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Space–time DG for the wave equation: quasi-Trefftz and sparse versions

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Initial–boundary value problem

First-order initial–boundary value problem (Dirichlet): find $(v, \boldsymbol{\sigma})$ s.t.

$$\begin{cases} \nabla v + \partial_t \boldsymbol{\sigma} = \mathbf{0} & \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1},\ n \in \mathbb{N}, \\ \nabla \cdot \boldsymbol{\sigma} + \dfrac{1}{c^2} \partial_t v = f & \text{in } Q, \\ v(\cdot,0) = v_0, \quad \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega, \\ v(\mathbf{x},\cdot) = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

From $-\Delta u + c^{-2}\partial_t^2 u = f$, choose $v = \partial_t u$ and $\boldsymbol{\sigma} = -\nabla u$.

Velocity $c = c(\mathbf{x})$ piecewise smooth. $\Omega \subset \mathbb{R}^n$ Lipschitz bounded.

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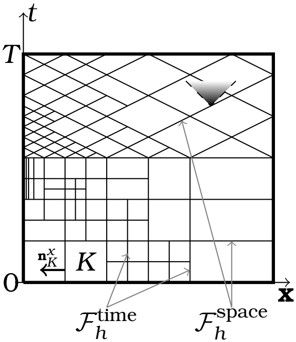
Extensions:

- Neumann $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ & Robin $\frac{\vartheta}{c} v - \boldsymbol{\sigma} \cdot \mathbf{n} = g$ BCs
- more general coeff.'s $-\nabla \cdot (\rho^{-1} \nabla u) + G\, \partial_t^2 u = 0$
- Maxwell equations
- elasticity
- 1^{st} order hyperbolic systems...

Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on Q.
Assume: $c = c(\mathbf{x})$ smooth in each element.



Assume: each face $F = \partial K_1 \cap \partial K_2$
with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c|\mathbf{n}_F^x| < n_F^t$, $\qquad F \subset \mathcal{F}_h^{\text{space}}$,
  or
- time-like: $n_F^t = 0$, $\qquad F \subset \mathcal{F}_h^{\text{time}}$.

Usual DG notation with averages $\{\!\{\cdot\}\!\}$,
$\mathbf{n}^x$-normal space jumps $[\![\cdot]\!]_{\mathbf{N}}$, $\quad n^t$-time jumps $[\![\cdot]\!]_t$.
Lateral boundary $\mathcal{F}_h^{\partial} := \partial\Omega \times [0, T]$.

DG elemental equation and numerical fluxes

Multiply PDEs with test field $(w, \boldsymbol{\tau})$ & integrate by parts on $K \in \mathcal{T}_h$:

$$-\int_K \left(v\Big(\nabla\cdot\boldsymbol{\tau} + c^{-2}\partial_t w\Big) + \boldsymbol{\sigma}\cdot\Big(\nabla w + \partial_t \boldsymbol{\tau}\Big)\right)\mathrm{d}V$$
$$+\int_{\partial K}\left((v\,\boldsymbol{\tau} + \boldsymbol{\sigma}\,w)\cdot\mathbf{n}_K^x + \Big(\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + c^{-2} v\,w\Big)\, n_K^t\right)\mathrm{d}S = \int_K f w\,\mathrm{d}V.$$

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Approximate skeleton traces of $(v, \boldsymbol{\sigma})$ with numerical fluxes $(\widehat{v}_h, \widehat{\boldsymbol{\sigma}}_h)$, defined as $\qquad \alpha, \beta \in L^\infty(\mathcal{F}_h^{\mathrm{time}} \cup \mathcal{F}_h^{\partial})$

$$\widehat{v}_h := \begin{cases} v_h^- \\ v_0 \\ \{\!\{v_h\}\!\} + \beta[\![\boldsymbol{\sigma}_h]\!]_{\mathbf{N}} \\ g \end{cases} \quad \widehat{\boldsymbol{\sigma}}_h := \begin{cases} \boldsymbol{\sigma}_h^- & \text{on } \mathcal{F}_h^{\mathrm{space}} \cup \mathcal{F}_h^T \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^0 \\ \{\!\{\boldsymbol{\sigma}_h\}\!\} + \alpha[\![v_h]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\mathrm{time}} \\ \boldsymbol{\sigma}_h - \alpha(v-g)\mathbf{n}_\Omega^x & \text{on } \mathcal{F}_h^{\partial} \end{cases}$$

"upwind in time, elliptic-DG in space".
$\alpha = \beta = 0 \to$ KRETZSCHMAR–S.–T.–W., $\quad \alpha\beta \geq \frac{1}{4} \to$ MONK–RICHTER.

Space–time DG formulation

Substitute the fluxes in the elemental equation,
choose discrete space $\mathbf{V}_p(\mathcal{T}_h)$, sum over $K \to$ write $\mathbf{x}t$-DG as:

$$\text{Seek } (v_h, \boldsymbol{\sigma}_h) \in \mathbf{V}_p(\mathcal{T}_h) \text{ s.t.,} \quad \forall (w, \boldsymbol{\tau}) \in \mathbf{V}_p(\mathcal{T}_h),$$

$$\mathcal{A}(v_h, \boldsymbol{\sigma}_h; w, \boldsymbol{\tau}) = \ell(w, \boldsymbol{\tau}) \qquad \text{where}$$

$$\begin{aligned}
\mathcal{A}(v_h, \boldsymbol{\sigma}_h; w, \boldsymbol{\tau}) :=& - \sum_{K \in \mathcal{T}_h} \int_K \Big(v_h \big(\nabla \cdot \boldsymbol{\tau} + c^{-2} \partial_t w \big) + \boldsymbol{\sigma}_h \cdot \big(\nabla w + \partial_t \boldsymbol{\tau} \big) \Big) \, \mathrm{d}V \\
&+ \int_{\mathcal{F}_h^{\text{space}}} \Big(\frac{v_h^- [\![w]\!]_t}{c^2} + \boldsymbol{\sigma}_h^- \cdot [\![\boldsymbol{\tau}]\!]_t + v_h^- [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \boldsymbol{\sigma}_h^- \cdot [\![w]\!]_{\mathbf{N}} \Big) \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{\text{time}}} \Big(\{\!\{ v_h \}\!\} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \{\!\{ \boldsymbol{\sigma}_h \}\!\} \cdot [\![w]\!]_{\mathbf{N}} + \alpha [\![v_h]\!]_{\mathbf{N}} \cdot [\![w]\!]_{\mathbf{N}} + \beta [\![\boldsymbol{\sigma}_h]\!]_{\mathbf{N}} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \Big) \, \mathrm{d}S \\
&+ \int_{\Omega \times \{T\}} (c^{-2} v_h w + \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} (\boldsymbol{\sigma}_h \cdot \mathbf{n}_\Omega + \alpha v_h) w \, \mathrm{d}S, \\
\ell(w, \boldsymbol{\tau}) :=& \int_Q f w \, \mathrm{d}V + \int_{\Omega \times \{0\}} (c^{-2} v_0 w + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} g (\alpha w - \boldsymbol{\tau} \cdot \mathbf{n}_\Omega) \, \mathrm{d}S.
\end{aligned}$$

This is an "ultra-weak" variational formulation (UWVF).

