A space–time quasi-Trefftz DG method for the wave equation with smooth coefficients

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Part I

Quasi-Trefftz spaces for linear PDEs
Trefftz methods

Consider a boundary value problem

\[ \mathcal{L}u = 0 \quad \text{in} \ D \subset \mathbb{R}^d, \quad \mathcal{B}u = g \quad \text{on} \ \partial D. \]

A Trefftz scheme is a discretisation whose trial (& test) functions \( v_h \) are solutions of the PDE \( \mathcal{L}v_h = 0 \) in each element of a mesh.

This works well for PDEs that

- are linear
- are homogeneous (source term is 0)
- have constant coefficients

\{ v : \mathcal{L}v = 0 \} \text{ is linear space}

Trefftz functions are “easy” to build

Examples:

- Laplace equation \( \Delta u = 0 \) \rightarrow \text{harmonic polynomials,} \quad (\mathbf{d} \in \mathbb{R}^n, |\mathbf{d}| = 1)
- Helmholtz equation \( \Delta u + k^2 u = 0 \) \rightarrow \text{plane waves} \quad e^{ik\mathbf{d} \cdot \mathbf{x}}
- wave equation \( -\Delta u + c^{-2} \partial_t^2 u = 0 \) \rightarrow \text{plane waves} \quad f(\mathbf{d} \cdot \mathbf{x} - ct).
Quasi-Trefftz methods

What happens if the PDE has smooth coefficients? We typically don’t know how to construct discrete Trefftz space.

Quasi-Trefftz idea:
use discrete functions that are approximate solution of the PDE \( \mathcal{L}v_h \approx 0 \) (in each mesh element \( K \)).

More precisely: quasi-Trefftz functions \( v_h \) satisfy

\[
(D^i \mathcal{L}v_h)(x_K) = 0 \quad \forall i \in \mathbb{N}_0^n, \ |i| \leq q, \quad \text{for a given } x_K \in K, \ q \in \mathbb{N}.
\]

Instead of \( \mathcal{L}v_h = 0 \) in \( K \), this only requires that the degree-\( q \) Taylor polynomial (centred at a given point \( x_K \)) of \( \mathcal{L}v_h \) is 0.

\[\Rightarrow\text{ Small residual: } \mathcal{L}v_h(x) = \mathcal{O}(|x - x_K|^{q+1}), \quad x \in K.\]

Which kind of functions are these?
Polynomial quasi-Trefftz approximation

Let $m$ be the order of the linear PDE operator $\mathcal{L}$. We use degree-$p$ polynomials: for $p \in \mathbb{N}$

$$\mathcal{Q}_p^T(K) := \left\{ v_h \in \mathbb{P}^p(K) : (D^i \mathcal{L} v_h)(\mathbf{x}_K) = 0 \ \forall \mathbf{i} \in \mathbb{N}_0^n, \ |\mathbf{i}| \leq p - m \right\}.$$ 

Taylor polynomials of PDE solutions are quasi-Trefftz

Let $\mathcal{L} = \sum_{|j| \leq m} \alpha_j \mathcal{D}^j$ for $\alpha_j \in C_{\max\{p-m,0\}}(K)$, $\mathcal{L}u = 0$ for $u \in \mathcal{C}^{p+1}(K)$. Then $T_{\mathbf{x}_K}^{p+1}[u] \in \mathcal{Q}_p^T(K)$.

$h$-approximation estimates follow for any (linear, smooth-coeff.) PDE:

$\mathcal{L}$ and $u$ as above, $K$ star-shaped wrt $\mathbf{x}_K$, $r_K := \sup_{\mathbf{x} \in K} |\mathbf{x} - \mathbf{x}_K|$

$$\inf_{P \in \mathcal{Q}_p^T(K)} |u - P|_{C^q(K)} \leq \frac{dp+1-q}{(p+1-q)!} r_K^{p+1-q} |u|_{C^{p+1}(K)} \forall q \leq p.$$
Quasi-Trefftz methods

\( \mathcal{Q}_L^p \) has same approximation orders as full polynomial space \( \mathbb{P}^p \) but much fewer DOFs. Typically, on \( K \subset \mathbb{R}^d \):

\[
\dim(\mathcal{Q}_L^p) = \mathcal{O}_{p \to \infty}(p^{d-1}) \ll \dim(\mathbb{P}^p) = \mathcal{O}_{p \to \infty}(p^d).
\]

To approximate a BVP we also need:

- a (DG) variational formulation,
- a basis of \( \mathcal{Q}_L^p \).

