WAVES2026 — — MONTRÉAL — — 22–26.6.2026

Trefftz methods with evanescent plane waves

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A century of Trefftz methods: 1926–2026

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Trefftz Workshop · 7–9.9.2026 · Vienna
C. Lehrenfeld & P. Stocker
https://trefftz2026.univie.ac.at/

Trefftz for Helmholtz

Trefftz methods are popular for linear time-harmonic (acoustic, el.magn., elastic) waves

E.g.: UWVF, TDG, PWDG, DEM, VTCR, WBM, LS, PUM... mostly in DG / DD setting

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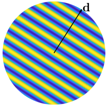
Typical basis functions are propagative plane waves (PPWs):

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{R}^n \quad \mathbf{d}\cdot\mathbf{d} = 1$$



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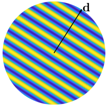

- PPWs are complex exponentials:
  easy & cheap to manipulate, evaluate, differentiate, integrate. . .
  → preferred against other Trefftz functions (e.g. circular waves)
- PPWs approximate Helmholtz solutions with better rates vs DOFs than polynomials
  CESSENAT, DESPRÉS 1998, Taylor-based, h
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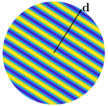

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So... what's the issue?

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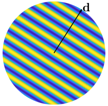

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So...what's the issue? Instability!

A negative result — PPW instability

Take $\Omega = B_1 \subset \mathbb{R}^n$ the unit disc/ball, $n \in \{2,3\}$. Choose your favourite

- wavenumber $\kappa > 0$
- norm on Ω (e.g. $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\frac{\|\cdot\|_{H^1(\Omega)}}{\|\mathrm{PW}\|_{H^1(\Omega)}}$)
- target relative accuracy $0 < \delta < 1$ (e.g. 0.5, 1% or 10^{-10})
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Then we can give you an explicit $u = u_{\kappa,\delta,M}(\mathbf{x})$ such that:

$$u \in C^\infty(\mathbb{R}^n), \qquad \Delta u + \kappa^2 u = 0, \qquad \|u\|_{\text{your favourite}} = 1$$

$$\begin{array}{c}\forall P \in \mathbb{N},\\ \forall \boldsymbol{\mu} \in \mathbb{C}^P\\ \forall \mathbf{d}_1, \dots, \mathbf{d}_P \in \mathbb{R}^n\\ |\mathbf{d}_p| = 1\end{array} \quad \text{with} \quad \left\| u - \sum_{p=1}^{P} \mu_p e^{i\kappa \mathbf{d}_p \cdot \mathbf{x}} \right\|_{\text{your favourite}} \le \delta \quad \Longrightarrow$$

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Every PPW combination with accuracy δ has huge coefficient vector: cancellation!
If $M > (\text{machine precision})^{-1}$, we can't represent u in computer arithmetic with PPWs

PPW approximation is unstable!

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Stability = existence of small coeff.s approximation (ADCOCK, HUYBRECHS 2019–20)
Requires regularization (oversampling + SVD truncation)

Part I

Evanescent plane waves

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Evanescent plane waves (EPW): $e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d}\in\mathbb{C}^n \qquad \mathbf{d}\cdot\mathbf{d} = d_1^2+\cdots+d_n^2 = 1$

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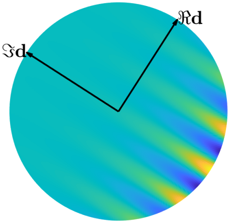
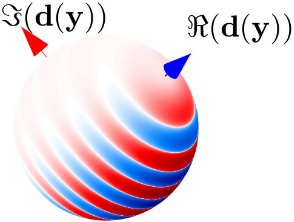
- Complex $\mathbf{d}$!
- Helmholtz solutions
- Idea from WBM (wave-based method) by Wim Desmet, Elke Deckers etc (Leuven)
  EPWs often used in modelling, not so much in numerics
- Complex exponentials: cheap computations, exact quadrature...





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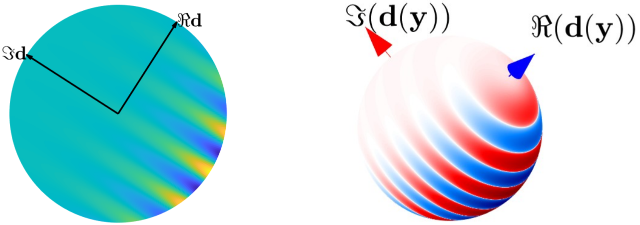
- $e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = e^{i\kappa \Re\mathbf{d}\cdot\mathbf{x}}\, e^{-\kappa \Im\mathbf{d}\cdot\mathbf{x}}$
  $\Re\mathbf{d}$: propagation direction
  $\Im\mathbf{d}$: evanescence direction
- $|e^{i\kappa \mathbf{d}\cdot\mathbf{x}}| = e^{-\kappa \Im\mathbf{d}\cdot\mathbf{x}}$ essentially localised, need normalisation, easy e.g. in L^∞

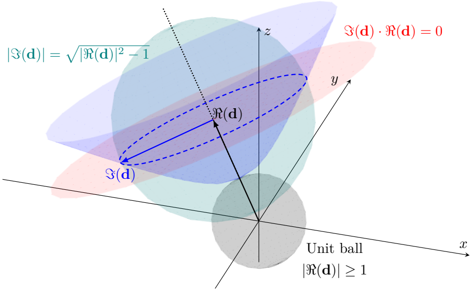


Evanescent plane waves: parametrisation

$e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$ $\quad \mathbf{d} = \Re\mathbf{d} + i\Im\mathbf{d} \in \mathbb{C}^n$

$(\Delta + \kappa^2)e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = 0 \iff$

$$\mathbf{d}\cdot\mathbf{d} = 1 \iff \begin{cases} |\Re\mathbf{d}|^2 - |\Im\mathbf{d}|^2 = 1 \\ \Re\mathbf{d}\cdot\Im\mathbf{d} = 0 \end{cases}$$



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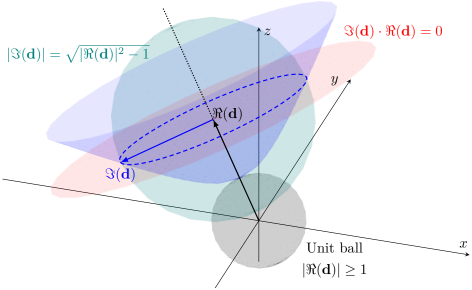