Coercivity in DG semi-norm

Key property, from integration by parts:

$$\mathcal{A}(w,\boldsymbol{\tau};w,\boldsymbol{\tau}) \geq |||(w,\boldsymbol{\tau})|||^2_{\mathrm{DG}}$$

where

$$|||(w,\boldsymbol{\tau})|||^2_{\mathrm{DG}} := \frac{1}{2}\left\|\left(\frac{1-\gamma}{n_F^t}\right)^{1/2} c^{-1}[\![w]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{space}})} + \frac{1}{2}\left\|\left(\frac{1-\gamma}{n_F^t}\right)^{1/2}[\![\boldsymbol{\tau}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{space}})^n}$$
$$+\frac{1}{2}\left\|c^{-1}w\right\|^2_{L^2(\mathcal{F}_h^0\cup\mathcal{F}_h^T)} + \frac{1}{2}\left\|\boldsymbol{\tau}\right\|^2_{L^2(\mathcal{F}_h^0\cup\mathcal{F}_h^T)^n}$$
$$+\left\|\alpha^{1/2}[\![w]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})^n} + \left\|\beta^{1/2}[\![\boldsymbol{\tau}]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})} + \left\|\alpha^{1/2}w\right\|^2_{L^2(\mathcal{F}_h^{\partial})}$$

$\gamma := \frac{\|c\|_{C^0(F)}|\mathbf{n}_F^x|}{n_F^t} \in [0,1) \sim$ distance between space-like face F & char. cone.

In general, $|||(w,\boldsymbol{\tau})|||_{\mathrm{DG}}$ is only a semi-norm.

Special case: space–time Trefftz method

Assume c is constant in $K \subset \mathbb{R}^{n+1}$.
Consider homogeneous wave eq. $-\Delta u + c^{-2}\partial_t^2 u = 0$ in K.

Can choose Trefftz space of polynomials of deg. $\le p$ on element K:

$$\mathbb{U}^p(K) := \left\{u \in \mathbb{P}^p(K),\ -\Delta u + c^{-2}\partial_t^2 u = 0\right\},$$

$$\mathbb{W}^p(K) := \left\{(v, \boldsymbol{\sigma}) = (\partial_t u, -\nabla u),\ u \in \mathbb{U}^{p+1}(K)\right\}.$$

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- Basis functions easily constructed, e.g. $b_{j,\ell}(\mathbf{x}, t) = (\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j$.
- Taylor $T^{p+1}[u] \in \mathbb{U}^p(K) \Rightarrow$ orders of approximation in h are for free.

  Much better accuracy for fewer DOFs:
  $\dim\left(\mathbb{U}^p(K)\right) = \mathcal{O}_{p\to\infty}(p^n) \quad \ll \quad \dim\left(\mathbb{P}^p(K)\right) = \mathcal{O}_{p\to\infty}(p^{n+1})$.
- With Trefftz test fields, volume terms in $\mathbf{x}t$-DG bilinear form vanish: quadrature on n-dimensional faces only.
- $|||\cdot|||_{\mathrm{DG}}$ is a norm: stability and error analysis. (M., Perugia 2018)

Global, implicit and explicit schemes

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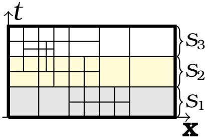

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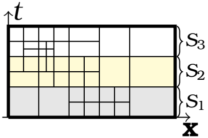
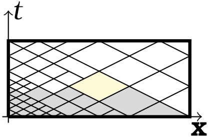


3 If mesh is suitably chosen, DG solution can be computed with a sequence of local systems:
explicit method, allows parallelism!



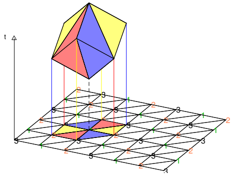
"Tent pitching" method of ÜNGÖR–SHEFFER, MONK–RICHTER, GOPALAKRISHNAN–MONK–SEPÚLVEDA, . . .

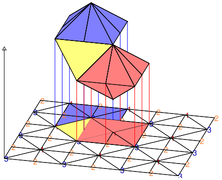
Trefftz requires quadrature on faces only: easier tent-pitching.

Versions 1–2–3 are algebraically equivalent (on the same mesh).

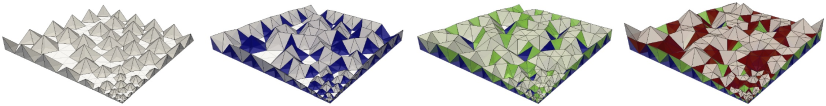
Tent-pitched elements

Tent-pitched elements/patches obtained from regular space meshes in 2+1D give parallelepipeds or octahedra+tetrahedra:





More complicated shapes from unstructured meshes:



(from GOPALAKRISHNAN, SCHÖBERL, WINTERSTEIGER 2016)

Simplices around a tent pole can be merged in macroelement.

Trefftz requires quadrature on faces only:
only the shape of space elements matters.

Bibliography

Proposed $\mathbf{x}t$-DG formulation comes from:

- (Monk, Richter 2005), linear symmetric hyperbolic systems, tent-pitched meshes, $\mathbb{P}^p$ spaces, $\alpha\beta \geq \frac{1}{4}$
- (Kretzschmar, Schnepp et al. 2014–16) Maxwell eq.s, Trefftz
- (M., Perugia 2018) Trefftz error analysis
- (Perugia, Schoeberl, Stocker, Wintersteiger 2020) Trefftz & tents

This presentation:

- (Imbert-Gérard, M., Stocker 2020 — arXiv:2011.04617) pw-smooth c, quasi-Trefftz
- (Bansal, M., Perugia, Schwab 2021) tensor-product grids, corner singularities, sparse version

Related works:

- (Barucq, Calandra, Diaz, Shishenina 2020) elasticity
- (Gómez, M. 2021 — arXiv:2106.04724) Schrödinger

Part I

Quasi-Trefftz $\mathbf{x}t$-DG

Imbert-Gérard, Moiola, Stocker

Trefftz doesn't like smooth coefficients

Homogeneous wave equation $-\Delta u + c^{-2}\partial_t^2 u = 0$, $\quad c =$wavespeed.

