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Sobolev spaces on non-Lipschitz sets with application to BIEs on fractal screens

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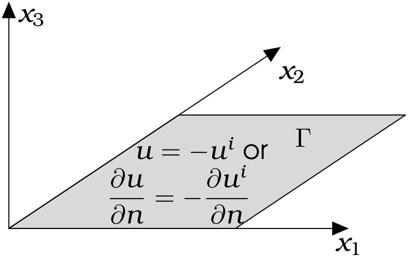
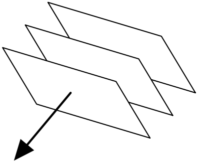
Motivation: acoustic scattering by fractal screen

Γ bounded open subset of $\{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{R}^n$, $n = 1, 2$

$$(\Delta + k^2)u = 0$$

$$D := \mathbb{R}^{n+1} \setminus \{\overline{\Gamma} \times \{0\}\}$$





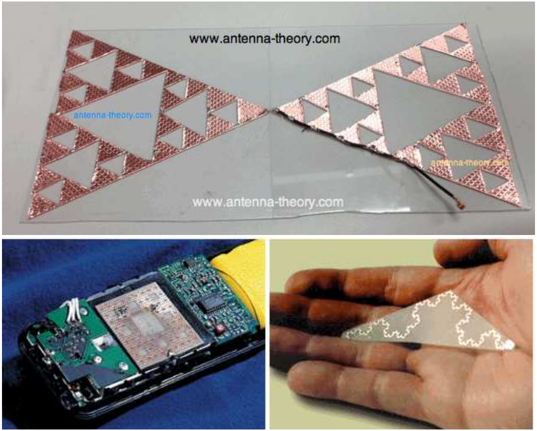
$$u^i = \sum_{j=1}^N a_j \mathrm{e}^{\mathrm{i}k\mathbf{d}_j\cdot\mathbf{x}}$$

$$|\mathbf{d}_j| = 1, k > 0$$

u satisfies Sommerfeld radiation condition (SRC) at infinity
(i.e. $\partial_r u - iku = o\left(r^{-(n-1)/2}\right)$ uniformly as $r = |\mathbf{x}| \to \infty$).

Classical problem when Γ is Lipschitz.
What happens for arbitrary (e.g. fractal) Γ?

Fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

Fractal antennas are a popular topic in engineering:
Wideband/multiband, compact, cheap, metamaterials, cloaking. . .
Not analysed by mathematicians.

Example: Dirichlet scattering

Given $g_D \in H^{1/2}(\Gamma)$, we can write 3 different BVPs:
Find $u \in C^2(D) \cap \{u \in L^2_{\mathrm{loc}}(D), \nabla u \in L^2_{\mathrm{loc}}(D)^n\}$ s.t.

P $\begin{cases} \Delta u + k^2 u = 0 & \text{in } D \\ \gamma^\pm(u)|_\Gamma = g_D \\ \text{SRC} \end{cases}$

P′ $\begin{cases} \text{P} \\ [u] = \gamma^+ u - \gamma^- u = 0 \end{cases}$

P″ $\begin{cases} \text{P}' \\ [\partial u/\partial x_n] \in \overline{\mathcal{D}(\Gamma)}^{H^{1/2}(\mathbb{R}^n)} \end{cases}$

P $\Rightarrow$ P′ if $\left\{ {u \in H^{1/2}(\mathbb{R}^n) \atop \operatorname{supp} u \subset \partial\Gamma} \right\} = \{0\}$

P′ $\Rightarrow$ P″ if $\mathcal{D}(\Gamma)$ dense in $\left\{ {u \in H^{-1/2}(\mathbb{R}^n) \atop \operatorname{supp} u \subset \overline{\Gamma}} \right\}$

P″ $\Rightarrow$ P′ $\Rightarrow$ P

Jump $[u]$ is supported in $\overline{\Gamma}$, while $\gamma^\pm$ is restricted to Γ only.
P′ is equivalent to BIE ($-S_k \phi = g_D$, S_k coercive in $\overline{\mathcal{D}(\Gamma)}^{H^{-1/2}}$).
P″ is uniquely solvable.
P $\iff$ P′ $\iff$ P″ holds if Γ is C^0.
Swap $\pm 1/2$ for Neumann problem. (Chandler–Wilde, Hewett 2013)

Questions

- For which Γ and s are conditions above satisfied?
  $\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \partial\Gamma\} = \{0\}$,
  $\mathcal{D}(\Gamma)$ dense in $\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Gamma}\}$
- Given Γ, which other $\widetilde{\Gamma} \neq \Gamma$ give the same scattered fields $\forall u^i$?
- When is Γ "inaudible", i.e. $u^s = 0$ for all g_D?
  Can a screen with zero mass scatter waves?
- Does it matter whether Γ is open or closed?
- Where do Galerkin (BEM) solutions converge to?