$\mathbf{d}$ parametrised by:

- $\mathbf{p} = \frac{\Re\mathbf{d}}{|\Re\mathbf{d}|} \in \mathbb{S}^{n-1}$: propagation direction
- $\mathbf{e} = \frac{\Im\mathbf{d}}{|\Im\mathbf{d}|} \in \mathbb{S}^{n-1}$: evanescence direction $\perp \mathbf{p}$
- $\eta = |\Im\mathbf{d}| \in [0,\infty)$: evanescence strength

  $\eta = 0 \iff$ EPW is PPW $\qquad \zeta = |\Re\mathbf{d}| = \sqrt{1+\eta^2}$ $\qquad \mathbf{d} = \sqrt{1+\eta^2}\,\mathbf{p} + i\,\eta\,\mathbf{e}$

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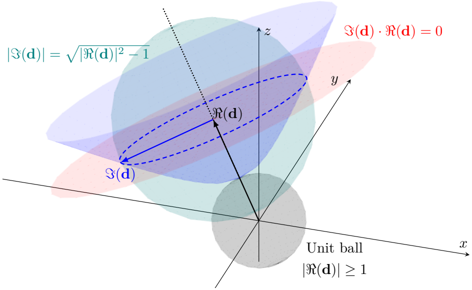

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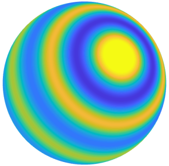
Parameter vector $\mathbf{y} := (\mathbf{p}, \mathbf{e}, \eta) \in Y := \mathbb{S}^{n-1}\times\mathbb{S}^{n-2}\times[0,\infty)$, $\qquad \mathrm{EW}_{\mathbf{y}}(\mathbf{x}) := e^{i\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}}$

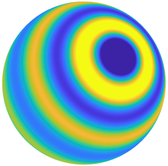
In 2D: $\mathbf{p} \in \mathbb{S}^1 \sim \theta \in [0, 2\pi)$, $\quad \mathbf{e} \sim \pm 1$
In 3D: use Euler angles of rotation from reference direction $\mathbf{d}_\uparrow = (i\eta,\ 0,\ \sqrt{1+\eta^2}) \to \mathbf{d}$

Approximation of 3D fundamental solution $u(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}-\mathbf{s}|}}{4\pi|\mathbf{x}-\mathbf{s}|}$ on B_1

$\mathrm{dist}(\mathbf{s}, \Omega) = \lambda$



$\mathrm{dist}(\mathbf{s}, \Omega) = \lambda/2$



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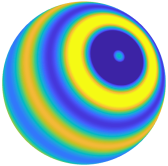

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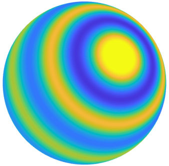

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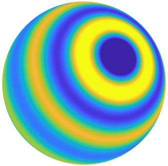

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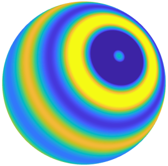
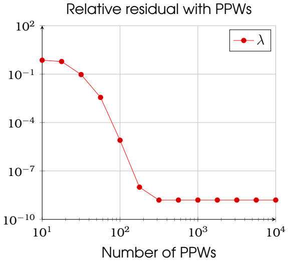


Approximation by equispaced PPWs: 

$\kappa = 10, \quad \mathrm{diam}(\Omega) \approx 3.18\lambda$



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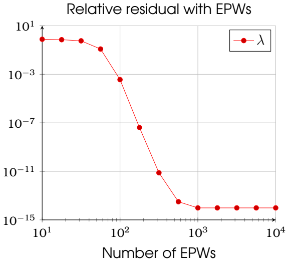

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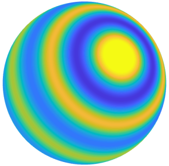

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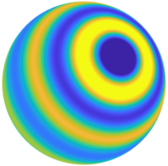

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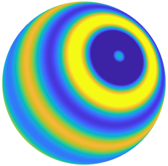

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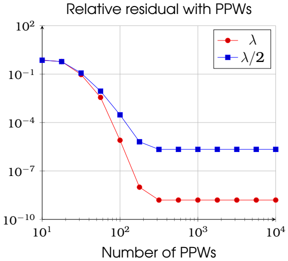

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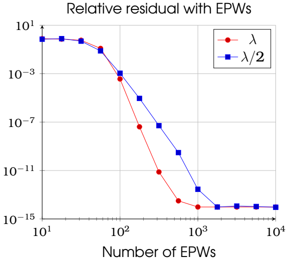

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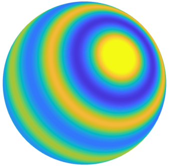

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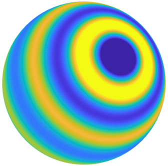

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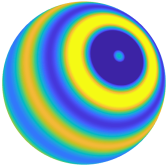
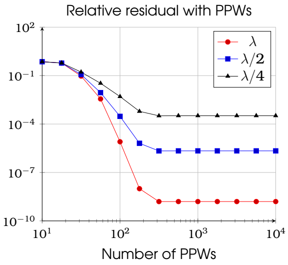


Approximation by equispaced PPWs: 

$\kappa = 10, \quad \mathrm{diam}(\Omega) \approx 3.18\lambda$



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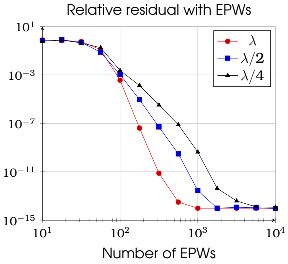

Approximation of 3D fundamental solution $u(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}-\mathbf{s}|}}{4\pi|\mathbf{x}-\mathbf{s}|}$ on B_1

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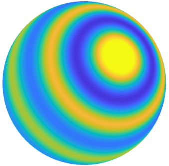

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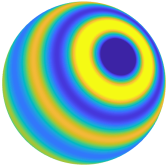

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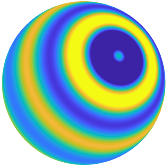

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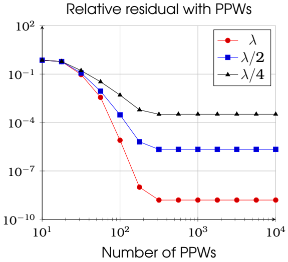