Trefftz-DG is clear for piecewise-constant c:
basis functions are polynomial local solution of wave eq.

How to extend to piecewise-smooth $c = c(\mathbf{x})$?
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Helmholtz equation: $\Delta u + k^2 u = 0$.

- Constant wavenumber $k \in \mathbb{R} \rightarrow$ plane waves $b_J(\mathbf{x}) = \mathrm{e}^{ik\mathbf{d}_j\cdot\mathbf{x}}$, $|\mathbf{d}_j| = 1$.
- Smooth wavenumber $k = k(\mathbf{x})$

IMBERT-GÉRARD, ≈2013: generalised plane waves $b_J(\mathbf{x}) = \mathrm{e}^{P_j(\mathbf{x})}$ s.t.

$$D^{\boldsymbol{i}}(\Delta b_J + k^2 b_J)(\mathbf{x}_K) = 0 \quad \forall |\boldsymbol{i}| < q$$

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– Basis construction, implementation, analysis are complicated.

Our goal: extend this idea to wave equation, without pain!

Quasi-Trefftz space

Define wave operator $\square_G u := \Delta u - G\partial_t^2 u$, $G(\mathbf{x}) = c^{-2}$ smooth.
Fix $(\mathbf{x}_K, t_K) \in K \subset \mathbb{R}^{n+1}$. Define quasi-Trefftz (polynomial) space

$$\mathbb{QU}^p(K) := \left\{ u \in \mathbb{P}^p(K) : \quad D^{\boldsymbol{i}} \square_G u(\mathbf{x}_K, t_K) = 0, \quad \forall |\boldsymbol{i}| \leq p-2 \right\}$$

$$\mathbb{QW}^p(K) := \left\{ (\partial_t u, -\nabla u), u \in \mathbb{QU}^{p+1}(K) \right\}$$

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Theorem: approximation properties

If $u \in C^{p+1}(K)$, $\square_G u = 0$, $0 \le j \le p$, K star-shaped wrt $(\mathbf{x}_K, t_K)$

$$\inf_{P \in \mathbb{QU}^p(K)} \|u - P\|_{C^j(K)} \le h^{p+1-j} \frac{n^{p+1-j}}{(p+1-j)!} |u|_{C^{p+1}(K)}$$

Main idea: Taylor polynomial $T^{p+1}_{(\mathbf{x}_K, t_K)}[u] \in \mathbb{QU}^p(K)$.

In condition "$|\boldsymbol{i}| \le q$", why $q = p-2$?
If $q < p-2$, space is too big, larger than Trefftz for constant G.
If $q > p-2$, space loses approximation properties.

Generalised Trefftz basis

The local discrete space is clear.
How to construct a basis for it?

Choose two $\mathbf{x}$-only polynomial basis:

$$\{\widehat{b}_J\}_{J=1,\dots,\binom{p+n}{n}} \text{ for } \mathbb{P}^p(\mathbb{R}^n), \qquad \{\widetilde{b}_J\}_{J=1,\dots,\binom{p-1+n}{n}} \text{ for } \mathbb{P}^{p-1}(\mathbb{R}^n).$$

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Construct a basis for $\mathbb{QU}^p(K)$ "evolving" $\widehat{b}_J$ and $\widetilde{b}_J$ in time:

$$\left\{ b_J \in \mathbb{QU}^p(K) : \begin{array}{lll} b_J(\cdot, t_K) = \widehat{b}_J, & \partial_t b_J(\cdot, t_K) = 0, & \text{for } J \le \binom{p+n}{n} \\ b_J(\cdot, t_K) = 0, & \partial_t b_J(\cdot, t_K) = \widetilde{b}_{J-\binom{p+n}{n}}, & \text{for } \binom{p+n}{n} < J \end{array} \right\}$$

for $J = 1, \dots, \binom{p+n}{n} + \binom{p-1+n}{n}$.

We prove that this defines a basis and show how to compute $\{b_J\}$.

Computation of basis coefficients

Fix $n = 1$ (for simplicity). Denote $G(x) = \sum_{m=0}^{\infty} g_m (x - x_K)^m$. $g_0 > 0$.
Monomial expansion of basis element:

$$b_J(x,t) = \sum_{i_x + i_t \leq p} a_{i_x, i_t} (x - x_K)^{i_x} (t - t_K)^{i_t},$$

Cauchy conditions $(b_J(\cdot, t_K), \partial_t b_J(\cdot, t_K))$ determine $a_{i_x,0}$, $a_{i_x,1}$.

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To be in $\mathbb{QU}^p$, coeff.s have to satisfy: for $i_x + i_t \le p - 2$

$$\partial_x^{i_x} \partial_t^{i_t} \Box_G b_J(x_K, t_K) = (i_x + 2)!\, i_t!\, a_{i_x+2, i_t} - \sum_{j_x=0}^{i_x} i_x!\, (i_t + 2)!\, g_{i_x - j_x}\, a_{j_x, i_t+2} \stackrel{!}{=} 0$$

Linear system for coeff.s a_{i_x, i_t}.

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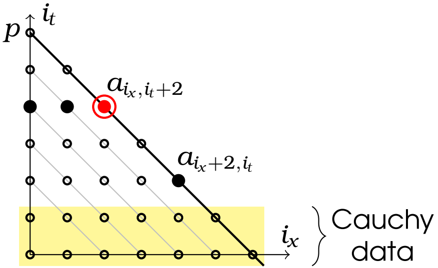
Linear system for coeff.s a_{i_x, i_t}.

Compute a_{i_x, i_t+2} ◉
from coefficients ● :

first loop across diagonals ↗,
then along diagonals ↖.



Basis construction: algorithm — $n = 1$

Data: $(g_m)_{m\in\mathbb{N}_0}$, x_K, t_K, p.

Choose favourite polynomial bases $\{\widehat{b}_J\}$, $\{\widetilde{b}_J\}$ in $\mathbf{x}$,
$\rightarrow$ coeff's $a_{k_x,0}$, $a_{k_x,1}$.