To try to answer these questions we need to learn more about Sobolev spaces on non-Lipschitz sets.

Many results available (Maz'ya, Triebel, Polking, Adams, Hedberg,...) but not entirely clear/satisfactory/useful for us.

Part II

Definitions and duality

Basic definitions I: Sobolev spaces on $\mathbb{R}^n$

For $k \in \mathbb{N}_0$, $$W^k := \{u \in L^2(\mathbb{R}^n) : \partial^{\boldsymbol{\alpha}} u \in L^2(\mathbb{R}^n),\ \forall |\boldsymbol{\alpha}| \le k\},$$
$$\|u\|^2_{W^k} := \sum_{|\boldsymbol{\alpha}|\le k} \int_{\mathbb{R}^n} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}.$$

For $s \in \mathbb{R}$, $$H^s := \{u \in \mathcal{S}^*(\mathbb{R}^n) : \hat{u} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \|u\|_{H^s} < \infty\},$$
$$\|u\|^2_{H^s} := \int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)^s \, |\widehat{u}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi}.$$

- For $k \in \mathbb{N}_0$, $H^k = W^k$ with equivalent norms.
- For $t > s$, $H^t \subset H^s$ (continuous embedding, norm 1).
- $(H^s)^* = H^{-s}$, with duality pairing

$$\langle u, v\rangle_{H^{-s}\times H^s} := \int_{\mathbb{R}^n} \hat{u}(\boldsymbol{\xi})\overline{\hat{v}(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}.$$

- $H^s \subset C(\mathbb{R}^n)$ for $s > n/2$ (Sobolev embedding theorem).
  $\delta_{\mathbf{x}_0} \in H^s \iff s < -n/2$ $(\langle \delta_{\mathbf{x}_0}, \phi\rangle = \overline{\phi(\mathbf{x}_0)})$.

Basic definitions II: Sobolev sp. on subsets of $\mathbb{R}^n$

Notation: $\Gamma \subset \mathbb{R}^n$ open, $F \subset \mathbb{R}^n$ closed, $K \subset \mathbb{R}^n$ compact.

$$\widetilde{H}^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s} \qquad\qquad (\mathcal{D}(\Gamma) := C_0^\infty(\Gamma) \subset C^\infty(\mathbb{R}^n))$$

$$H_F^s := \{u \in H^s : \operatorname{supp} u \subset F\} = \{u \in H^s : u(\varphi) = 0 \ \forall \varphi \in \mathcal{D}(F^c)\}$$

$$H^s(\Gamma) := \{u|_\Gamma : u \in H^s\}$$

$$H_0^s(\Gamma) := \overline{\mathcal{D}(\Gamma)|_\Gamma}^{H^s(\Gamma)} \qquad\qquad \text{(notation from McLean)}$$

"Global" and "local" spaces:

$$\widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}} \subset H^s \subset \mathcal{D}^*(\mathbb{R}^n), \qquad\qquad H_0^s(\Gamma) \subset H^s(\Gamma) \subset \mathcal{D}^*(\Gamma)$$

When Γ is Lipschitz it holds that

- $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$
- $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$
- $H^{\pm 1/2}_{\partial\Gamma} = \{0\}$
- $\{H^s(\Gamma)\}_{s\in\mathbb{R}}$ and $\{\widetilde{H}^s(\Gamma)\}_{s\in\mathbb{R}}$ are interpolation scales

Duality

Theorem (Chandler-Wilde and Hewett, 2013)

Let Γ be any open subset of $\mathbb{R}^n$ and let $s \in \mathbb{R}$.
Then $(H^s(\Gamma))^* = \widetilde{H}^{-s}(\Gamma)$ and $(\widetilde{H}^s(\Gamma))^* = H^{-s}(\Gamma)$ with equal norms and $\langle u, w\rangle_{H^{-s}(\Gamma)\times\widetilde{H}^s(\Gamma)} = \langle U, w\rangle_{H^{-s}\times H^s}$ for any $U \in H^{-s}, U|_\Gamma = u$.

Well-known for Lipschitz but not in general case.