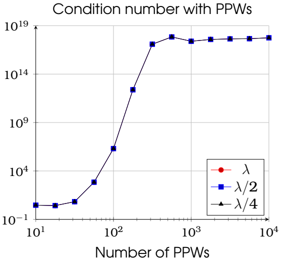

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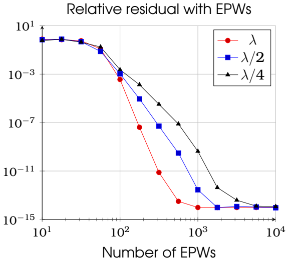

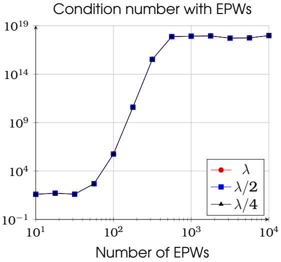

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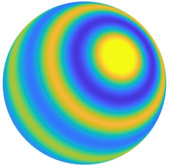

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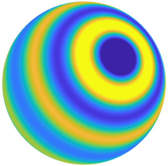

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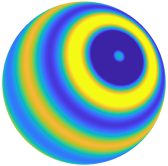

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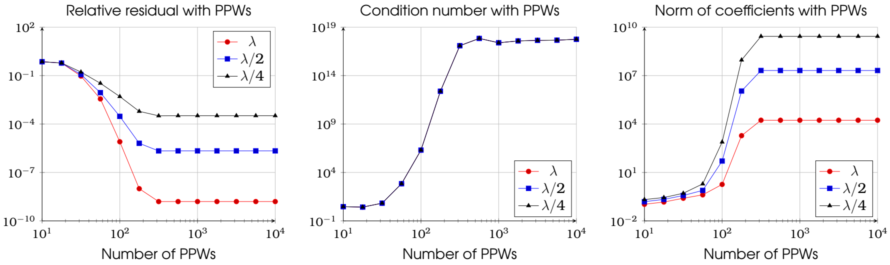

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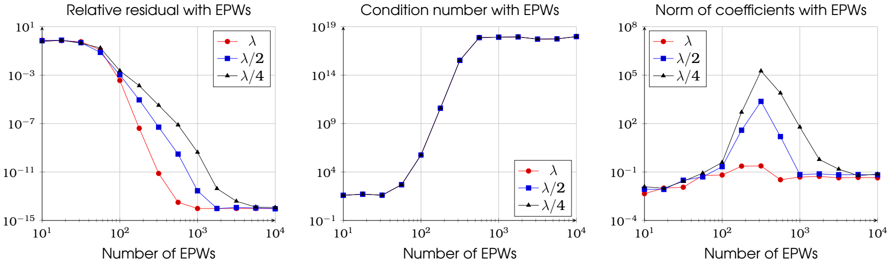

Herglotz functions & EPW Herglotz representation

Herglotz functions are continuous superposition of PPWs: (COLTON, KRESS...)

$$u(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} v(\mathbf{d})\, e^{i\kappa \mathbf{d}\cdot\mathbf{x}}\, d\mathbf{d} \qquad \text{for} \quad v \in L^2(\mathbb{S}^{n-1})$$

Only some Helmholtz solutions $u \in C^\infty(\mathbb{R}^n)$ are Herglotz:
$L^2(\mathbb{S}^{n-1}) \ni v \mapsto u$ has dense image (WECK 2004) and is compact, but is not surjective

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Idea: Define the EPW version of Herglotz functions:

$$u(\mathbf{x}) = (Tv)(\mathbf{x}) := \int_Y v(\mathbf{y})\, \underbrace{e^{i\kappa \mathbf{d}(\mathbf{y})\cdot\mathbf{x}}}_{\mathrm{EW}_{\mathbf{y}}(\mathbf{x})}\, w^2(\mathbf{y})\, d\mathbf{y} \qquad \text{for} \quad v \in L^2_{w^2}(Y)$$

Weight $w(\mathbf{y}) = e^{-\kappa\eta}\eta^{\frac{2n-5}{4}}$ is a normalisation, $Y = \mathbb{S}^{n-1} \times \mathbb{S}^{n-2} \times [0,\infty)$ is unbounded in η

Herglotz representation on the disc and the ball

Theorem: Helmholtz solutions on B_1 are EPW superposition

For Ω the disc/ball B_1, $T : \mathcal{A} \subset L^2_{w^2}(Y) \to \mathcal{B} := \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\}$ is invertible.

In particular, for all Helmholtz solutions $u \in H^1(B_1)$, there is a density $v = T^{-1}u$ such that

$$u(\mathbf{x}) = \int_Y v(\mathbf{y})\ \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\ w^2(\mathbf{y})\ \mathrm{d}\mathbf{y}, \qquad \|v\|_{L^2_{w^2}(Y)} \leq C\|u\|_{H^1(B_1)}$$

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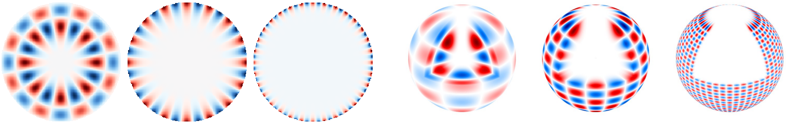
$$2D: \quad b_\ell(\mathbf{x}) = \beta_\ell\ J_\ell(\kappa r)\ \mathrm{e}^{\mathrm{i}\ell\vartheta}$$

$$3D: \quad b_\ell^m(\mathbf{x}) = \beta_\ell\, j_\ell(\kappa|\mathbf{x}|)\ Y_\ell^m\left(\tfrac{\mathbf{x}}{|\mathbf{x}|}\right)$$

$\beta_\ell = H^1_\kappa(B_1)$-normalisation



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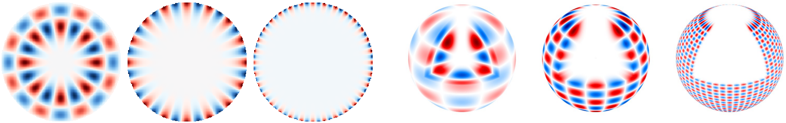
o.n. basis a_ℓ^m of $\mathcal{A}$ s.t. T is diagonal & continuous