For each J (i.e. for each basis function), construct b_J as follows:

for $\ell = 2$ *to* p *(loop across diagonals* $\nearrow$*)* **do**
  for $i_t = 0$ *to* $\ell - 2$ *(loop along diagonals* $\nwarrow$*)* **do**
    set $i_x = \ell - i_t - 2$ and compute

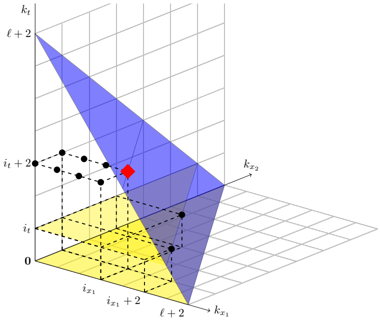
$$a_{i_x,i_t+2} = \frac{(i_x+2)(i_x+1)}{(i_t+2)(i_t+1)g_0} a_{i_x+2,i_t} - \sum_{j_x=0}^{i_x-1} \frac{g_{i_x-j_x}}{g_0} a_{j_x,i_t+2}$$

  end
end

$$b_J(x,t) = \sum_{0<k_x+k_t\le p} a_{k_x,k_t}(x-x_K)^{k_x}(t-t_K)^{k_t}$$

Basis construction: algorithm — $n > 1$

In higher space dimensions $n > 1$,
with $G(\mathbf{x}) = \sum_{\boldsymbol{i}_\mathbf{x}} (\mathbf{x} - \mathbf{x}_K)^{\boldsymbol{i}_\mathbf{x}} g_{\boldsymbol{i}_\mathbf{x}}$,
the algorithm is the same
with a further inner loop:



for $\ell = 2$ *to* p *(loop across* $\{|\boldsymbol{i}_\mathbf{x}| + i_t = \ell - 2\}$ *hyperplanes,* $\nearrow$*)* **do**

  for $i_t = 0$ *to* $\ell - 2$ *(loop across constant-t hyperplanes* $\uparrow$*)* **do**

    for $\boldsymbol{i}_\mathbf{x}$ *with* $|\boldsymbol{i}_\mathbf{x}| = \ell - i_t - 2$ **do**

$$a_{\boldsymbol{i}_\mathbf{x}, i_t+2} = \sum_{l=1}^{n} \frac{(i_{x_l}+2)(i_{x_l}+1)}{(i_t+2)(i_t+1)g_\mathbf{0}} a_{\boldsymbol{i}_\mathbf{x}+2\mathbf{e}_l, i_t} - \sum_{\boldsymbol{j}_\mathbf{x} < \boldsymbol{i}_\mathbf{x}} \frac{g_{\boldsymbol{i}_\mathbf{x}-\boldsymbol{j}_\mathbf{x}}}{g_\mathbf{0}} a_{\boldsymbol{j}_\mathbf{x}, i_t+2}$$

    end

  end

end

Quasi-Trefftz $\mathbf{x}t$-DG

Use $\prod_{K\in\mathcal{T}_h}\mathbb{QW}^p(K)$ with $\mathbf{x}t$-DG for IBVP with piecewise-smooth c.

Use idea of (Imbert-Gèrard, Monk 2017): add volume penalty term

$$\sum_{K\in\mathcal{T}_h}\int_K \mu_1(\nabla\cdot\boldsymbol{\sigma}+c^{-2}\partial_t v)(\nabla\cdot\boldsymbol{\tau}+c^{-2}\partial_t w)+\mu_2(\partial_t\boldsymbol{\sigma}+\nabla v)\cdot(\partial_t\boldsymbol{\tau}+\nabla w).$$

- Coercivity in DG norm (with volume terms)
- Well-posedness
- Quasi-optimality
- Error bounds (high-order h-convergence, optimal rates, explicit)

$$|||(v,\boldsymbol{\sigma})-(v_h,\boldsymbol{\sigma}_h)|||_{\mathrm{DG}}\le C\sup_{K\in\mathcal{T}_h} h_{K,c}^{p+1/2}\,|u|_{C_c^{p+2}(K)}\,.$$

Same DOF saving as for Helmholtz or constant c ($\mathcal{O}(p^n)$ vs $\mathcal{O}(p^{n+1})$).

More general IBVPs

Everything extends to 2 piecewise-smooth material parameters ρ, G:

$$\nabla v + \rho \partial_t \boldsymbol{\sigma} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + G \partial_t v = 0,$$

Wavespeed is $c = (\rho G)^{-1/2}$.

Second-order version:

$$-\nabla \cdot \left(\frac{1}{\rho} \nabla u \right) + G\, \partial_t^2 u = 0 \qquad \left(v = \partial_t u,\ \boldsymbol{\sigma} = -\frac{1}{\rho} \nabla u \right).$$

Basis coefficient algorithm needs some more terms.

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Basis coefficient algorithm needs some more terms.

If the 1st-order IBVP does not come from a 2nd-order one, we use

$$\mathbb{QT}^p(K) := \left\{ (w, \boldsymbol{\tau}) \in \mathbb{P}^p(K)^{n+1} \,\middle|\, \begin{array}{r} D^{\boldsymbol{i}}(\nabla w + \rho \partial_t \boldsymbol{\tau})(\mathbf{x}_K, t_K) = \mathbf{0} \\ D^{\boldsymbol{i}}(\nabla \cdot \boldsymbol{\tau} + G \partial_t w)(\mathbf{x}_K, t_K) = 0 \\ \forall |\boldsymbol{i}| \le p - 1 \end{array} \right\}$$

This space is only slightly larger ($\approx \frac{n+1}{2} \times$, still $\mathcal{O}_{p\to\infty}(p^n)$ DOFs) and allows the same analysis.

Numerics

- Implemented in NGSolve.
- Both Cartesian and tent-pitched meshes.
- Volume penalty term not needed in computations.
- DG flux coefficients $\alpha^{-1} = \beta = c$, but even $\alpha = \beta = 0$ works.
- Good conditioning.
- Monomial bases $\{\widehat{b}_J\}, \{\widetilde{b}_J\}$ outperform Legendre/Chebyshev.

Numerics 1: convergence

Compare quasi-Trefftz, full polynomials, Trefftz ($c|_K = c(\mathbf{x}_K)$) spaces

$$\begin{aligned}
\mathbb{QW}^p(\mathcal{T}_h) &:= \big\{(w,\boldsymbol{\tau}) \in \mathbf{H}(\mathcal{T}_h):\ w|_K = \partial_t u,\ \boldsymbol{\tau}|_K = -\nabla u,\ u \in \mathbb{QU}^{p+1}(K)\big\}\\
\mathbb{Y}^p(\mathcal{T}_h) &:= \big\{(w,\boldsymbol{\tau}) \in \mathbf{H}(\mathcal{T}_h):\ w|_K = \partial_t u,\ \boldsymbol{\tau}|_K = -\nabla u,\ u \in \mathbb{P}^{p+1}(K)\big\}\\
\mathbb{W}^p(\mathcal{T}_h) &:= \big\{(w,\boldsymbol{\tau}) \in \mathbf{H}(\mathcal{T}_h):\ w|_K = \partial_t u,\ \boldsymbol{\tau}|_K = -\nabla u,\ u \in \mathbb{P}^{p+1}(K),\\
&\qquad -\Delta u + c^{-2}(\mathbf{x}_K)\partial_t^2 u = 0 \text{ in } K\big\}.
\end{aligned}$$

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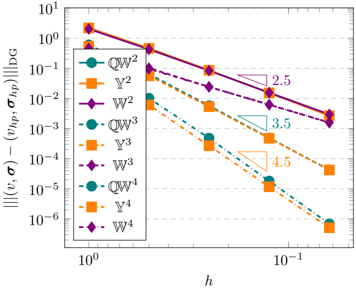
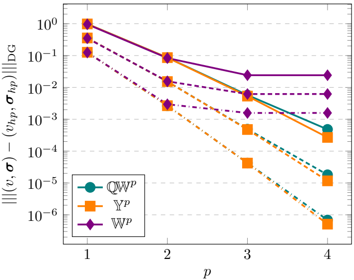
DG-norm error: optimal order in h, exponential in p.

