$a_k \le \left(\sum_{i=0}^{k} \tau e^{-2\beta(t_k-t_i)}c_i^2\right)^{1/2} + \sum_{i=0}^{k} \tau e^{-\beta(t_k-t_i)}b_i\,.$$

which concludes the proof. ■

A.2 Representation of linear functionals

We provide here a couple of representations, to be used in Theorem 3.9 and in Corollary 5.10. The first, related to unilateral gradient flows, is already stated, without proof, in [17]. The second follows from [13, Lemma 4.4]. In the next Lemmas we assume that Ω is an open, bounded set in $\mathbb{R}^n$.

Lemma A.2 *Let $\zeta \in H^{-1}(\Omega)$. If*

$$\sup \big\{ \langle \zeta, \xi \rangle : \xi \in H_0^1(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \big\} < +\infty \tag{77}$$

then ζ is a (locally finite) Radon measure whose positive part belongs to $L^2(\Omega)$.

Proof. We introduce the indicator functions $I_B, I_+ : H_0^1 \to [0, +\infty]$ given by

$$I_B(\xi) = \begin{cases} 0 & \|\xi\|_{L^2} \leq 1 \\ +\infty & \text{otherwise,} \end{cases} \qquad I_+(\xi) = \begin{cases} 0 & \xi \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

By (77)

$$\sup_{\xi \in H_0^1} \langle \zeta, \xi \rangle - \big(I_+(\xi) + I_B(\xi) \big) < +\infty.$$

In other terms, ζ belongs to the proper domain of the Legendre transform $(I_+ + I_B)^*$ in H^{-1}. In order to characterize the proper domain, let us write by inf-convolution, e.g. §15.1 in [8],

$$(I_+ + I_B)^*(\zeta) = \min_{\varphi \in H^{-1}} I_+^*(\varphi) + I_B^*(\zeta - \varphi).$$

Clearly, if $(I_+ + I_B)^*(\zeta) < +\infty$ there exists $\mu \in H^{-1}$ such that $I_+^*(\mu) + I_B^*(\zeta - \mu) < +\infty$ and hence $I_+^*(\mu) < +\infty$ and $I_B^*(\zeta - \mu) < +\infty$. Choosing $\xi = \lambda \hat{\xi}$, for $\lambda \geq 0$ and $\hat{\xi} \geq 0$, yields

$$\lambda \langle \mu, \hat{\xi} \rangle \leq \sup\{ \langle \mu, \xi \rangle : \xi \in H_0^1,\, \xi \geq 0 \} = I_+^*(\mu) < +\infty \quad \text{for every } \lambda \geq 0,$$

thus $\langle \mu, \hat{\xi} \rangle \leq 0$ for every $\hat{\xi} \geq 0$ in H_0^1. By Riesz-Markov Theorem it follows that μ is a negative Radon measure. Further, since

$$I_B^*(\zeta - \mu) = \sup\{ \langle \zeta - \mu, \xi \rangle : \xi \in H_0^1,\, \|\xi\|_{L^2} \leq 1 \} < +\infty,$$

the functional $\zeta - \mu$ can be extended from H_0^1 to the whole L^2 (by Hahn-Banach Theorem) and thus it can be represented as an element $f \in L^2$ (by Riesz's representation Theorem). In summary, we write $\zeta = \mu + f\mathcal{L}$, where μ is a negative Radon measure, f is an L^2-function and $\mathcal{L}$ is the Lebesgue measure. Write $\mu = \mu_{ac} + \mu_s$ where μ_{ac} and μ_s are, respectively, absolutely continuous and singular with respect to $\mathcal{L}$. Then $\mu_{ac} = -m\mathcal{L}$ (by Radon-Nikodym Theorem) where $m \in L^1$ and $m \geq 0$. Hence

$$\zeta^+ = (f - m)^+ \mathcal{L} + \mu_s^+ = (f - m)^+ \mathcal{L} = (f - m) \mathcal{L}_{|A}$$

where $A = \{f - m \geq 0\}$. In A we have $f \geq m \geq 0$ and thus $m \in L^2(A)$. It follows that $\zeta^+ \in L^2$. ■

Lemma A.3 *Let $\zeta \in (H^1(\Omega))^*$. If*

$$\sup \big\{ \langle \zeta, \xi \rangle : \xi \in H^1(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \big\} < +\infty \tag{78}$$

then ζ is a finite Radon measure whose positive part belongs to $L^2(\Omega)$. Moreover

$$\sup \big\{ \langle \zeta, \xi \rangle : \xi \in H^1(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \big\} = \|\zeta^+\|_{L^2}. \tag{79}$$

Corollary A.4 *Let $\zeta \in (H^1(\Omega))^*$ and let*

$$\xi_M \in \operatorname{argmax} \left\{ \langle \zeta, \xi \rangle : \xi \in H^1(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \right\},$$

then ζ is a finite Radon measure and $\zeta^+ = \xi_M \, \|\zeta^+\|_{L^2}$. In particular, the positive part ζ^+ belongs to $H^1(\Omega)$.