$T : a_\ell^m \to \tau_\ell b_\ell^m, \qquad 0 < \tau_- \le |\tau_\ell| \le \tau_+ < \infty \quad \forall \ell$

a_ℓ^m explicit up to normalisation



EPWs are a continuous frame for the Helmholtz solution space $\mathcal{B}$. $\quad T =$ synthesis operator

Weyl expansion

Jacobi–Anger approach relies on expansion in orthonormal basis: works only for $\Omega = B_1$

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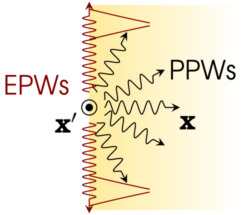
Expand 2D Helmholtz fundamental solution $\Phi(\mathbf{x}, \mathbf{x}')$ in half plane with PPWs+EPWs:

$$\mathrm{PW}_\theta(\mathbf{x}) := \mathrm{EW}_{(\theta,\pm 1,0)} = e^{i\kappa(\cos\theta\, x_1 + \sin\theta\, x_2)}$$

$$\mathrm{EW}_{(\theta,\varphi,\eta)}(\mathbf{x}) := e^{i\kappa\sqrt{1+\eta^2}\binom{\cos\theta}{\sin\theta}\cdot\mathbf{x}} e^{\varphi\kappa\eta\binom{-\sin\theta}{\cos\theta}\cdot\mathbf{x}}$$

$$\mathbf{y} = (\theta, \varphi, \eta) \in Y := [0, 2\pi) \times \{\pm 1\} \times [0, \infty)$$



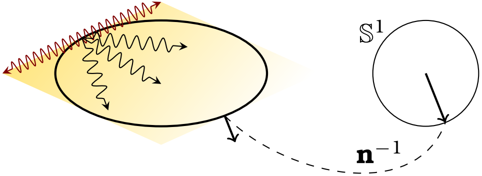
$$x_1 > x_1'$$

$$\Phi(\mathbf{x}, \mathbf{x}') = \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x} - \mathbf{x}'|)$$

$$= \frac{i}{4\pi} \int_{-\pi/2}^{\pi/2} \mathrm{PW}_\theta(\mathbf{x} - \mathbf{x}')\, d\theta + \frac{1}{4\pi} \int_0^\infty \sum_{\varphi\in\{\pm 1\}} \frac{\mathrm{EW}_{(\varphi\frac{\pi}{2},\varphi,\eta)}(\mathbf{x} - \mathbf{x}')}{\sqrt{1+\eta^2}}\, d\eta$$

PW expansion in convex domains

For smooth convex $\Omega \subset \mathbb{R}^2$ with strictly positive curvature,
combine (rotated) expansion of $\Phi(\mathbf{x}, \mathbf{y})$ with $\mathcal{S}$ingle+$\mathcal{D}$ouble layer representation



$$u(\mathbf{x}) = \mathcal{S}\gamma_N u(\mathbf{x}) - \mathcal{D}\gamma_D u(\mathbf{x})$$
$$= \int_{\partial\Omega} [\Phi(\mathbf{x}, \mathbf{y})\partial_{\mathbf{n}} u(\mathbf{y}) - \partial_{\mathbf{n}} \Phi(\mathbf{x}, \mathbf{y}) u(\mathbf{y})] \, \mathrm{d}s(\mathbf{y})$$

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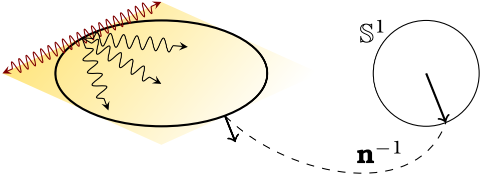

We can write Helmholtz solution u as superposition of PPWs and EPWs

$$u(\mathbf{x}) = \int_0^{2\pi} (\mathcal{Q}^P u)(\theta) \, \mathrm{PW}_\theta(\mathbf{x}) \, \mathrm{d}\theta + \int_Y (\mathcal{Q}^E u)(\mathbf{y}) \, \mathrm{EW}_\mathbf{y}(\mathbf{x}) \, w^2(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

EPW weight $\quad w^2(\mathbf{y}) = \left(\left| \mathrm{EW}_\mathbf{y}(\mathbf{n}^{-1}(\theta - \varphi\pi/2)) \right|^2 (1+\eta^2)^\gamma \, \mathrm{curv}(\mathbf{n}^{-1}(\theta - \varphi\pi/2)) \right)^{-1} \quad \gamma < 1/2$

with bounded solution-to-PW-coefficient operators ($\mathcal{Q}^\bullet$ can be written explicitly)

$$\mathcal{Q}^P : H^{\frac{3}{2}+\epsilon}(\Omega) \to L^2(\mathbb{S}^1), \qquad \mathcal{Q}^E : H^{\frac{3}{2}+\epsilon}(\Omega) \to L^2_{w^2}(Y)$$

See (GALANTE, PhD dissertation 2026)

How to select $\{\mathbf{y}_p\}_{p=1}^P \in Y$ and discrete EPW basis $\{\mathrm{e}^{\mathrm{i}\kappa\mathbf{d}(\mathbf{y}_p)\cdot\mathbf{x}}\}_p$?

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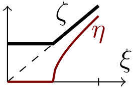

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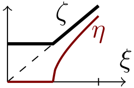

- The discrete Trefftz space is

$$\mathbb{T}_P(K) = \mathrm{span}\left\{\frac{\mathrm{EW}_p}{\|\mathrm{EW}_p\|_{L^\infty(K)}}\right\}_{p=1}^P \quad \text{with} \quad \mathrm{EW}_p(\mathbf{x}) = e^{i\kappa\zeta_p\binom{\cos\theta_p}{\sin\theta_p}\cdot\mathbf{x}} e^{-\kappa\eta_p\varphi_p\binom{-\sin\theta_p}{\cos\theta_p}\cdot\mathbf{x}}$$

If $P \le 2\kappa\,\mathrm{diam}(K) \Rightarrow$ only PPWs Otherwise, $P - 2\kappa\,\mathrm{diam}(K)$ out of P DOFs are EPWs

Part II

Ultra-weak variational formulation

Ultra-weak variational formulation (UWVF)

Consider 2D Helmholtz impedance BVP $\begin{cases} \Delta u + \kappa^2 u = 0 & \text{on } \Omega \subset \mathbb{R}^n \\ \partial_{\mathbf{n}} u - \imath\kappa\vartheta u = g & \in L^2(\partial\Omega) \end{cases}$

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Quasi-uniform mesh $\mathcal{T} = \{K\}$, Trefftz space $V_h = \prod_{K \in \mathcal{T}} \mathbb{T}_p(K)$, extend $0 < \vartheta \in L^\infty(\bigcup_K \partial K)$

Robin traces: $\gamma_\pm^K : \mathbb{T}_p(K) \to L^2(\partial K)$, $\gamma_\pm^K v = \pm\partial_{\mathbf{n}_K} v - \imath\kappa\vartheta v$

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UWVF (Cessenat, Després 1998)

Find $u_h \in V_h$ s.t.