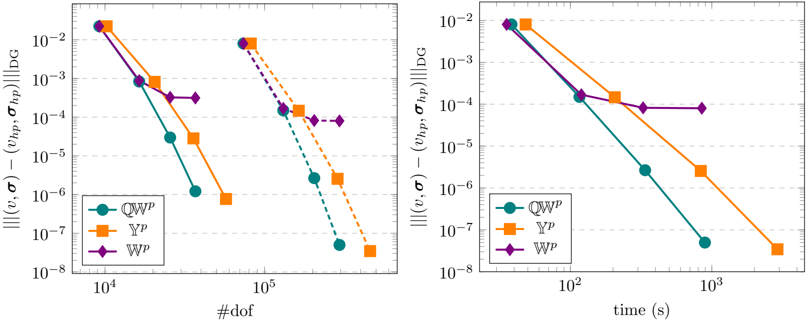




$$n = 2, \quad G = (x_1 + x_2 + 1)^{-1}, \quad u = (x_1 + x_2 + 1)^{2.5} e^{-\sqrt{7.5}t}, \quad Q = (0,1)^3.$$

Numerics 2: DOF & computational time

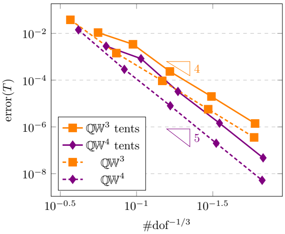
Quasi-Trefftz wins > 1 order of magnitude against full polynomials:

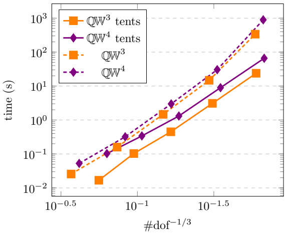


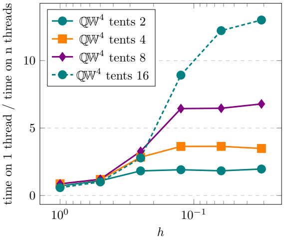
$h = 2^{-3}, 2^{-4}, \quad p = 1, 2, 3, 4.$

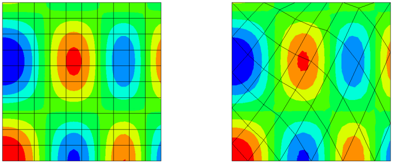
$n = 2, \quad G = x_1 + x_2 + 1, \quad u = \mathrm{Ai}(-x_1 - x_2 - 1)\cos(\sqrt{2}t), \quad Q = (0,1)^3.$

($n = 2$) Final-time error, computational time (sequential), speedup: (#dof$^{-1/3} \sim h$)



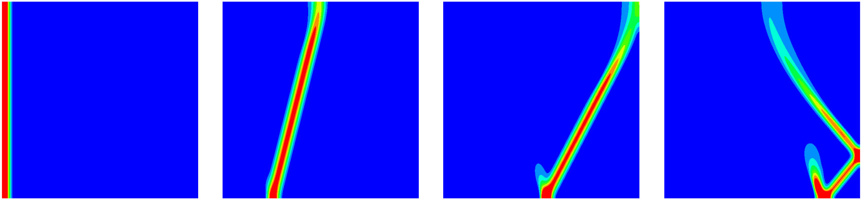


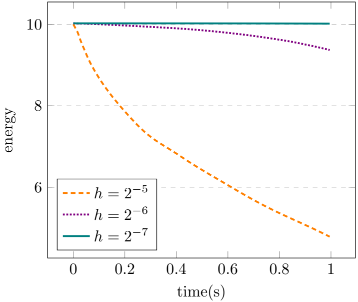




Numerics 4: energy conservation

Plane wave through medium with $G = 1 + x_2$ in $(0, 1)^3$:





$$\mathcal{E} = \frac{1}{2}\int_\Omega (c^{-2}v^2 + |\boldsymbol{\sigma}|^2)\,\mathrm{d}S$$

DG scheme is (provably) dissipative.
For $p = 3$, $h = 2^{-7}$, only 0.076% loss.

Part 1: summary

Quasi-Trefftz DG:

- Extend Trefftz scheme to piecewise-smooth coefficients.
  Basis are PDE solution "up to given order in h".
- Simple construction of basis functions:
  same "Cauchy data" at element centre as for Trefftz.
- Use in $\mathbf{x}t$-DG , stability and error analysis.
  High orders of convergence in h,
  much fewer DOFs than standard polynomial spaces.

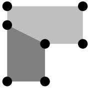
(Imbert-Gérard, M., Stocker, arXiv:2011.04617, 2020)

Part II

$\mathbf{x}t$-DG with point singularities

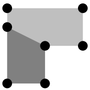
Bansal, Moiola, Perugia, Schwab

Wave solutions on polygons are singular



Fix $n = 2$.
Piecewise-constant c, on polygonal partition of Ω.
Denote by $\{\mathbf{c}_i\}_{i=1,\ldots,M}$ the vertices of this partition.

Wave solutions on polygons are singular



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Even for smooth initial conditions & source term, homogeneous BCs, the IBVP solution in polygon$\times(0,T)$ lives in corner-weigthed spaces:

$$(v, \boldsymbol{\sigma}) = \big(\partial_t u, -\nabla u\big) \in C^{k_t-1}\big([0,T]; H^{k_x+1,2}_{\boldsymbol{\delta}}(\Omega)\big) \times C^{k_t}\big([0,T]; H^{k_x,1}_{\boldsymbol{\delta}}(\Omega)^2\big)$$

$$\|u\|^2_{H^{k,\ell}_{\boldsymbol{\delta}}(\Omega)} := \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{m=\ell}^{k} \int_\Omega \Big(\prod_{i=1}^{M} |\mathbf{x} - \mathbf{c}_i|^{\delta_i} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^2 \\ \alpha_1+\alpha_2=m}} |D^{\boldsymbol{\alpha}} u|^2\Big)$$

Kokotov, Plamenevskiĭ 1999–2004 $\to$ Müller, Schwab 2015–18.