In the rest of the talk, for the wave eq. only.

Missing:

- \( p \)-estimates,
- estimates in Sobolev norms.
Part II

Space–time DG for the wave equation
Initial–boundary value problem

Wave eq.: \(-\Delta u + c^{-2} \partial_t^2 u = 0\). Set \(v = \partial_t u\) and \(\sigma = -\nabla u\).

First-order initial–boundary value problem (Dirichlet): find \((v, \sigma)\) s.t.

\[
\begin{aligned}
\nabla v + \partial_t \sigma &= 0 \\
\nabla \cdot \sigma + \frac{1}{c^2} \partial_t v &= 0 \\
v(\cdot, 0) &= v_0, \quad \sigma(\cdot, 0) = \sigma_0 \\
v(x, \cdot) &= g
\end{aligned}
\]

in \(Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}, \ n \in \mathbb{N},\)
in \(Q,
\)
on \(\Omega,
\)
on \(\partial \Omega \times (0, T).
\)

Velocity \(c = c(x)\) piecewise smooth. \(\Omega \subset \mathbb{R}^n\) Lipschitz bounded.

- Neumann \(\sigma \cdot n = g\) & Robin \(\frac{\partial}{c} v - \sigma \cdot n = g\) BCs
- more general coeff.'s \(-\nabla \cdot (\rho^{-1} \nabla u) + G \partial_t^2 u = 0\)

Extensions:
- Maxwell equations
- elasticity
- \(1^{st}\) order hyperbolic systems...
Introduce space–time polytopic mesh $\mathcal{T}_h$ on $\Omega$. Assume: $c = c(\mathbf{x})$ smooth in each element.

Assume: each face $F = \partial K_1 \cap \partial K_2$ with normal $(\mathbf{n}^x_F, n^t_F)$ is either

- space-like: $c|\mathbf{n}^x_F| < n^t_F$
- or
- time-like: $n^t_F = 0$

Usual DG notation with averages $\{ \cdot \}$, $\mathbf{n}^x$-normal space jumps $[ \cdot ]_N$, $n^t$-time jumps $[ \cdot ]_t$.

Lateral boundary $\mathcal{F}^\partial_h := \partial \Omega \times (0, T)$. 
Multiply PDEs with test field \((w, \tau)\) & integrate by parts on \(K \in \mathcal{T}_h\):

\[- \int_K \left( v \left( \nabla \cdot \tau + c^{-2} \partial_t w \right) + \sigma \cdot \left( \nabla w + \partial_t \tau \right) \right) dV \]

\[+ \int_{\partial K} \left( (v \tau + \sigma w) \cdot \mathbf{n}_K^x + (\sigma \cdot \tau + c^{-2} v w) n_K^t \right) dS = 0.\]

This is an “ultra-weak” variational formulation (UWVF).

Approximate skeleton traces of \((v, \sigma)\) with numerical fluxes \((\hat{v}_h, \hat{\sigma}_h)\), defined as

\[\hat{v}_h := \begin{cases} v_h & \text{on } \mathcal{F}_h^{\text{space}} \cup \mathcal{F}_h^T \\ v_0 & \text{on } \mathcal{F}_h^0 \\ \{v_h\} + \beta [\sigma_h]_N & \text{on } \mathcal{F}_h^{\text{time}} \\ g & \text{on } \mathcal{F}_h^\partial \end{cases}\]

\[\hat{\sigma}_h := \begin{cases} \sigma_h & \text{on } \mathcal{F}_h^{\text{space}} \cup \mathcal{F}_h^T \\ \sigma_0 & \text{on } \mathcal{F}_h^0 \\ \{\sigma_h\} + \alpha [v_h]_N & \text{on } \mathcal{F}_h^{\text{time}} \\ \sigma_h - \alpha (v - g) n_\Omega^x & \text{on } \mathcal{F}_h^\partial \end{cases}\]

“upwind in time, elliptic-DG in space”.

\[\alpha = \beta = 0 \rightarrow \text{Kretzschmar–S.–T.–W.}, \quad \alpha \beta \geq \frac{1}{4} \rightarrow \text{Monk–Richter}.\]
Space–time DG formulation