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \vartheta^{-1}\, \gamma_-^K u_h\, \overline{\gamma_-^K v_h} - \sum_{K_1\in\mathcal{T}} \sum_{K_2\in\mathcal{T}} \int_{\partial K_1 \cap \partial K_2} \vartheta^{-1}\, \gamma_-^{K_1} u_h\, \overline{\gamma_+^{K_2} v_h}$$
$$= \sum_{K\in\mathcal{T}} \int_{\partial K\cap\partial\Omega} \vartheta^{-1}\, g\, \overline{\gamma_+^K v_h} \qquad \forall v_h \in V_h$$

Can be written as DG (Gabard '07, Buffa Monk '08, Gittelson Hiptmair Perugia '09)

Consistent with BVP, coercive, well-posed, quasi-optimal in skeleton norm(s), sparse

Extensions to: Dirichlet, Neumann, Q-BCs, DtN, multiple scattering, waveguides, quasi-periodic, Maxwell, elasticity, Friedrichs, sources, piecewise-constant media...

Stable approximations

General message from (Adcock, Huybrechs 2019–20)

If bounded-coefficient approximations exist in discrete space,
then computer-arithmetic stable Galerkin approximation is possible. . .

"Quasi-optimality with roundoff":

$$\left\| u - u_h[\boldsymbol{\mu}_{\text{Galerkin}}] \right\| \lesssim \inf_{\boldsymbol{\mu} \in \mathbb{C}^{N^{\text{trial}}}} \left(\left\| u - u_h[\boldsymbol{\mu}] \right\| + \epsilon \left\| \boldsymbol{\mu} \right\|_{\mathbb{C}^{N^{\text{trial}}}} \right)$$

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"Quasi-optimality with roundoff":

$$\left\| u - u_h[\boldsymbol{\mu}_{\text{Galerkin}}] \right\| \lesssim \inf_{\boldsymbol{\mu} \in \mathbb{C}^{N^{\text{trial}}}} \left(\left\| u - u_h[\boldsymbol{\mu}] \right\| + \epsilon \left\| \boldsymbol{\mu} \right\|_{\mathbb{C}^{N^{\text{trial}}}} \right)$$

Applies to redundant "basis"

Ill-conditioning is unavoidable, but stable solutions are possible

Analysis based on frame theory

Also: (Herremans, Huybrechs 2026)

Oversampled regularised UWVF

Oversampling = test space larger than trial   ($P_{\text{test}} = 1.1 P_{\text{trial}}$ on same mesh works well)

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UWVF matrix:

$$\mathbf{A}_{\ell,m} = \mathbf{D}_{\ell,m} - \mathbf{C}_{\ell,m} = \underbrace{\sum_{K\in\mathcal{T}} \int_{\partial K} \vartheta^{-1}\, \gamma_-^K \varphi_m\, \overline{\gamma_-^K \varphi_\ell}}_{\mathbf{D} = \operatorname{diag}_{K\in\mathcal{T}}(\mathbf{D}_K) \text{ block diagonal}} - \underbrace{\sum_{K_1\in\mathcal{T}} \sum_{K_2\in\mathcal{T}} \int_{\partial K_1 \cap \partial K_2} \vartheta^{-1}\, \gamma_-^{K_1} \varphi_m\, \overline{\gamma_+^{K_2} \varphi_\ell}}_{\mathbf{C}}$$

In standard UWVF with test=trial, $\mathbf{D}^{-1}\mathbf{C}$ is a contraction

$\rightarrow$ instead of $(\mathbf{D} - \mathbf{C})\mathbf{u} = \mathbf{b}$ solve $(\mathbf{I} - \mathbf{D}^{-1}\mathbf{C})\mathbf{u} = \mathbf{D}^{-1}\mathbf{b}$

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$\to$ instead of $(\mathbf{D}-\mathbf{C})\mathbf{u} = \mathbf{b}$ solve $(\mathbf{I}-\mathbf{D}^{-1}\mathbf{C})\mathbf{u} = \mathbf{D}^{-1}\mathbf{b}$

(Barucq, Bendali, Diaz, Tordeux 2021)

Local SVD for each block: $\mathbf{D}_K = \mathbf{U}_K \,\mathrm{diag}(\sigma_1, \sigma_2, \ldots)\, \mathbf{V}_K^* \in \mathbb{C}^{N_K^{\text{test}} \times N_K^{\text{trial}}}$

Local truncated pseudoinverse: $\boldsymbol{\Sigma}_{K,\epsilon}^\dagger = \mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_q^{-1}, 0, \ldots, 0), \quad \frac{\sigma_q}{\sigma_1} \geq \epsilon > 0$

Global truncated pseudoinverse: $\mathbf{D}_\epsilon^\dagger = \mathrm{diag}_{K\in\mathcal{T}}(\mathbf{V}_K \boldsymbol{\Sigma}_{K,\epsilon}^\dagger \mathbf{U}_K^*)$

Linear system: $(\mathbf{D}-\mathbf{C})\mathbf{u} = \mathbf{b} \quad \to \quad (\mathbf{I}-\mathbf{D}_\epsilon^\dagger\mathbf{C})\mathbf{u}_\epsilon = \mathbf{D}_\epsilon^\dagger \mathbf{b}$

Part III

Numerical results

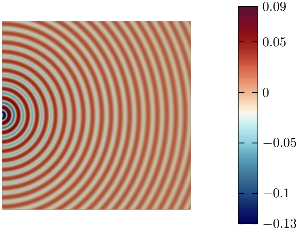
EPW+UWVF implementation

- julia code
- 2D polygonal domains
- Quasi-uniform triangular mesh
- Same number P of PWs on each element
- No quadrature used for UWVF matrix
- Oversampling ratio 10%
- Elementwise SVD truncation $\epsilon = 10^{-14}$
- 2D quasi-random Sobol sampling $(\theta_p, \xi_p \in (0, 2\pi) \times (0, 1))$
- Relative error measured in (κ-weighted) $H^1(\Omega)$ norm