– This means $v(\cdot,t) \notin H^2(\Omega), \boldsymbol{\sigma}(\cdot,t) \notin H^1(\Omega)^2$.

\+ Diffraction singularities are confined (in space) to the corners $\mathbf{c}_i$ and have smooth time-dependence.

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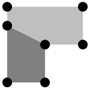

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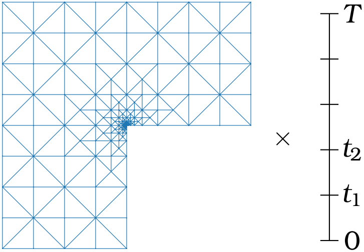
+ Diffraction singularities are confined (in space) to the corners $\mathbf{c}_i$ and have smooth time-dependence.

$\to$ Suggests local mesh refinement in space only.

Locally-refined product meshes

Locally-refined mesh in space $\times$ quasi-uniform mesh in time:



Space-like faces are horizontal.

To avoid short time steps,
corner elements will be "tall&thin":
$\rightarrow$implicit method.

Can't use Trefftz spaces as they
requires some $\mathbf{x}t$-shape regularity.

$$\mathbf{V_p}(\mathcal{T}_h) = \prod_{K=K_{\mathbf{x}}\times I_n \in \mathcal{T}_h} \left(\mathbb{P}^{p^v_{x,K}}(K_{\mathbf{x}}) \otimes \mathbb{P}^{p^v_{t,K}}(I_n)\right) \times \left(\mathbb{P}^{p^\sigma_{x,K}}(K_{\mathbf{x}}) \otimes \mathbb{P}^{p^\sigma_{t,K}}(I_n)\right)^2.$$

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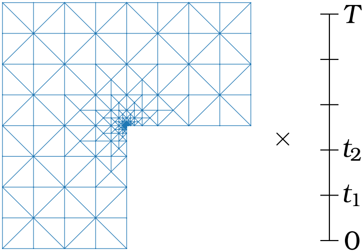

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DG semi-norm is not a norm on $\mathbf{V}_{\mathbf{p}}(\mathcal{T}_h)$:
"coercivity analysis" is not enough for well-posedness.

Well-posedness

In general, assume that "PDEs map local discrete space into itself":

$$\Big(\nabla\cdot\boldsymbol{\tau}_h + c^{-2}\partial_t w_h,\ \nabla w_h + \partial_t\boldsymbol{\tau}_h\Big) \in \mathbf{V_p}(\mathcal{T}_h) \qquad \forall (w_h, \boldsymbol{\tau}_h)\in \mathbf{V_p}(\mathcal{T}_h).$$

Holds, e.g., for $\mathbf{V_p}(\mathcal{T}_h)$ with $\quad |p^{\boldsymbol{\sigma}}_{x,K} - p^{v}_{x,K}| \le 1, \quad p^{\boldsymbol{\sigma}}_{t,K} = p^{v}_{t,K}.$

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This ensures that the method is well-posed:

- Assume $\mathcal{A}((v_h, \boldsymbol{\sigma}_h), (w_h, \boldsymbol{\tau}_h)) = 0 \qquad \forall (w_h, \boldsymbol{\tau}_h) \in \mathbf{V_p}(\mathcal{T}_h)$.
- $0 = \mathcal{A}((v_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\sigma}_h)) = |||(v_h, \boldsymbol{\sigma}_h)|||^2_{\mathrm{DG}}$
  $\Rightarrow$ jump and boundary traces of $(v_h, \boldsymbol{\sigma}_h)$ vanish.
- After IBP, only volume terms are left in $\mathcal{A}((v_h, \boldsymbol{\sigma}_h), (w_h, \boldsymbol{\tau}_h))$:
  $0 = \mathcal{A}((v_h, \boldsymbol{\sigma}_h), (w_h, \boldsymbol{\tau}_h)) =$
  $-\sum_{K\in\mathcal{T}_h} \int_K \Big(\big(\nabla \cdot \boldsymbol{\sigma}_h + c^{-2}\partial_t v_h\big) w_h + \big(\nabla v_h + \partial_t \boldsymbol{\sigma}_h\big) \cdot \boldsymbol{\tau}_h\Big)\, \mathrm{d}V$
- Choose $w_h = \nabla \cdot \boldsymbol{\sigma}_h + c^{-2}\partial_t v_h$ and $\boldsymbol{\tau}_h = \nabla v_h + \partial_t \boldsymbol{\sigma}_h$:
  $(v_h, \boldsymbol{\sigma}_h)$ solves homogeneous IBVP.
- $\Rightarrow (v_h, \boldsymbol{\sigma}_h) = (0, \mathbf{0})$.

Quasi-optimality and unconditional stability

Under the same assumption,
DG norm of error is controlled by error of L^2-projection on $\mathbf{V_p}(\mathcal{T}_h)$:

$$|||(v,\boldsymbol{\sigma})-(v_h,\boldsymbol{\sigma}_h)|||_{\mathrm{DG}} \;\le\; (3+p^{\boldsymbol{\sigma}}_{x,\angle})|||(v,\boldsymbol{\sigma})-(\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma})|||_{\mathrm{DG}+}$$

Here $|||\cdot|||_{\mathrm{DG}+}$ is a skeleton seminorm, stronger than $|||\cdot|||_{\mathrm{DG}}$.

It includes $\left\|\alpha^{-1/2}(\boldsymbol{\sigma}-\Pi_{L^2}\boldsymbol{\sigma})\cdot\mathbf{n}_x\right\|_{L^2(F_t,L^1(F_{\mathbf{x}}))}$ terms on time-like faces of corner elements, to accomodate $H^{1,1}_{\boldsymbol{\delta}}$ arguments.

$p^{\boldsymbol{\sigma}}_{x,\angle}$ is the polynomial degree in $\mathbf{x}$ used in corner elements

(from inverse & trace estimates for $H^{1,1}_{\boldsymbol{\delta}}$)

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$$\frac{1}{2}\left\|c^{-1}(v-v_h)\right\|_{L^2(\Omega\times\{t_n\})}+\frac{1}{2}\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\|_{L^2(\Omega\times\{t_n\})^2}\le$$
$$|||(v,\boldsymbol{\sigma})-(v_h,\boldsymbol{\sigma}_h)|||_{\mathrm{DG}}\;\le\;(3+p^{\boldsymbol{\sigma}}_{x,\angle})|||(v,\boldsymbol{\sigma})-(\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma})|||_{\mathrm{DG}^+}$$

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Bound controls also $L^2(\Omega)$ error at discrete times.