Substitute the fluxes in the elemental equation, add volume penalty term as in (IMBERT-GÈRARD, MONK 2017), choose discrete space $\mathbf{V}_p \subset H^1(T_h)^{1+n}$, sum over $K \to$ write $xt$-DG:

$$\text{Seek } (v_h, \sigma_h) \in \mathbf{V}_p \text{ s.t., } \forall (w, \tau) \in \mathbf{V}_p,$$

$$A(v_h, \sigma_h; w, \tau) = \ell(w, \tau) \quad \text{where ...}$$

$$A(v_h, \sigma_h; w, \tau) := - \sum_{K \in T_h} \int_K \left(v_h (\nabla \cdot \tau + c^{-2} \partial_t w) + \sigma_h \cdot (\nabla w + \partial_t \tau)\right) dV$$

$$+ \int_{\mathcal{F}_{\text{space}}} \left(\frac{v_h^- \left[w\right]_t}{c^2} + \sigma_h^- \cdot [\tau]_t + v_h^- [\tau]_N + \sigma_h^- \cdot [w]_N\right) dS$$

$$+ \int_{\mathcal{F}_{\text{time}}} \left(\{v_h\} [\tau]_N + \{\sigma_h\} \cdot [w]_N + \alpha [v_h]_N \cdot [w]_N + \beta [\sigma_h]_N [\tau]_N\right) dS$$

$$+ \int_{\Omega \times \{T\}} (c^{-2} v_h w + \sigma_h \cdot \tau) dS + \int_{\mathcal{F}_h^\partial} (\sigma_h \cdot \mathbf{n}_\Omega + \alpha v_h) w dS$$

$$+ \sum_{K \in T_h} \int_K \left(\mu_1 (\nabla \cdot \sigma + c^{-2} \partial_t v)(\nabla \cdot \tau + c^{-2} \partial_t w) + \mu_2 (\partial_t \sigma + \nabla v) \cdot (\partial_t \tau + \nabla w)\right) dV,$$

$$\ell(w, \tau) := \int_{\Omega \times \{0\}} (c^{-2} v_0 w + \sigma_0 \cdot \tau) dS + \int_{\mathcal{F}_h^\partial} g(\alpha w - \tau \cdot \mathbf{n}_\Omega) dS.$$
Coercivity in DG skeleton norm

Key property, from integration by parts, is coercivity in DG norm:

\[ A(w, \tau; w, \tau) \geq \|\| (w, \tau) \|\|^2_{DG} \quad \forall (w, \tau) \in \prod_{K \in \mathcal{T}_h} H^1(K)^{n+1} \]

\[ \|\| (w, \tau) \|\|^2_{DG} := \frac{1}{2} \left\| \left( \frac{1 - \gamma}{n_F^t} \right)^{1/2} c^{-1} [w]_t \right\|^2_{L^2(\mathcal{F}_{h, \text{space}}^\text{space})} + \frac{1}{2} \left( \frac{1 - \gamma}{n_F^t} \right)^{1/2} \left\| [\tau]_t \right\|^2_{L^2(\mathcal{F}_{h, \text{space}}^\text{space})^n} \]

\[ + \frac{1}{2} \left\| c^{-1} w \right\|^2_{L^2(\mathcal{F}_{h, \text{time}}^0 \cup \mathcal{F}_{h, \text{time}}^T)^n} + \frac{1}{2} \left\| \tau \right\|^2_{L^2(\mathcal{F}_{h, \text{time}}^0 \cup \mathcal{F}_{h, \text{time}}^T)^n} \]

\[ + \left\| \alpha^{1/2} [w]_N \right\|^2_{L^2(\mathcal{F}_{h, \text{time}})^n} + \left\| \beta^{1/2} [\tau]_N \right\|^2_{L^2(\mathcal{F}_{h, \text{time}})^n} + \left\| \alpha^{1/2} w \right\|^2_{L^2(\mathcal{F}_{h, \text{time}})^n} \]

\[ + \sum_{K \in \mathcal{T}_h} \left( \left\| \mu_1^{1/2} (c \nabla \cdot \tau + c^{-1} \partial_t w) \right\|^2_{L^2(K)} + \left\| \mu_2^{1/2} (\nabla w + \partial_t \tau) \right\|^2_{L^2(K)^n} \right) \]

\[ \gamma := \frac{\|c\|_{C^0(F)} |n_F^x|}{n_F^t} \in [0, 1) \sim \text{distance between space-like face } F \text{ & char. cone}. \]

▶ Well-posedness and quasi-optimality \hspace{1cm} (\forall \text{ discrete spaces})

\[ \|\| (v, \sigma) - (v_h, \sigma_h) \|\|_{DG} \leq 3 \inf_{(w, \tau) \in \mathbf{V}_p} \|\| (v, \sigma) - (w, \tau) \|\|_{DG} \]
Global, implicit and explicit $xt$-DG schemes

1. $xt$-DG formulation is global in space–time domain $\Omega$:
   - huge linear system!
   - Good for adaptivity, DD...