Main ideas of proof:

- H Hilbert, $V \subset H$ closed ssp, $\mathcal{H}$ unitary realisation of H^*, then $(V^{a,\mathcal{H}})^\perp = \{\psi \in \mathcal{H}, \langle \psi, \phi\rangle = 0 \ \forall \phi \in V\}^\perp$ is unitary realisation of V^*
- $H^{-s}_{\Gamma^c} = \{u \in H^{-s} : u(\psi) = \langle u, \psi\rangle = 0 \ \forall \psi \in \mathcal{D}(\Gamma)\} = (\widetilde{H}^s(\Gamma))^{a,H^{-s}}$
- Restriction operator $|_\Gamma$ is unitary isomorphism $|_\Gamma : (H^{-s}_{\Gamma^c})^\perp \to H^{-s}(\Gamma)$ (from identification of $H^{-s}(\Gamma)$ with $H^{-s}/H^{-s}_{\Gamma^c}$)
- Choose $V = \widetilde{H}^s(\Gamma)$, $H = H^s$, $\mathcal{H} = H^{-s}$

Sobolev space questions

We will address the following questions:

- When does $E \subset \mathbb{R}^n$ support non-zero $u \in H^s$?
- When is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?
- When is $H^s(\Gamma) = H^s_0(\Gamma)$?
- For which spaces is $|_\Gamma$ an isomorphism?
- When are $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ interpolation scales?
- What's the limit of a sequence of Galerkin solutions to a variational problem on prefractals?

Part III

s-nullity

s-nullity

Definition

Given $s \in \mathbb{R}$ we say that a set $E \subset \mathbb{R}^n$ is s-null if there are no non-zero elements of H^s supported in E.

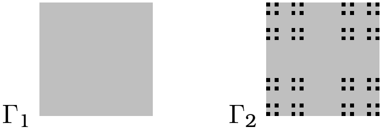
(I.e. if $H^s_F = \{0\}$ for every closed set $F \subset E$.)

Other terminology exists: "$(-s)$-polar" (Maz'ya, Littman), "set of uniqueness for H^s" (Maz'ya, Adams/Hedberg).

Relevance of s-nullity

For the screen scattering problem:

- For a compact screen K to be audible we need $H_K^{\pm 1/2} \neq \{0\}$.
- For the solution of the classical Dirichlet/Neumann BVP to be unique we need $H_{\partial\Gamma}^{\pm 1/2} = \{0\}$.
- Two screens Γ_1 and Γ_2 give the same scattered field for all incident waves if and only if $\Gamma_1 \ominus \Gamma_2$ is $\pm 1/2$-null.



For general Sobolev space results:

- $H^s_{F_1} = H^s_{F_2} \iff F_1 \ominus F_2$ is s-null.
- $\widetilde{H}^s(\Gamma_1) = \widetilde{H}^s(\Gamma_2) \iff \Gamma_1 \ominus \Gamma_2$ is $(-s)$-null.
- If $\mathrm{int}(\overline{\Gamma}) \setminus \Gamma$ is not $(-s)$-null then $\widetilde{H}^s(\Gamma) \subsetneqq H^s_{\overline{\Gamma}}$.
- We'll see many more uses of nullity…

s-nullity: basic results

- A subset of an s-null set is s-null.
- If E is s-null and $t > s$ then E is t-null.
- If E is s-null then has empty interior.
- If $s > n/2$ then E is s-null $\iff \operatorname{int}(E) = \emptyset$.
- For $s < -n/2$ there are no non-empty s-null sets.

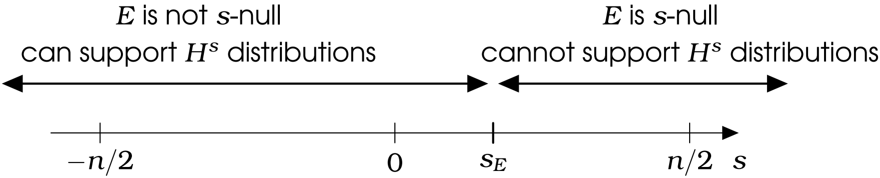
Non-trivial results:

- The union of finitely many s-null closed sets is s-null.
- The union of countably many s-null Borel sets is s-null if $s \leq 0$.

Union of non-closed s-null sets for $s > 0$ is not s-null:
counterexample is $E_1 = \mathbb{Q}^n, E_2 = \mathbb{R}^n \setminus \mathbb{Q}^n, s > n/2$.

Nullity threshold

For every $E \subset \mathbb{R}^n$ with $\mathrm{int}(E) = \emptyset$ there exists $s_E \in [-n/2, n/2]$ such that E is s-null for $s > s_E$ and not s-null for $s < s_E$.
We call s_E the nullity threshold of E.



Q1: Given $E \subset \mathbb{R}^n$, can we determine s_E?

Q2: Given $s \in [-n/2, n/2]$, can we find some $E \subset \mathbb{R}^n$ for which $s_E = s$?

Q3: When is E s_E-null? (i.e. is the maximum regularity attained?)

We study separately sets with zero and positive Lebesgue measure.