L^2-projection & Galerkin error bounds

To obtain concrete error bound, we need approximation bounds for the $L^2(K)$ projection on $\mathbb{P}^{p_x}(K_{\mathbf{x}}) \times \mathbb{P}^{p_t}(t_{n-1}, t_n)$, in Bochner norms, via Peetre–Tartar lemma[1]:

$$\|\varphi - \Pi_{L^2}\varphi\|_{L^2(I_n;L^2(K_{\mathbf{x}}))} + h_n\,|\varphi - \Pi_{L^2}\varphi|_{H^1(I_n;L^2(K_{\mathbf{x}}))} + h_{K_{\mathbf{x}}}\,|\varphi - \Pi_{L^2}\varphi|_{L^2(I_n;H^1(K_{\mathbf{x}}))}$$
$$\lesssim h_n^{s_t+1}\,|\varphi|_{H^{s_t+1}(I_n;L^2(K_{\mathbf{x}}))} + h_{K_{\mathbf{x}}}^{s_x+1}\,|\varphi|_{L^2(I_n;H^{s_x+1}(K_{\mathbf{x}}))},$$

and similarly for weighted spaces.

---

[1] $A : X \to Y$ injective, $T : X \to Z$ compact, $\|x\|_X \lesssim \|Ax\|_Y + \|Tx\|_Z \;\Rightarrow\; \|x\|_X \lesssim \|Ax\|_Y$.
Here, $X = H^{s_t+1}(I; L^2(K_{\mathbf{x}})) \cap L^2(I; H^{s_x+1}(K_{\mathbf{x}})) \overset{T}{\hookrightarrow} L^2(K)$,

$$X \xrightarrow{A=(\Pi_{L^2},\partial_t^{s_t+1},D_{\mathbf{x}}^{s_x+1})} (\mathbb{P}^{s_x}(K_{\mathbf{x}}) \otimes \mathbb{P}^{s_t}(I)) \times L^2(K) \times L^2(K)^{s_x+2}$$

L^2-projection & Galerkin error bounds

To obtain concrete error bound, we need approximation bounds for the $L^2(K)$ projection on $\mathbb{P}^{p_x}(K_{\mathbf{x}}) \times \mathbb{P}^{p_t}(t_{n-1}, t_n)$, in Bochner norms, via Peetre–Tartar lemma[1]:

$$\|\varphi - \Pi_{L^2}\varphi\|_{L^2(I_n;L^2(K_{\mathbf{x}}))} + h_n\,|\varphi - \Pi_{L^2}\varphi|_{H^1(I_n;L^2(K_{\mathbf{x}}))} + h_{K_{\mathbf{x}}}\,|\varphi - \Pi_{L^2}\varphi|_{L^2(I_n;H^1(K_{\mathbf{x}}))} \lesssim h_n^{s_t+1}\,|\varphi|_{H^{s_t+1}(I_n;L^2(K_{\mathbf{x}}))} + h_{K_{\mathbf{x}}}^{s_x+1}\,|\varphi|_{L^2(I_n;H^{s_x+1}(K_{\mathbf{x}}))},$$

and similarly for weighted spaces.

For smooth solutions + quasi-uniform meshes + uniform degree p:

$$\left\|c^{-1}(v - v_h)\right\|_{L^2(\Omega\times\{t_n\})} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega\times\{t_n\})^2} \lesssim h^{p+\frac{1}{2}}$$

$\frac{1}{2}$-order suboptimal: h^{p+1} from numerics.

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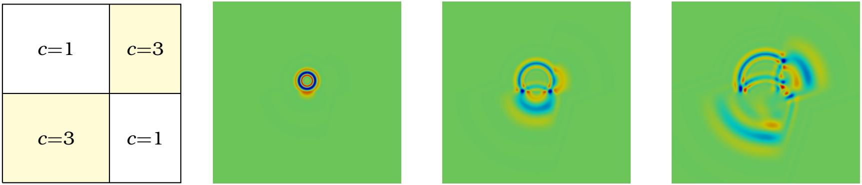
[1] $A: X \to Y$ injective, $T: X \to Z$ compact, $\|x\|_X \lesssim \|Ax\|_Y + \|Tx\|_Z \;\Rightarrow\; \|x\|_X \lesssim \|Ax\|_Y$.
Here, $X = H^{s_t+1}(I; L^2(K_{\mathbf{x}})) \cap L^2(I; H^{s_x+1}(K_{\mathbf{x}})) \overset{T}{\hookrightarrow} L^2(K)$,

$$X \xrightarrow{A=(\Pi_{L^2},\partial_t^{s_t+1},D_{\mathbf{x}}^{s_x+1})} (\mathbb{P}^{s_x}(K_{\mathbf{x}}) \otimes \mathbb{P}^{s_t}(I)) \times L^2(K) \times L^2(K)^{s_x+2}$$

Error bounds: singular solutions & graded meshes

- $(v,\boldsymbol{\sigma}) \in C^{k-1}\big([0,T]; H^{k+1,2}_{\delta}(\Omega)\big) \times C^{k}\big([0,T]; H^{k,1}_{\delta}(\Omega)^2\big)$, $k_x \geq 1,\ k_t \geq 2$,
- graded mesh $\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}}$ in $\mathbf{x}$ (GASPOZ–MORIN), max size $h_{\mathbf{x}}$, refinement of uniform $\mathcal{T}^{\mathbf{x}}_0$ with $\#\mathcal{T}^{\mathbf{x}}_{h_{\mathbf{x}}} - \#\mathcal{T}^{\mathbf{x}}_0 \leq C h_{\mathbf{x}}^{-2}$
- $h_{\mathbf{x}} \sim h_t \sim h$
- uniform polynomial degrees p (in $\mathbf{x}\&t$, $v\&\boldsymbol{\sigma}$, K)
- numerical flux parameters $\alpha^{-1} = \beta = c\frac{h_{F_{\mathbf{x}}}}{h_{\mathbf{x}}} = c\frac{\text{local}}{\text{global}}$

$$\Rightarrow \quad \left\| c^{-1}(v - v_h) \right\|_{L^2(\Omega\times\{t_n\})} + \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_{L^2(\Omega\times\{t_n\})^2} \lesssim h^{\min\{k-\frac{1}{2},\,p+\frac{1}{2}\}}$$



Again, numerics on L-shape give h^{p+1} rates.