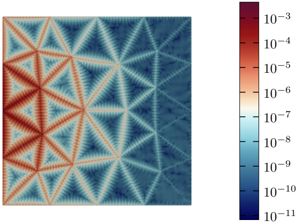
Point source

$$\Omega = (0,1) \times (-\tfrac{1}{2}, \tfrac{1}{2}), \qquad \kappa = 128, \qquad u(\mathbf{x}) = \Phi\big(\mathbf{x}, (-\tfrac{\lambda}{10}, 0)\big), \qquad \text{41 triangles}, \qquad P = 815$$

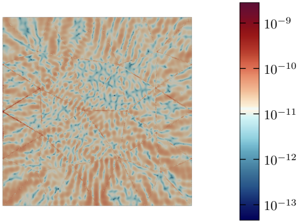
Real part of the solution $\Re(u)$

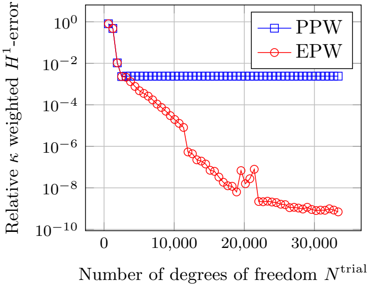


Pointwise error for PPW



Pointwise error for EPW

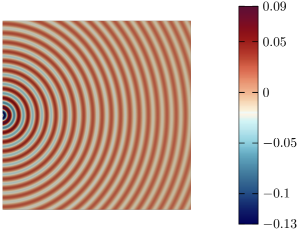




Point source

$\Omega = (0,1) \times (-\frac{1}{2}, \frac{1}{2})$, $\kappa = 128$, $u(\mathbf{x}) = \Phi\big(\mathbf{x}, (-\frac{\lambda}{10}, 0)\big)$, 41 triangles, $P = 815$

Real part of the solution $\Re(u)$



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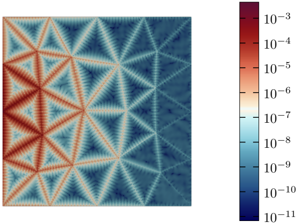

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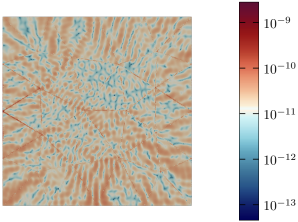
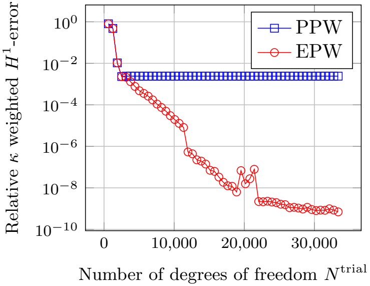
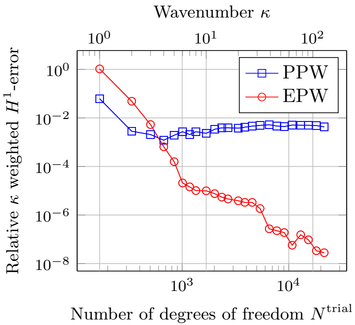






Increase κ

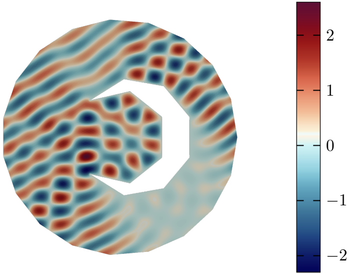
Increase P linearly: $P = 4\kappa$

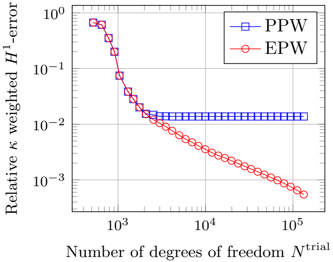
Source dist. $= \frac{\lambda}{10} \searrow 0$

Low-frequency sound-soft scattering

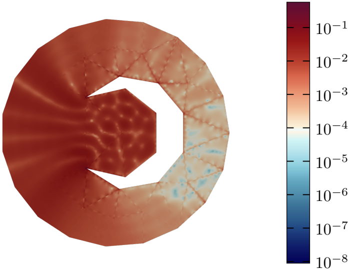
Robin BCs on truncation, $\kappa = 16$, $\mathrm{diam}(\Omega) = 4$, 64 triangles, $P = 2080$

Real part of the solution $\Re(u)$

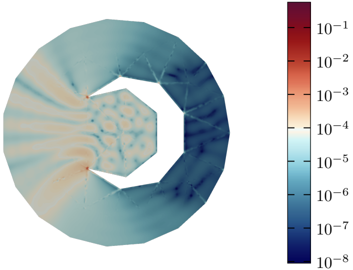




Pointwise error for PPW



Pointwise error for EPW



High-frequency, trapping, sound-soft scattering

Robin truncation

$\kappa = 1024$

$\mathrm{diam}(\Omega) = 2\pi = 1024\lambda$

1472 triangles

$h = 0.2 \approx 32.6\lambda$

$P = 512$

DOFs $= 753\,664$

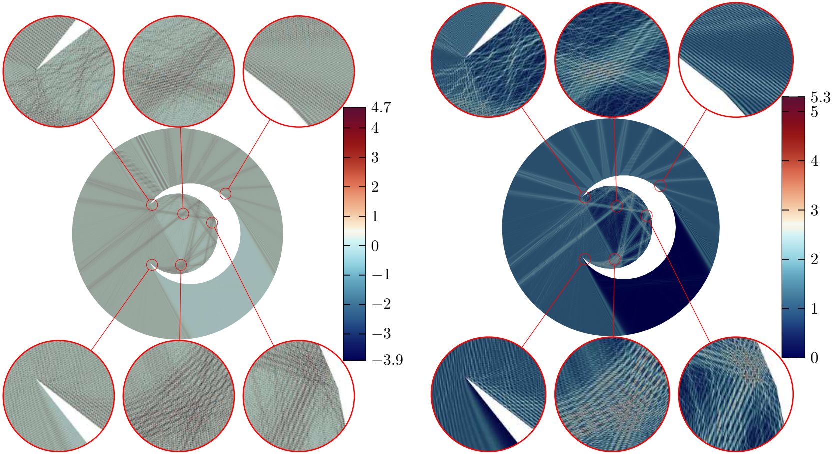
< 1DOFs/wavelength

assembly time $< 1'$
incl. SVD, 64 threads

2231 GMRES iter.s

GMRES time $2h37'$

tested against BEM



$\Re\{u\}$ $|u|$

References

EPWs + TDG/UWVF: Galante, Moiola, Parolin · coming soon!
See short version in Waves2026 abstract · arXiv:2604.18175