Zero Lebesgue measure $\Rightarrow s_K \in [-n/2, 0]$

Let $K \subset \mathbb{R}^n$ be non-empty and compact. Then:

- $H^0_K = L^2(K) = \{0\} \iff m(K) = 0$.
- If $m(K) = 0$ then $H^s_K = \{0\}$ for $s \geq 0$ (i.e. $s_K \leq 0$).
- If K is countable then $s_K = -n/2$ $(H^s_K = \{0\} \Leftrightarrow s \geq -n/2)$.

Theorem

If $m(K) = 0$, then $s_K = \dfrac{\dim_{\mathrm{H}} K - n}{2}$.

$(\dim_{\mathrm{H}}$ =Hausdorff dimension, m =Lebesgue measure)

$$\dim_{\mathrm{H}} K = \inf\left\{d > 0 : H^{(d-n)/2}_K = \{0\}\right\}$$

This does not tell us if K is s_K-null; examples of both cases are possible.

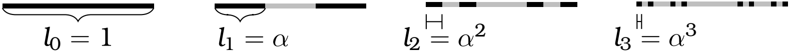
Sharpens previous results by Littman (1967) and Triebel (1997).

Examples

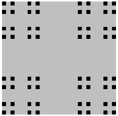
Let $\Gamma \subset \mathbb{R}^n$ be non-empty and open.

- If Γ is C^0 then $s_{\partial\Gamma} \in [-1/2, 0]$.
- If Γ is $C^{0,\alpha}$ for some $0 < \alpha < 1$ then $s_{\partial\Gamma} \in [-1/2, -\alpha/2]$ (sharp).
- If Γ is Lipschitz then $s_{\partial\Gamma} = -1/2$ (and $H^{-1/2}_{\partial\Gamma} = \{0\}$).
- If K is boundary of Koch snowflake, $s_K = \frac{\log 2}{\log 3} - 1 \approx -0.37$.

For $0 < \alpha < 1/2$ let $\mathcal{C}_\alpha \subset [0,1]$ be the Cantor set with $l_j = \alpha^j$, $j \in \mathbb{N}_0$:



Let $\mathcal{C}_\alpha^2 := \mathcal{C}_\alpha \times \mathcal{C}_\alpha \subset \mathbb{R}^2$ denote the associated "Cantor dust":



$$s_{\mathcal{C}_\alpha^n} = -\frac{n}{2}\left(1 + \frac{\log 2}{\log \alpha}\right) \in \left(-\frac{n}{2}, 0\right)$$

Choose $\alpha = 2^{-n/(2s+n)}$ to have $s_{\mathcal{C}_\alpha^n} = s$.

Can also define "thin" Cantor dusts which have $s_K = -n/2$

Capacity

Our proofs rely on the following equivalence, which follows from results by Grusin 1962, Littman 1967, Adams and Hedberg 1996 and Maz'ya 2011:

Theorem

For $s > 0$, K compact, $H_K^{-s} = \{0\} \iff \mathrm{cap}_s(K) = 0$, where

$$\mathrm{cap}_s(K) := \inf\{\|u\|_{H^s}^2 : u \in C_0^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } K\}.$$

This allows us to apply well-known results relating $\mathrm{cap}_s(E)$ to $\dim_{\mathrm{H}}(E)$ (see e.g. Adams and Hedberg 1996).
Requires relating different set capacities.

Positive Lebesgue measure $\Rightarrow s_K \in [0, n/2]$

Theorem (Polking, 1972)

There exists a compact set K for which $s_K = n/2$. Also, $H_K^{n/2} \neq \{0\}$.

Maximal nullity threshold is achieved.
Proof is constructive: "Swiss cheese set".
Also "open minus countable-dense" (e.g. $\mathbb{R}^n \setminus \mathbb{Q}^n$) are not $n/2$-null.

Open question: Do there exist sets K for which $s_K \in (0, n/2)$?

Our contribution:

Theorem

$\forall s_* \in (0, 1/2)$ the "fat" Cantor set $\mathcal{C}_{\alpha,\beta} \subset \mathbb{R}$ with

$$\alpha \in (0, 2^{-1/(1-2s_*)}), \quad \beta \in (0, 1-2\alpha), \quad l_j = \frac{1}{2^j}\left(1 - \beta\frac{1-(2\alpha)^j}{1-2\alpha}\right)$$

has nullity threshold $s_{\mathcal{C}_{\alpha,\beta}} \geq s_*$ (and $\chi_{\mathcal{C}_{\alpha,\beta}} \in H^{s_*}$).

Sketch of proof

Write $E = \mathcal{C}_{\alpha,\beta}$. Want $\|\chi_E\|_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s \, |\widehat{\chi_E}(\xi)|^2 \, \mathrm{d}\xi < \infty$.

Suffices to show $|\xi|^{2s} |\widehat{\chi_E}(\xi)|^2 = \mathcal{O}\left(|\xi|^{-(1+\varepsilon)}\right)$, $\xi \to \infty$.