2. If mesh is partitioned in time-slabs $\Omega \times (t_{j-1}, t_j)$
   then matrix is block lower-triangular:
   - sequentially solve a system for each slab
   - implicit method.

3. If mesh is “tent-pitched”, DG solution
   is computed with a sequence of local systems:
   - explicit method, allows parallelism!
   ÜNGÖR–SHEFFER, MONK–RICHTER...

Versions 1–2–3 are algebraically equivalent (on the same mesh).
Related works on $\mathbf{xt}$-DG formulations

Proposed $\mathbf{xt}$-DG formulation comes from:

- **MONK, RICHTER** 2005, linear symmetric hyperbolic systems, tent-pitched meshes, $\mathbb{P}^p$ spaces, $\alpha \beta \geq \frac{1}{4}$
- **KRETZSCHMAR, SCHNEPP ET AL.** 2014–16 Maxwell eq.s, Trefftz
- **M., PERUGIA** 2018 Trefftz error analysis
- **PERUGIA, SCHOEBERL, STOCKER, WINTERSTEIGER** 2020 Trefftz & tents
  pw-smooth $c$, quasi-Trefftz

Related works:

- **BANSAL, M., PERUGIA, SCHWAB** 2021 corner sing.s, sparse grids
- **BARUCQ, CALANDRA, DIAZ, SHISHENINA** 2020 Trefftz + elasticity
- **GÓMEZ, M.** 2021 Trefftz + Schrödinger

Many other $\mathbf{xt}$-DG formulations for waves exist!
Part III

Quasi-Trefftz bases for the wave equation
Quasi-Trefftz space

Define wave operator

\[ \square_{G} u := \Delta u - G \partial_{t}^2 u, \quad G(x) = c^{-2} \text{ smooth.} \]

Fix \((x_K, t_K) \in K \subset \mathbb{R}^{n+1}.\)

Quasi-Trefftz (polynomial) space:

\[ \mathcal{Q}U^p(K) := \{ u \in P^p(K) : D^i \square_G u(x_K, t_K) = 0, \quad \forall |i| \leq p - 2 \} \]

\[ \mathcal{Q}W^p(K) := \{ (\partial_t u, -\nabla u), u \in \mathcal{Q}U^{p+1}(K) \} \]

- Taylor polynomials of smooth wave solutions belong to \(\mathcal{Q}U^p(K)\)
- \(xt\)-DG is quasi-optimal

It follows that \(xt\)-DG converges with optimal rates in DG norm:

\[ \|\|(v, \sigma) - (v_h, \sigma_h)\|_{DG} \leq C \sup_{K \in \mathcal{T}_h} h_{K,c}^{p+1/2} |u|_{C^{p+2}(K)} \]
Quasi-Trefftz basis

The local discrete space is clear. How to construct a basis for it? Use the following fact:

\[ u \in \mathbb{QUP}(K) \] is determined by \( u(\cdot, t_K) \) and \( \partial_t u(\cdot, t_K) \).

Choose two \( x \)-only polynomial basis:

\[ \{ \widehat{b}_J \}_{J=1,\ldots,(p+n)} \text{ for } \mathbb{P}^p(\mathbb{R}^n), \quad \{ \tilde{b}_J \}_{J=1,\ldots,(p-1+n)} \text{ for } \mathbb{P}^{p-1}(\mathbb{R}^n). \]

Construct a basis for \( \mathbb{QUP}(K) \) “evolving” \( \widehat{b}_J \) and \( \tilde{b}_J \) in time:

\[
\begin{align*}
\{ b_J \in \mathbb{QUP}(K) : & \quad b_J(\cdot, t_K) = \widehat{b}_J, \quad \partial_t b_J(\cdot, t_K) = 0, \quad \text{for } J \leq (\frac{p+n}{n}) \\
& \quad b_J(\cdot, t_K) = 0, \quad \partial_t b_J(\cdot, t_K) = \tilde{b}_{J-(\frac{p+n}{n})}, \quad \text{for } (\frac{p+n}{n}) < J \}
\end{align*}
\]

for \( J = 1, \ldots, (\frac{p+n}{n}) + (\frac{p-1+n}{n}). \)

We prove that this defines a basis and show how to compute \( \{ b_J \} \).
Computation of basis coefficients

Fix \( n = 1 \) (for simplicity). Denote \( G(x) = \sum_{m=0}^{\infty} g_m (x-x_K)^m \). \( g_0 > 0 \).