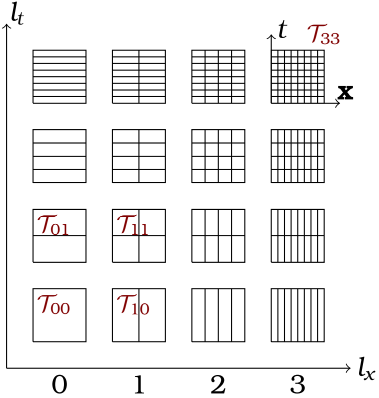
Sparse $\mathbf{x}t$-DG

Want to use a sparse grid approach in space–time.

Take initial mesh $\mathcal{T}_{0,0}$ of size $h_{0,x}$, $h_{0,t}$.
For $(l_x, l_t) \in \mathbb{N}_0^2$,
denote $\mathcal{T}_{l_x,l_t}$ a refinement of $\mathcal{T}_{0,0}$ with

$$h_{l_x,x} = 2^{-l_x} h_{0,x}, \quad h_{l_t,t} = 2^{-l_t} h_{0,t},$$

$\mathbf{w}_{l_x,l_t}$ = corresponding DG solution
(same polynomial space $\forall$ element).



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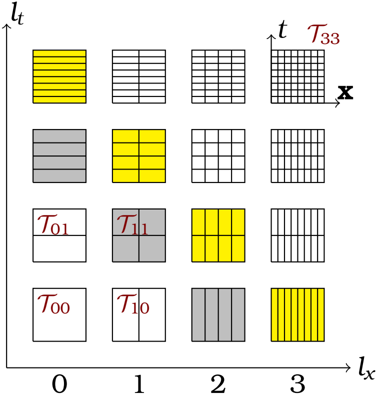
Combination formula:

$$\widehat{\mathbf{w}}_L := + \sum_{l=0}^{L} \mathbf{w}_{l,L-l} - \sum_{l=0}^{L-1} \mathbf{w}_{l,L-1-l}$$



Combines fine-in-t–coarse-in-$\mathbf{x}$ & fine-in-$\mathbf{x}$–coarse-in-t discretizations.
Never use fine-in-t–fine-in-$\mathbf{x}$.

Sparse vs full $\mathbf{x}t$-DG: accuracy and #DOFs

We observe comparable accuracy for full-tensor $\mathbf{w}_{L,L}$ and sparse $\widehat{\mathbf{w}}_L$:

$$\|(v, \boldsymbol{\sigma}) - \mathbf{w}_{L,L}\|_{L^2(\Omega\times\{T\})} \approx \|(v, \boldsymbol{\sigma}) - \widehat{\mathbf{w}}_L\|_{L^2(\Omega\times\{T\})} .$$

Consistent with sparse grid theory, which we can't apply here.

So why is it convenient?

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So why is it convenient? Same accuracy but cheaper!

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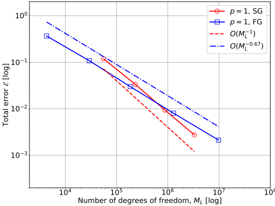
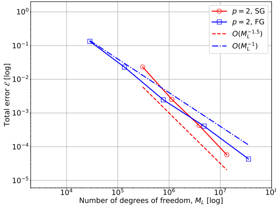
$p = 1 \rightarrow$





$\leftarrow p = 2$

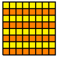
Singular solution on L-shape, mesh locally refined in $\mathbf{x}$.

$\rightarrow$ #DOFs is not where sparse scheme wins...

Sparse vs full $\mathbf{x}t$-DG: complexity

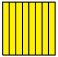
Not only #DOFs differ but also sizes & numbers of linear systems.

Full-tensor $\mathbf{w}_{L,L}$ requires:



$O(2^L)\times$ solves of size $O(2^{2L})$

Sparse $\widehat{\mathbf{w}}_L$ requires:



$\mathcal{O}(1)\times$ solves of size $\mathcal{O}(2^{2L})$

$\mathcal{O}(2)\times$ solves of size $\mathcal{O}(2^{2(L-1)})$

⋮

$\mathcal{O}(2^L)\times$ solves of size $\mathcal{O}(1)$

Total complexity is the same as single elliptic solve in $\Omega(\subset \mathbb{R}^2)$ $\times$ logarithmic terms.

Includes CFL-violating solves: requires unconditionally stable formulation.

Part 2: summary

- Unconditionally stable $\mathbf{x}t$-DG formulation,
  discrete functions are tensor-product polynomials.
- Well-posedness and error control
  also for solutions with point singularities.
- $h^{p+\frac{1}{2}}$ convergence rates
  for smooth solutions and quasi-uniform meshes,
  for singular solutions and refined meshes.
- Sparse version: same accuracy, fewer DOFs, lower complexity.

Main future work: sparse $\mathbf{x}t$-DG error analysis.

(Bansal, M., Perugia, Schwab, IMA JNA, 2021)

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Thank you!

Quasi-optimality

In non-Trefftz case, assume

$$\Big(\nabla\cdot\boldsymbol{\tau}_h + c^{-2}\partial_t w_h,\ \nabla w_h + \partial_t\boldsymbol{\tau}_h\Big) \in \mathbf{V_p}(\mathcal{T}_h) \qquad \forall (w_h,\boldsymbol{\tau}_h)\in\mathbf{V_p}(\mathcal{T}_h);$$

Then

$$\begin{aligned}
&|(\Pi_{L^2}v, \Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)|^2_{\mathrm{DG}(Q_n)}\\
&= \mathcal{A}_{\mathrm{DG}(Q_n)}\big((\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h); (\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)\big)\\
&= \mathcal{A}_{\mathrm{DG}(Q_n)}\big((\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v,\boldsymbol{\sigma}); (\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)\big)\\
&\le 2C_{\infty|2}\,|(\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v,\boldsymbol{\sigma})|_{\mathrm{DG}(Q_n)^+}\,|(\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)|_{\mathrm{DG}(Q_n)}\,.
\end{aligned}$$

Last ineq. uses inverse inequality on corner elements and cancellation of volume terms due to choice of L^2 projection.

$$\begin{aligned}
&\tfrac{1}{2}\left\|c^{-1}(v-v_h)\right\|_{L^2(\Omega\times\{t_n\})} + \tfrac{1}{2}\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\|_{L^2(\Omega\times\{t_n\})^2}\\
&\le |(v,\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)|_{\mathrm{DG}(Q_n)}\\
&\le |(v,\boldsymbol{\sigma}) - (\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma})|_{\mathrm{DG}(Q_n)} + |(\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma}) - (v_h,\boldsymbol{\sigma}_h)|_{\mathrm{DG}(Q_n)}\\
&\le (1+2C_{\infty|2})\,|(v,\boldsymbol{\sigma}) - (\Pi_{L^2}v,\Pi_{L^2}\boldsymbol{\sigma})|_{\mathrm{DG}(Q_n)^+}\,.
\end{aligned}$$