Trick: $\widehat{\chi_E}(\xi) = \frac{1}{2}\left(1 - \mathrm{e}^{\mathrm{i}a\xi}\right)\widehat{\chi_E}(\xi) = \frac{1}{2}(\chi_E \widehat{-\chi}_{E+a})(\xi)$, $\quad a = \frac{\pi}{|\xi|}$.

$$\begin{aligned}\Rightarrow |\widehat{\chi_E}(\xi)| &\leq \frac{1}{2\sqrt{2\pi}} \|\chi_E - \chi_{E+a}\|_{L^1} \\ &\leq \frac{1}{2\sqrt{2\pi}} \left(\|\chi_E - \chi_{E_j}\|_{L^1} + \|\chi_{E_j} - \chi_{E_j+a}\|_{L^1} + \|\chi_{E_j+a} - \chi_{E+a}\|_{L^1} \right)\end{aligned}$$

where E_j is jth iteration in construction of E. Pick largest j such that $a = \frac{\pi}{|\xi|} \leq \mathrm{Gap}_j$ (smallest gap between subintervals of E_j). Find that

$$|\widehat{\chi_E}(\xi)| \leq C(\alpha,\beta) |\xi|^{-1-\frac{\log 2}{\log \alpha}}.$$

So $\chi_E = \chi_{\mathcal{C}_{\alpha,\beta}} \in H^s(\mathbb{R})$ for $s < \frac{1}{2} + \frac{\log 2}{\log \alpha} \in (-\frac{1}{2}, \frac{1}{2})$, thus $H^s_E \neq \{0\}$.

Open question: What is $s_{\mathcal{C}_{\alpha,\beta}}$ for the fat Cantor set?

Nullity of Cartesian products of sets

Let $N \in \mathbb{N}$, $s_j \in \mathbb{R}$, $n_j \in \mathbb{N}$, and $u_j \in H^{s_j}(\mathbb{R}^{n_j})$ for $j = 1, \dots, N$. Then

$$u_1 \otimes \cdots \otimes u_N \in H^s(\mathbb{R}^{n_1+\cdots+n_N}), \quad \text{for } s < s_* := \begin{cases} \min\limits_{j=1,\dots,N} s_j & \text{if } s_j \geq 0 \ \forall j, \\ \sum\limits_{j \text{ s.t. } s_j<0} s_j & \text{otherwise.} \end{cases}$$

If $u_1 \notin H^s(\mathbb{R}^{n_1})$, then $u_1 \otimes \cdots \otimes u_N \notin H^t(\mathbb{R}^{n_1+\cdots+n_N})$ for any $t > s$.

Let $n_1, n_2 \in \mathbb{N}$, and let $E_1 \subset \mathbb{R}^{n_1}$ and $E_2 \subset \mathbb{R}^{n_2}$ be Borel. Then

$$s_- \leq s_{E_1 \times E_2} \leq s_+, \quad \text{where}$$

$$s_- := \min\{s_{E_1},\ s_{E_2},\ s_{E_1} + s_{E_2}\},$$

$$s_+ := \begin{cases} \min\{s_{E_1},\ s_{E_2}\} & \text{if } m(E_1 \times E_2) = 0, \\ \min\{s_{E_1} + \frac{n_2}{2},\ s_{E_2} + \frac{n_1}{2}\} & \text{if } m(E_1 \times E_2) > 0. \end{cases}$$

$s_- \neq s_+$ is needed because s_{E_1}, s_{E_2} do not determine $s_{E_1 \times E_2}$:

$$\exists E_j \subset \mathbb{R} \text{ such that } s_{E_1} = s_{E_2} = s_{E_3} = s_{E_1 \times E_2} = -1/2 \neq s_{E_3 \times E_3} = -1.$$

Bessel potentials and the general case $1 < p < \infty$

Almost all of our results generalise to $1 < p < \infty$.

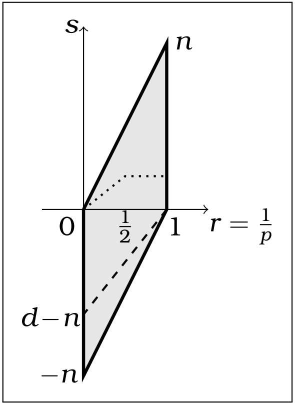
$$\begin{aligned}
H^{s,p} &:= \{u \in \mathcal{S}^*(\mathbb{R}^n) : \mathcal{J}_s u \in L^p(\mathbb{R}^n)\},\\
\|u\|_{H^{s,p}} &:= \|\mathcal{J}_s u\|_{L^p(\mathbb{R}^n)},\\
\mathcal{J}_s u(\mathbf{x}) &:= \mathcal{F}^{-1}\Big((1+|\boldsymbol{\xi}|^2)^{s/2}\hat{u}(\boldsymbol{\xi})\Big)(\mathbf{x}) = (J_s * u)(\mathbf{x}),
\end{aligned}$$

$$J_s(\mathbf{x}) = \begin{cases} \dfrac{2^{1+s/2}}{(2\pi)^{n/2}\Gamma(-s/2)} |\mathbf{x}|^{-\frac{n+s}{2}} K_{\frac{n+s}{2}}(|\mathbf{x}|), & s \neq 0, 2, 4, \dots, \\ (1-\Delta)^{s/2}\delta_0, & s = 0, 2, 4, \dots. \end{cases}$$