Proof. If $\zeta = 0$ there is nothing to prove because the identity $\zeta^+ = \xi_M \|\zeta^+\|_{L^2}$ becomes trivial. Otherwise, since $\xi \geq 0$ we have, by the previous Lemma and by density of smooth functions

$$\begin{aligned} \|\zeta^+\|_{L^2} = \langle \zeta, \xi_M \rangle &= \sup \left\{ \langle \zeta, \xi \rangle : \xi \in H^1(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \right\} \\ &= \sup \{ \langle \zeta, \xi \rangle : \xi \in C^\infty,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \} \\ &\leq \sup \{ \langle \zeta^+, \xi \rangle : \xi \in L^2,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \} = \|\zeta^+\|_{L^2}. \end{aligned}$$

It is now enough to note that $\xi = \zeta^+ / \|\zeta^+\|_{L^2}$ is the unique maximizer in $L^2(\Omega)$. ∎

A.3 Continuous dependence and differentiability

Finally, we collect, for the readers convenience, few results from [22] adapted to our notation and framework; the first follows from [22, Lemma 2.2] (which in turn is based on a general regularity result proved in [20, Theorem 1.1]), the second from [22, Lemma 2.4] while the last from [22, Lemma 2.3].

Lemma A.5 *Let $g \in C^1([0,T]; W^{1,\bar{p}}(\Omega, \mathbb{R}^2))$ for $\bar{p} > 2$. For $t \in [0,T]$ and $v \in \mathcal{V}$ denote $u(t,v) = \operatorname{argmin}\{\mathcal{F}(t, \cdot, v) : u \in \mathcal{U}\}$. There exists $C > 0$ and $2 < \tilde{p} < \bar{p}$ such that: for every $2 \leq p < \tilde{p}$, every $t_1, t_2 \in [0,T]$ and every $v_1, v_2 \in \mathcal{V}$ it holds*

$$\|u(t_2, v_2) - u(t_1, v_1)\|_{W^{1,p}} \leq C \big(\|g(t_2) - g(t_1)\|_{W^{1,p}} + \|g\|_{L^\infty(0,T; W^{1,p})} \, \|v_2 - v_1\|_{L^q} \big)$$

where $1/q = 1/p - 1/\tilde{p}$. We remark that $C > 0$ depends only on the linear elastic density W, on $\eta > 0$ and on Ω; in particular, it is independent of the boundary condition.

Lemma A.6 *If $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ then $\mathcal{F}(t, u, \cdot)$ is Gateaux differentiable (with respect to $H^1(\Omega)$) and*

$$\partial_v \mathcal{F}(t, u, v)[\xi] = 2 \int_\Omega v \xi \, W(Du + Dg(t)) \, dx + G_c \int_\Omega (v - 1)\xi + \nabla v \cdot \nabla \xi \, dx \qquad \forall \, \xi \in H^1(\Omega).$$

Note that the above integrals makes sense thanks to the fact that, for $\Omega \subset \mathbb{R}^2$, $\xi \in L^q$ for any $1 \leq q < +\infty$ while, by assumption, $W(Du + Dg(t)) \in L^p$ for some $p > 1$.

Lemma A.7 *If $v \in W^{1,2}(0,T; H^1)$ and $u(t) \in \operatorname{argmin}\{\mathcal{F}(t, u, v(t)) : u \in \mathcal{U}\}$ then the energy $t \mapsto \mathcal{F}(t, u(t), v(t))$ is a.e. differentiable in $(0,T)$ and the following chain rule holds:*

$$\dot{\mathcal{F}}(t, u(t), v(t)) = \partial_t \mathcal{F}(t, u(t), v(t)) + \partial_v \mathcal{F}(t, u(t), v(t)) \, [\dot{v}(t)] \, . \tag{80}$$

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References

[1] A.Abdollahi and I.Arias. Phase-field modeling of crack propagation in piezoelectric and ferroelectric materials with different electromechanical crack conditions. *J. Mech. Phys. Solids*, 60(12):2100–2126, 2012.

[2] R. Alessi, J.-J. Marigo, and S. Vidoli. Gradient damage models coupled with plasticity and nucleation of cohesive cracks. *Arch. Ration. Mech. Anal.*, 214(2):575–615, 2014.

[3] M. Ambati, T. Gerasimov, and L. De Lorenzis. A review on phase-field models of brittle fracture and a new fast hybrid formulation. *Comp. Mech.*, 55(2):383–405, 2015.

[4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[5] L. Ambrosio and V.M. Tortorelli. On the approximation of free discontinuity problems. *Boll. Un. Mat. Ital. B (7)*, 6(1):105–123, 1992.

[6] M. Artina, M. Fornasier, S. Micheletti, and S. Perotto. Anisotropic mesh adaptation for crack detection in brittle materials. *SIAM J. Sci. Comput.*, 37(4):633–659, 2015.