References

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- EPW approximation 2D: Parolin, Huybrechs, Moiola · M2AN 2023
- EPW approximation 3D, Weyl expansion: Galante · PhD thesis 2026
- Abstract approximation framework: Adcock, Huybrechs · SiRev 2019 + JFAA 2020
  Herremans, Huybrechs · arXiv 2026
- Christoffel sampling: Cohen, Migliorati · SMAI-JCM 2017
- Trefftz-for-Helmholtz survey: Hiptmair, Moiola, Perugia · LN-CSE 2016
- Original UWVF: Cessenat, Després · SiNum 1998
- More on UWVF/TDG: Barucq, Gabard, Gittelson, Hiptmair, Huttunen, Imbert-Gérard Kaipio, Kapita, Luostari, Moiola, Monforte, Monk, Perugia, Selgas, Sirday, Tordeux. . .
- WBM (Trefftz with EPWs): Deckers,. . . , Desmet · WaveMot 2014
- Birthday boy: Trefftz · 1926

Summary

- PPW-based Trefftz for Helmholtz: simple, cheap, can be accurate but unstable
- Evanescent PW-based Trefftz can cure instability
- All Helmholtz solutions are EPW superpositions: $u = \int_Y v \,\mathrm{EW}$ on disc/ball

  $u = \int_0^{2\pi} v^P \,\mathrm{PW} + \int_Y v^E \,\mathrm{EW}$ on strictly convex
- Simple recipe to select P-dimensional PW basis

  Sample evanescence strength as $\zeta = \max\left\{1, \frac{(P\xi/2)^{\frac{1}{n-1}}}{\kappa \,\mathrm{diam}(K)}\right\}$, with $\xi \sim \mathcal{U}(0,1)$
- Solve with oversampling and (elementwise) regularisation
- Ill-conditioning is not the point: the key is small-coefficient representation

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Much more to do:

- Discrete space analysis
- General convex elements
- Iterative solvers
- Presence of evanescent modes in BVPs
- Maxwell, elasticity
- . . .

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- ...

Thank you!

Slides available on `https://euler.unipv.it/moiola/#slides`

Circular & spherical waves

Separable Helmholtz solutions in polar and spherical coordinates:

$$2D: \quad b_\ell(\mathbf{x}) = \beta_\ell\, J_\ell(\kappa r)\, e^{i\ell\vartheta} \qquad \ell \in \mathbb{Z}, \qquad \mathbf{x} = (r,\vartheta) \in B_1$$

$$3D: \quad b_\ell^m(\mathbf{x}) = \beta_\ell\, j_\ell(\kappa|\mathbf{x}|)\, Y_\ell^m(\mathbf{x}/|\mathbf{x}|) \qquad \ell, m \in \mathbb{Z},\ |m| \le \ell, \qquad \mathbf{x} \in B_1$$

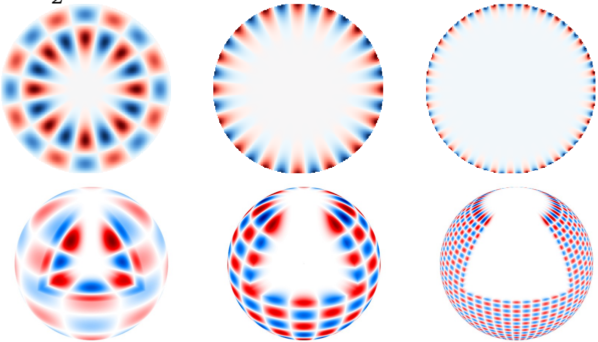
β_ℓ = normalisation in $H^1_\kappa(B_1)$ norm $\qquad \beta_\ell \sim \kappa(\frac{2}{e\kappa})^{|\ell|}|\ell|^{|\ell|+\frac{n-2}{2}}$ for $|\ell| \to \infty$

Orthonormal basis of $\mathcal{B} = \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\}$

$\ell = \frac{\kappa}{2}$ "bulk" $\qquad \ell = \kappa \qquad \ell > \kappa$ "evanescent"

2D

3D, $m = \frac{\ell}{2}$



b_ℓ and b_ℓ^m are Herglotz functions with density

$v(\theta) = \beta_\ell \frac{e^{i\ell\theta}}{2\pi i^\ell}$,

$v(\mathbf{d}) = \beta_\ell \frac{Y_\ell^m(\mathbf{d})}{4\pi i^\ell}$:

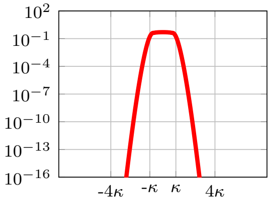
$\|v\|_{L^2(\mathbb{S}^{n-1})} \sim |\ell|^{|\ell|}$

Expansion of PPW in Fourier modes

Jacobi–Anger expansion: $\mathbf{d} \in \mathbb{R}^n, \mathbf{d}\cdot\mathbf{d} = 1$

$$e^{i\kappa\mathbf{d}\cdot\mathbf{x}} = \begin{cases} \sum_{\ell\in\mathbb{Z}} \left(i^\ell e^{-i\ell\theta_\mathbf{d}} \beta_\ell^{-1}\right) b_\ell(\mathbf{x}) \\ 4\pi \sum_{\ell=0}^{\infty} i^\ell \beta_\ell^{-1} \sum_{m=-\ell}^{\ell} \overline{Y_\ell^m(\mathbf{d})}\, b_\ell^m(\mathbf{x}) \end{cases}$$

$\mathbf{d} = (\cos\theta_\mathbf{d}, \sin\theta_\mathbf{d})$



Mode number ℓ (2D)