Monomial expansion of basis element:

\[
b_J(x, t) = \sum_{i_x + i_t \leq p} a_{i_x, i_t} (x-x_K)^{i_x} (t-t_K)^{i_t},
\]

Cauchy conditions \((b_J(\cdot, t_K), \partial_t b_J(\cdot, t_K))\) determine \( a_{i_x,0}, a_{i_x,1} \).

\( b_J \in \mathcal{QU}^p \) if coeff.s satisfy:

\[
\partial_x^{i_x} \partial_t^{i_t} \bigtriangleup_G b_J(x_K, t_K) = (i_x + 2)! i_t! \ a_{i_x+2, i_t} - \sum_{j_x=0}^{i_x} i_x! (i_t + 2)! g_{i_x-j_x} \ a_{j_x, i_t+2} = 0
\]

Linear system for coeff.s \( a_{i_x, i_t} \).

Compute \( a_{i_x, i_t+2} \) from coefficients \( \bullet \):

first loop across diagonals \( \nearrow \),
then along diagonals \( \searrow \).
Data: \((g_m)_{m \in \mathbb{N}_0}, x_K, t_K, p\).

Choose favourite polynomial bases \(\{\widehat{b}_J\}, \{\widetilde{b}_J\}\) in \(x, t\),
→ coeff’s \(a_{k_x,0}, a_{k_x,1}\).

For each \(J\) (i.e. for each basis function), construct \(b_J\) as follows:

\[
\text{for } \ell = 2 \text{ to } p \quad \text{(loop across diagonals ↑)} \text{ do}
\]

\[
\text{for } i_t = 0 \text{ to } \ell - 2 \quad \text{(loop along diagonals ↖) do}
\]

set \(i_x = \ell - i_t - 2\) and compute

\[
a_{i_x,i_t+2} = \frac{(i_x + 2)(i_x + 1)}{(i_t + 2)(i_t + 1)g_0} a_{i_x+2,i_t} - \sum_{j_x=0}^{i_x-1} \frac{g_{i_x-j_x}}{g_0} a_{j_x,i_t+2}
\]

end

end

\[
b_J(x, t) = \sum_{0 < k_x + k_t \leq p} a_{k_x,k_t} (x - x_K)^{k_x} (t - t_K)^{k_t}
\]
In higher space dimensions $n > 1$, with $G(x) = \sum_{i_x} (x - x_K)^i_x g_i_x$, the algorithm is the same with a further inner loop:

\[
\text{for } \ell = 2 \text{ to } p \quad (\text{loop across } \{|i_x| + i_t = \ell - 2\} \text{ hyperplanes, ↑}) \quad \text{do}
\]

\[
\text{for } i_t = 0 \text{ to } \ell - 2 \quad (\text{loop across constant-}t \text{ hyperplanes ↑}) \quad \text{do}
\]

\[
\text{for } i_x \text{ with } |i_x| = \ell - i_t - 2 \quad \text{do}
\]

\[
a_{i_x, i_t+2} = \sum_{l=1}^{n} \frac{(i_{x_l} + 2)(i_{x_l} + 1)}{(i_t + 2)(i_t + 1)} g_0 a_{i_x+2\ell, i_t} - \sum_{j_x < i_x} \frac{g_{i_x-j_x}}{g_0} a_{j_x, i_t+2}
\]

end

end

end
More general IBVPs

Everything extends to 2 piecewise-smooth material parameters \( \rho, G \):

\[
\nabla v + \rho \partial_t \sigma = 0, \quad \nabla \cdot \sigma + G \partial_t v = 0,
\]

Wavespeed is \( c = (\rho G)^{-1/2} \).

Second-order version:

\[-\nabla \cdot \left( \frac{1}{\rho} \nabla u \right) + G \partial_t^2 u = 0 \quad (v = \partial_t u, \, \sigma = -\frac{1}{\rho} \nabla u).\]

Basis coefficient algorithm needs some more terms.