Special case of Triebel–Lizorkin $H^{s,p} = F^s_{p,2}$;
$H^{s,2} = H^s$, $H^{0,p} = L^p$.
$\mathcal{J}_s\mathcal{J}_t = \mathcal{J}_{s+t}$; $\quad \mathcal{J}_t : H^{s,p} \xrightarrow{\sim} H^{s-t,p}$.

(s, p)-nullity

We say that $E \subset \mathbb{R}^n$ is (s,p)-null if $H_F^{s,p} = \{0\}$ for any closed $F \subset E$.
Nullity threshold function: $s_E(r) = \inf\{s, E \text{ is } (s, 1/r)\text{-null}\}$.



The graph of $s_E(r)$ lies in the parallelogram $n(r-1) \leq s_E(r) \leq nr$.

s_E is Lipschitz with $0 \leq s_E'(r) \leq n$ a.e. $r \in (0,1)$.

If $m(E) = 0$ then $s_E(r) = (n - \dim_{\mathrm{H}} E)(r-1)$.

Using Cantor sets, we have obtained a sharp characterisation of all possible threshold-nullity behaviours when $m(K) = 0$:
if $E \subset \mathbb{R}^n$, $m(E) = 0$, then there exists a Cantor dust $\mathcal{C} \subset \mathbb{R}^n$ s.t. $\forall (s,p)$ E is (s,p)-null iff $\mathcal{C}$ is.

Not clear which s_E are actually achieved for $m(E) > 0$ (except $s_E(r) = nr$).

Part IV

Zero trace spaces

Comparison of the "zero trace" subspaces of $\mathbb{R}^n$

Recall definitions: for open $\Gamma \subset \mathbb{R}^n$

$$\widetilde{H}^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s} \qquad \widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}} \subset H^s$$

$$H^s_{\overline{\Gamma}} := \{u \in H^s : \operatorname{supp} u \subset \overline{\Gamma}\}$$

When is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?

Classical result (e.g. McLean)

Let $\Gamma \subset \mathbb{R}^n$ be C^0. Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

For smooth ($C^{k,1}$) domains and $s > 1/2$, $s - 1/2 \notin \mathbb{N}$, these spaces are kernel of trace operators.
Intuition fails for negative s: if $s < -n/2$, $\delta_{\mathbf{x}_0} \in \widetilde{H}^s(\Gamma)$ for any $\mathbf{x}_0 \in \partial\Gamma$.

Theorem (negative example)

For every $n \in \mathbb{N}$, there exists a bounded open set $\Gamma \subset \mathbb{R}^n$ such that,

$$\forall s \geq -n/2 \qquad \widetilde{H}^s(\Gamma) \subsetneqq H^s_{\overline{\Gamma}},$$

$$\forall s > 0 \qquad \widetilde{H}^s(\Gamma) \subsetneqq \{u \in H^s : u = 0 \text{ a.e. in } \Gamma^c\} \subsetneqq H^s_{\overline{\Gamma}}.$$

Set Γ constructed using Cantor and Polking sets.

Zero trace spaces and $\mathrm{int}(\overline{\Gamma}) \setminus \Gamma$

We consider two classes of open sets.
First, open Γ that is a "nice domain minus small holes".

Lemma

If $\mathrm{int}(\overline{\Gamma})$ *is* C^0 *then* $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}} \iff \mathrm{int}(\overline{\Gamma}) \setminus \Gamma$ *is* $(-s)$*-null.*
(Holds more generally for Γ *s.t.* $\widetilde{H}^s(\mathrm{int}(\overline{\Gamma})) = H^s_{\overline{\Gamma}}$*.)*

Suppose that $\Gamma \subsetneq \mathrm{int}(\overline{\Gamma})$ and that $\mathrm{int}(\overline{\Gamma})$ is C^0. Then

$\exists \widetilde{s}_\Gamma \in [-n/2, n/2]$ s.t. $\widetilde{H}^{s_-}(\Gamma) = H^{s_-}_{\overline{\Gamma}}, \quad \widetilde{H}^{s_+}(\Gamma) \subsetneq H^{s_+}_{\overline{\Gamma}} \quad \forall s_- < \widetilde{s}_\Gamma < s_+.$

If $m(\mathrm{int}(\overline{\Gamma}) \setminus \Gamma) = 0$ then $\widetilde{s}_\Gamma = \frac{n - \dim_{\mathrm{H}}(\mathrm{int}(\overline{\Gamma}) \setminus \Gamma)}{2}$.