The modulus of Fourier coefficient decays $\sim \beta_\ell^{-1} \sim |\ell|^{-|\ell|}$

In 2D: $|i^\ell e^{-i\ell\theta_\mathbf{d}} \beta_\ell^{-1}| = |\beta_\ell^{-1}| \sim |\ell|^{-|\ell|}$ indep. of $\theta_\mathbf{d}$

$\Rightarrow$ the approximation of $u = \sum_\ell \hat{u}_\ell b_\ell \in \mathcal{B}$
with $\hat{u}_\ell \neq 0$ for some $|\ell| \gg \kappa$
requires exponentially large coefficients

$$\begin{matrix} \forall \ell \in \mathbb{Z} \quad (|m| \le \ell) \\ \forall P \in \mathbb{N} \\ \forall \mathbf{d}_1 \ldots, \mathbf{d}_P \in \mathbb{S}^{n-1} \\ \forall \boldsymbol{\mu} \in \mathbb{C}^P \\ \forall \delta \in (0,1) \end{matrix} \quad \left\| b_\ell^{(m)}(\mathbf{x}) - \sum_{p=1}^{P} \mu_p e^{i\kappa\mathbf{d}_p\cdot\mathbf{x}} \right\|_{H^1(B_1)} \le \delta \quad \Longrightarrow \quad \|\boldsymbol{\mu}\|_{l^1(\mathbb{C}^P)} \ge (1-\delta) \underbrace{|\beta_\ell|}_{\sim|\ell|^{|\ell|}}$$

Complex-direction Jacobi–Anger & EPW Fourier expansion

Expand EPWs in Fourier modes. Generalised Jacobi–Anger expansion:

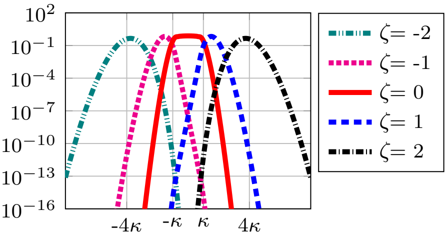
$$e^{i\kappa \mathbf{d}(\mathbf{y})\cdot\mathbf{x}} = \begin{cases} \sum_{\ell\in\mathbb{Z}} \left(i^\ell e^{-i\ell\theta}(\eta+\sqrt{\eta^2+1})^{\pm\ell}\beta_\ell^{-1}\right) b_\ell(\mathbf{x}) & \mathbf{y}=(\theta,\pm,\eta)\in[0,2\pi)\times\{\pm1\}\times[0,\infty) \\ 4\pi\sum_{\ell=0}^{\infty} i^\ell \sum_{m=-\ell}^{\ell}\left[\sum_{m'=-\ell}^{\ell}\overline{D_\ell^{m',m}(\boldsymbol{\theta},\psi)}\gamma_\ell^{m'} i^{-m'} P_\ell^{m'}(\sqrt{\eta^2+1})\right]\beta_\ell^{-1} b_\ell^m(\mathbf{x}) & \mathbf{y}=(\boldsymbol{\theta},\psi,\eta) \end{cases}$$

$D_\ell^{m',m}$ = Wigner matrix entry (spherical harmonic rotation)

P_ℓ^m = associated Legendre function (evaluated out of $[-1,1]$)

$\gamma_\ell^m = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}$

$\boldsymbol{\theta},\psi$ = Euler angles



Mode number ℓ (2D), $\kappa = 16$

◀ Absolute values of Fourier coefficients (2D)

$(\eta+\sqrt{\eta^2+1})^{\pm\ell}\beta_\ell^{-1} = e^{\ell\zeta}\beta_\ell^{-1}$ $\qquad \zeta = \pm\operatorname{arcsinh}\eta$

Looks promising!

We can hope to approximate large-ℓ Fourier modes with EPWs & small coefficients.

Invertibility of EPW Herglotz representation

We want to use the EPW Fourier expansion to prove invertibility of

$$\begin{aligned} T: \quad \mathcal{A} \subset L^2_{w^2}(Y) \quad &\to \quad \mathcal{B} := \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\} \\ v \quad &\mapsto \quad u(\mathbf{x}) = \int_Y v(\mathbf{y})\, \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\, w^2(\mathbf{y})\, \mathrm{d}\mathbf{y} \end{aligned}$$

Consider 2D case. $\mathbf{y} = (\theta, \pm, \eta) \in [0, 2\pi) \times \{\pm 1\} \times [0, \infty) = Y$

$w(\mathbf{y}) = \mathrm{e}^{-\kappa\eta}\eta^{-\frac{1}{4}}$, $a_\ell(\mathbf{y}) := \alpha_\ell(\eta + \sqrt{\eta^2+1})^{\pm\ell}\mathrm{e}^{\mathrm{i}\ell\theta} \in L^2_{w^2}(Y)$, α_ℓ = $L^2_{w^2}(Y)$-normalisation

$\{a_\ell, \ell \in \mathbb{Z}\}$ is orthonormal basis of $\mathcal{A} := \operatorname{span}\{a_\ell, \ell \in \mathbb{Z}\} \subsetneq L^2_{w^2}(Y)$

Jacobi Anger: $\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{\ell\in\mathbb{Z}} \tau_\ell \overline{a_\ell(\mathbf{y})} b_\ell(\mathbf{x})$ $\quad \forall \mathbf{x} \in B_1, \forall \mathbf{y} \in Y,$ $\quad \tau_\ell = \frac{\mathrm{i}^\ell}{\alpha_\ell \beta_\ell},$ $\quad 0 < \tau_- \le |\tau_\ell| \le \tau_+ < \infty \quad \forall \ell$

The operator $T : \mathcal{A} \to \mathcal{B}$ is diagonal in ONB $\{a_\ell\}, \{b_\ell\}$, bounded and invertible:

$$T : a_\ell \mapsto \sum_{\ell'} \tau_{\ell'} b_{\ell'} \int_Y a_\ell \overline{a_{\ell'}} w^2 = \tau_\ell b_\ell, \qquad \tau_- \|v\|_{\mathcal{A}} \le \|Tv\|_{\mathcal{B}} \le \tau_+ \|v\|_{\mathcal{A}} \qquad \forall v \in \mathcal{A}$$

Every Helmholtz solution is EPW superposition with small coefficients: $\|v\|_{\mathcal{A}} \le \tau_-^{-1} \|u\|_{\mathcal{B}}$