[7] J.-F. Babadjian and V. Millot. Unilateral gradient flow of the Ambrosio-Tortorelli functional by minimizing movements. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(4):779–822, 2014.

[8] H.H. Bauschke and P.L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[9] E. Bonetti, F. Freddi, and A. Segatti. An existence result for a model of complete damage in elastic materials with reversible evolution. *Contin. Mech. Thermodyn.*, 29(1):31–50, 2017.

[10] M.J. Borden, C.V. Verhoosel, M.A. Scott, T.J.R. Hughes, and C.M. Landis. A phase-field description of dynamic brittle fracture. *Comput. Methods Appl. Mech. Eng.*, 217:77–95, 2011.

[11] B. Bourdin, G.A. Francfort, and J.-J. Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids*, 48(4):797–826, 2000.

[12] A. Chambolle. An approximation result for special functions with bounded deformation. *J. Math. Pures Appl. (9)*, 83(7):929–954, 2004.

[13] V. Crismale and G. Lazzaroni. Viscous approximation of quasistatic evolutions for a coupled elastoplastic-damage model. *Calc. Var. Partial Differential Equations*, 55(1): art. 17, 2016.

[14] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences.* Springer-Verlag, Berlin, 1989.

[15] G. Dal Maso, G.A. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, 176(2):165–225, 2005.

[16] Iurlano F. A density result for gsbd and its application to the approximation of brittle fracture energies. *Calc. Var. Partial Differential Equation*, 51:315–342, 2014.

[17] U. Gianazza and G. Savaré. Some results on minimizing movements. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 18:57–80, 1994.

[18] H. Hahn. Über Annäherung an Lebesguesche Integrale durch Riemannsche Summen. *Wien. Ber.*, 123:713–743, 1914.

[19] V. Hakim and A. Karma. Laws of crack motion and phase-field models of fracture. *J. Mech. Phys. Solids*, 57(2):342–368, 2009.

[20] R. Herzog, C. Meyer, and G. Wachsmuth. Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions. *J. Math. Anal. Appl.*, 382(2):802–813, 2011.

[21] D. Knees and M. Negri. Convergence of alternate minimization schemes for phase-field fracture and damage. *Math. Models Methods Appl. Sci.*, (to appear).

[22] D. Knees, R. Rossi, and C. Zanini. A vanishing viscosity approach to a rate-independent damage model. *Math. Models Methods Appl. Sci.*, 23(4):565–616, 2013.

[23] C. Kuhn and R. Müller. A continuum phase field model for fracture. *Engineering Fracture Mechanics*, 77(18):3625–3634, 2010.

[24] C.J. Larsen, C. Ortner, and E. Süli. Existence of solutions to a regularized model of dynamic fracture. *Math. Models Methods Appl. Sci.*, 20(7):1021–1048, 2010.

[25] B. Li, C. Peco, D. Millán, I. Arias, and M. Arroyo. Phase-field modeling and simulation of fracture in brittle materials with strongly anisotropic surface energy. *International Journal for Numerical Methods in Engineering*, 102(3-4):711–727, 2015.

[26] C. Miehe, F. Welschinger, and M. Hofacker. Thermodynamically consistent phase-field models of fracture: variational principles and multi-field FE implementations. *Internat. J. Numer. Methods Engrg.*, 83(10):1273–1311, 2010.

[27] A. Mielke. Differential, energetic, and metric formulations for rate-independent processes. In *Nonlinear PDE's and applications*, volume 2028 of *Lecture Notes in Math.*, pages 87–170. Springer, Heidelberg, 2011.

[28] A. Mielke, R. Rossi, and G. Savaré. BV solutions and viscosity approximations of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 18(1):36–80, 2012.

[29] A. Mielke, R. Rossi, and G. Savaré. Balanced viscosity (BV) solutions to infinite-dimensional rate-independent systems. *J. Eur. Math. Soc. (JEMS)*, 18(9):2107–2165, 2016.

[30] A. Mielke and T. Roubíček. *Rate-Independent Systems: Theory and Application*, volume 193 of *Applied Mathematical Sciences*. Springer, New York, 2015.

[31] A. Mielke and S. Zelik. On the vanishing-viscosity limit in parabolic systems with rate-independent dissipation terms. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(1):67–135, 2014.

[32] M. Negri. Quasi-static rate-independent evolutions: characterization, existence, approximation and application to fracture mechanics. *ESAIM Control Optim. Calc. Var.*, 20(4):983–1008, 2014.

[33] R.H. Nochetto, G. Savaré, and C. Verdi. A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations. *Comm. Pure Appl. Math.*, 53(5):525–589, 2000.

[34] K. Pham, H. Amor, J.J. Marigo, and C. Maurini. Gradient damage models and their use to approximate brittle fracture. *International Journal of Damage Mechanics*, 20(4):618–652, 2011.

[35] T. Takaishi and M. Kimura. Phase field model for mode III crack growth in two dimensional elasticity. *Kybernetika*, 45(4):605–614, 2009.