If the 1st-order IBVP does not come from a 2nd-order one, we use

\[
\mathcal{QT}^p(K) := \left\{ (w, \tau) \in \mathbb{P}^p(K)^{n+1} \mid \begin{array}{l}
D^i(\nabla w + \rho \partial_t \tau)(\mathbf{x}_K, t_K) = 0 \\
D^i(\nabla \cdot \tau + G \partial_t w)(\mathbf{x}_K, t_K) = 0 \\
\forall |i| \leq p - 1
\end{array} \right\}
\]

This space is only slightly larger (\( \approx \frac{n+1}{2} \times \), still \( O_{p \to \infty}(p^n) \) DOFs) and allows the same analysis.
Numerics

- Implemented in NGSolve.
  https://github.com/PaulSt/NGSTrefftz
- Both Cartesian and tent-pitched meshes.
- Volume penalty term not needed in computations.
- DG flux coefficients $\alpha^{-1} = \beta = c$, but even $\alpha = \beta = 0$ works.
- Good conditioning.
- Monomial bases $\{\hat{b}_J\}$, $\{\tilde{b}_J\}$ outperform Legendre/Chebyshev.
Numerics 1: convergence

Compare quasi-Trefftz and full polynomials spaces

\[ \text{QW}^p(\mathcal{T}_h) := \Pi_K \{(\partial_t u, -\nabla u), \ u \in \text{QU}^{p+1}(K)\} \]
\[ \text{YP}(\mathcal{T}_h) := \Pi_K \{(\partial_t u, -\nabla u), \ u \in \mathbb{P}^{p+1}(K)\} \]

DG-norm error: optimal order in \( h \), exponential in \( p \).

\[
\begin{align*}
n = 2, \quad G &= (x_1 + x_2 + 1)^{-1}, \quad u = (x_1 + x_2 + 1)^{2.5} e^{-\sqrt{7.5}t}, \quad Q = (0, 1)^3.
\end{align*}
\]
Quasi-Trefftz wins $> 1$ order of magnitude against full polynomials:

$$h = 2^{-3}, 2^{-4}, \quad p = 1, 2, 3, 4.$$

$$n = 2, \quad G = x_1 + x_2 + 1, \quad u = \text{Ai}(-x_1 - x_2 - 1) \cos(\sqrt{2}t), \quad Q = (0, 1)^3.$$
(n = 2) Final-time error, computational time (sequential), speedup:
(#dof$^{-1/3}$ $\sim$ h)
Numerics 4: energy conservation

Plane wave through medium with $G = 1 + x_2$ in $(0, 1)^3$:

$$\mathcal{E} = \frac{1}{2} \int_{\Omega} (c^{-2} \mathbf{v}^2 + |\sigma|^2) \, dS$$

DG scheme is (provably) dissipative. For $p = 3$, $h = 2^{-7}$, only 0.076% loss.
Numerics 5: rough solutions

\[ v_0(x) = \sigma_0(x) = \max(0.25 - |x|, 0) = \mathbb{1} \quad \in H^1(\Omega) \setminus C^1(\Omega), \]

\[ G(x) = (1 + x)^{-2}, \quad \rho = 1, \quad c = 1 + x, \quad \text{on } \Omega = (-0.5, 0.5). \]

<table>
<thead>
<tr>
<th>( h )</th>
<th>( L^2(\Omega \times {T})^2 ) error</th>
<th>rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^{-6} )</td>
<td>0.020</td>
<td></td>
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<tr>
<td>( 2^{-7} )</td>
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<td>( 2^{-10} )</td>
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</tr>
</tbody>
</table>

\( \mathcal{QW}^0(T_h) \) (piecewise-constants) on uniform Cartesian meshes.

Optimal \( \mathcal{O}(h) \) convergence even for \( u \in H^2(T_h) \setminus C^2(T_h) \).
Quasi-Trefftz DG:

- Extend Trefftz scheme to piecewise-smooth coefficients. Basis are PDE solution “up to given order in $h$”.
- Simple construction of basis functions: same “Cauchy data” at element centre as for Trefftz.
- Use in $xt$-DG, stability and error analysis. High orders of convergence in $h$, much fewer DOFs than standard polynomial spaces.

If you use DG for linear PDEs, try quasi-Trefftz & save DOFs!

https://github.com/PaulSt/NGSTrefftz

Thank you!