If $\mathrm{int}(\overline{\Gamma}) \setminus \Gamma$ is a Lipschitz manifold, then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}} \iff s \leq 1/2$.
E.g. Γ a C^0 set minus a slit.

Sets with $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \le 1$

Second, we want to understand whether $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ for Γ "regular except at a few points", e.g. prefractal.

Theorem

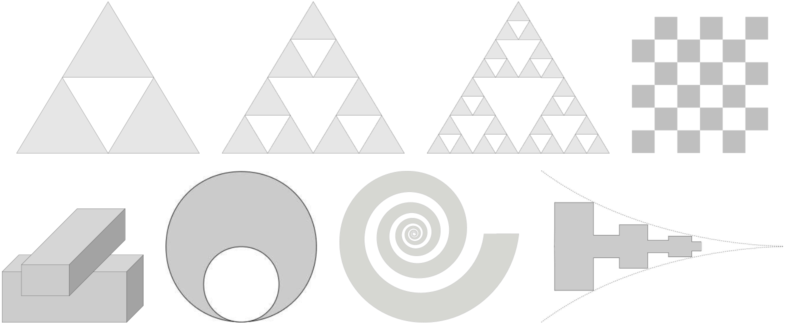
Fix $|s| \le 1$ if $n \ge 2$, $|s| \le 1/2$ if $n = 1$.
Let open $\Gamma \subset \mathbb{R}^n$ be C^0 except at $P \subset \partial\Gamma$, where P is closed, countable, with at most finitely many limit points in every bounded subset of $\partial\Gamma$.
Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

E.g. union of disjoint C^0 open sets, whose closures intersect only in P.

Proof uses sequence of special Tartar's cutoffs (for $n = 2$, easier for $n \ge 3$) for $s = 1$, then duality and interpolation.

Examples of sets with $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \le 1$

Examples of non-C^0 sets for which $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \le 1$:



Sierpinski triangle prefractal, (unbounded) checkerboard, double brick, inner and outer (double) curved cusps, spiral, Fraenkel's "rooms and passages".

Part V

Relations between different spaces

When is $H_0^s(\Gamma) = H^s(\Gamma)$?

What about relation between spaces with and without "zero trace"?

Recall: $H_0^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s(\Gamma)} \subset H^s(\Gamma) := \{u|_\Gamma : u \in H^s\} \subset \mathcal{D}^*(\Gamma)$.

Lemma

For open $\Gamma \subset \mathbb{R}^n$, $s \in \mathbb{R}$, $H_0^s(\Gamma) = H^s(\Gamma) \iff \widetilde{H}^{-s}(\Gamma) \cap H_{\partial\Gamma}^{-s} = \{0\}$.

Corollary

For any open $\emptyset \neq \Gamma \subsetneq \mathbb{R}^n$, *there exists* $0 \le s_0(\Gamma) \le n/2$ *such that*

$H_0^{s_-}(\Gamma) = H^{s_-}(\Gamma)$ *and* $H_0^{s_+}(\Gamma) \subsetneq H^{s_+}(\Gamma)$ *for all* $s_- < s_0(\Gamma) < s_+$.

- $s_0(\Gamma) \ge -s_{\partial\Gamma}$ (nullity threshold), with equality if Γ is C^0.
- $s_0(\Gamma) \ge (n - \dim_{\mathrm{H}}\partial\Gamma)/2$.
- If Γ is C^0, then $0 \le s_0(\Gamma) \le 1/2$.
- If Γ is $C^{0,\alpha}$ then $\alpha/2 \le s_0(\Gamma) \le 1/2$.
- If Γ is Lipschitz, then $s_0(\Gamma) = 1/2$.
- If $\Gamma = \mathbb{R}^n \setminus F$, F countable, $s_0(\Gamma) = n/2$.

All bounds on s_0 can be achieved. Improvement on Caetano 2000.

Relations between "global" and "local" spaces

The relations between subspaces of $\mathcal{D}^*(\mathbb{R}^n)$ and $\mathcal{D}^*(\Gamma)$ are described by the restriction operator $|_\Gamma : \mathcal{D}^*(\mathbb{R}^n) \to \mathcal{D}^*(\Gamma)$.

- $|_\Gamma : H^s(\mathbb{R}^n) \to H^s(\Gamma)$ is continuous with norm one;
- $|_\Gamma : (H^s_{\Gamma^c})^\perp \to H^s(\Gamma)$ is a unitary isomorphism ($H^s_{\Gamma^c} = \ker|_\Gamma$);

- For $s \geq 0$, $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is injective and has dense image; if $s \in \mathbb{N}_0$ then it is isomorphism;
- If Γ is finite union of disjoint Lipschitz open sets, $\partial\Gamma$ is bounded, $s > -1/2$, $s + 1/2 \notin \mathbb{N}$, then $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is isomorphism;

  Open question: for which s is $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ isomorphism?

- If Γ is bounded, or Γ^c is bounded with non-empty interior, then $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is a unitary isomorphism $\iff$ $s \in \mathbb{N}_0$ (equivalent to say that H^s norm is local only for $s \in \mathbb{N}_0$);
- If Γ^c is s-null, then $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is a unitary isomorphism.

(If one defines $H^s_{00}(\Gamma)$, $s \geq 0$, from interpolation of $H^k_0(\Gamma)$, $k \in \mathbb{N}_0$, then for sufficiently smooth Γ (e.g. Lipschitz) $H^s_{00}(\Gamma) = \widetilde{H}^s(\Gamma)|_\Gamma$.)

A warning about interpolation

It is well-known that, for $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$, and $s = s_0(1-\theta) + s_1\theta$,

$$\big(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)\big)_\theta = H^s(\mathbb{R}^n) \qquad \text{with equal norms.}$$

In McLean's book *Strongly Elliptic Systems and Boundary Integral Equations* it is claimed (in Theorem B.8) that the same holds for $H^s(\Gamma) := \{u|_\Gamma : u \in H^s(\mathbb{R}^n)\}$, for arbitrary open sets $\Gamma \subset \mathbb{R}^n$.

THIS RESULT IS FALSE!
The interpolation result only holds for Γ sufficiently smooth (e.g. Lipschitz) and even then, equality of norms does not hold in general.

Simple counterexamples:
for a cusp domain in $\mathbb{R}^2$, $\{H^s(\Gamma), 0 \le s \le 2\}$ is not interpolation scale;
for open interval in $\mathbb{R}$, no normalisation of $(\widetilde{H}^0(\Gamma), \widetilde{H}^1(\Gamma))_{1/2}$ can give norm equal to $\widetilde{H}^{1/2}(\Gamma)$.

Spaces on nested sets and FEM on fractals

Proposition

Consider a sequence of nested open sets $\{\Gamma_j\}_{j\in\mathbb{N}}$, $\Gamma_j \subset \Gamma_{j+1}$, and a collection of closed sets $\{F_j\}_{j\in\mathfrak{I}}$. Then

$$\widetilde{H}^s\Big(\bigcup_{j\in\mathbb{N}}\Gamma_j\Big) = \overline{\bigcup_{j\in\mathbb{N}}\widetilde{H}^s(\Gamma_j)}, \qquad H^s_{\bigcap_{j\in\mathfrak{I}}F_j} = \bigcap_{j\in\mathfrak{I}} H^s_{F_j}.$$

Together with Céa's Lemma, this allows to prove convergence of Galerkin methods on sets with fractal boundaries.

Example: Laplace–Dirichlet problem on $\Gamma =$Koch snowflake (open).

- $\Gamma_j =$ Lipschitz prefractal approximation of level j,
- $\{V_{j,k}\}_{j,k\in\mathbb{N}}$ nested FE spaces, $V_{j,k} \subset V_{j+1,k}$, $\widetilde{H}^1(\Gamma_j) = \overline{\bigcup_{k\in\mathbb{N}} V_{j,k}}$,
- $f \in H^{-1}(\mathbb{R}^2)$,
- $u_{jj} \in V_{j,j}$ solution of the FEM $\int_{\Gamma_j} \nabla u_{jj} \cdot \nabla v = \langle f, v\rangle\ \forall v \in V_{j,j}$,

Then u_{jj} converges in H^1 norm to $u \in \widetilde{H}^1(\Gamma)$, solution of

$$\int_\Gamma \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x} = \langle f, v\rangle\ \forall v \in \widetilde{H}^1(\Gamma).$$

Summary

We have studied (classical, fractional, Bessel-potential, Hilbert) Sobolev spaces on general open and closed subset of $\mathbb{R}^n$.

In particular we contributed to the questions:

- What are the duals of these spaces?
- When does $E \subset \mathbb{R}^n$ support non-zero $u \in H^s$?
- When is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?
- When is $H^s(\Gamma) = H^s_0(\Gamma)$?
- For which spaces is $|_\Gamma$ an isomorphism?

Some of these are relevant for screen scattering problems and Galerkin (FEM/BEM) methods on fractals.

Plenty of questions are still open!

Thank you!

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