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Spurious quasi-resonances in boundary integral equations for the Helmholtz transmission problem

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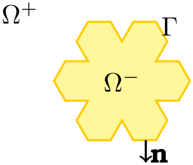
Helmholtz transmission problem (HTP)

$\Omega^- \subset \mathbb{R}^d$: penetrable homogeneous scatterer, bounded & Lipschitz, $d \in \{2, 3\}$
$\Gamma := \partial\Omega^-$
$\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$



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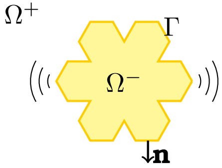

Given $k, n_i, n_o > 0$, $\mathbf{f} \in \mathcal{H}$

Find
$u \in H^1_{\text{loc}}(\Omega^- \cup \Omega^+)$
s.t.

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\begin{aligned}
(\Delta + k^2 n_i)u &= 0 && \text{in } \Omega^- \\
(\Delta + k^2 n_o)u &= 0 && \text{in } \Omega^+ \\
\gamma_C^- u &= \gamma_C^+ u + \mathbf{f} && \text{on } \Gamma \\
\partial_r u - \mathrm{i}k\sqrt{n_o}\,u &= o_{r\to\infty}(r^{\frac{1-d}{2}}) && \text{Sommerfeld r.c.}
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Cauchy trace $\gamma_C^\pm := (\underbrace{\gamma_D^\pm}_{\text{Dirichlet}}, \underbrace{\gamma_N^\pm}_{\text{Neumann}}) : H^1_{\text{loc}}(\Omega^\pm; \Delta) \to \mathcal{H} := H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

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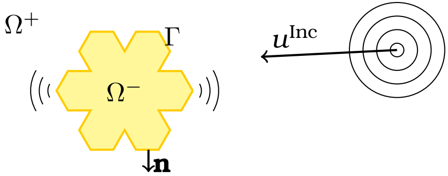

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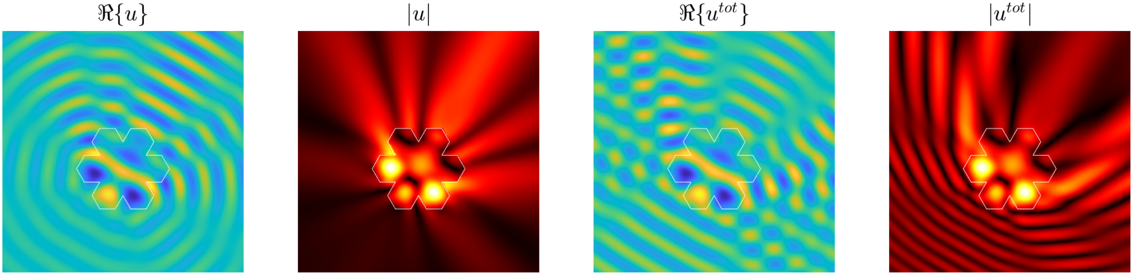
In scattering problems: $\mathbf{f} = \gamma_C^\pm u^{\mathrm{Inc}}$, with u^{Inc} the incident wave, $u|_{\Omega^-} = u^{\mathrm{Tot}}$, $u|_{\Omega^+} = u^{\mathrm{Scat}}$

Scattering problem



$$u^{\mathrm{Inc}}(\mathbf{x}) = \mathrm{e}^{ik\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{x} = (\cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3}), \quad k = 20, \quad n_i = \tfrac{1}{3}, \quad n_o = 1, \quad \operatorname{diam}\Omega^- = 1$$

Stability and wavenumber dependence

HTP is well-posed $\forall \mathbf{f} \in \mathcal{H}$
Well-defined (trace space) solution operator: $S_{io} : \mathcal{H} \to \mathcal{H}, \qquad \mathbf{f} \mapsto \gamma_C^- u$

How does S_{io} depend on wavenumber k?

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How does S_{io} depend on wavenumber k? (MOIOLA, SPENCE, M3AS 2019)

- If $n_i < n_o$ & Ω^- star-shaped wrt a ball: $\|S_{io}\|_{\mathcal{H}\to\mathcal{H}} \lesssim k, \quad \forall k > k_0$
- If $n_i > n_o$ & Ω^- C^∞ with curvature > 0: $\exists k_j \nearrow \infty$ s.t. $\|S_{io}\|_{\mathcal{H}\to\mathcal{H}} \geq C_N k_j^N, \quad \forall N > 0$

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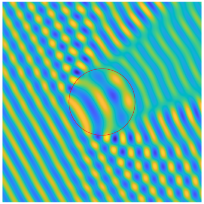
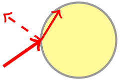
$n_i < n_o \quad \Rightarrow \quad \lambda_i > \lambda_o$
Longer inner wavelength

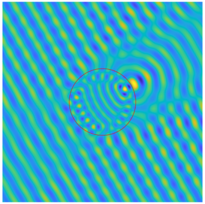
E.g. air bubble in water

$\forall$ rays eventually leave Ω^-:
stability for all $k > 0$



Snell's law:





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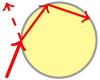

$n_i > n_o \quad \Rightarrow \quad \lambda_i < \lambda_o$
Shorter inner wavelength

E.g. fog droplets in air,
glass in air (lenses)

Total internal reflection,
creeping waves,
ray trapping:
quasi-resonances

Calderón projectors

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$$P^{\pm}_{i/o} := \tfrac{1}{2}I \pm \begin{bmatrix} K_{i/o} & -V_{i/o} \\ -W_{i/o} & -K'_{i/o} \end{bmatrix} \; : \; \mathcal{H} \to \mathcal{H}$$

$$(P^{\pm}_{i/o})^2 = P^{\pm}_{i/o}, \qquad P^{-}_{i/o} + P^{+}_{i/o} = I$$

$$\ker P^{\pm}_{i/o} = \mathrm{range}(P^{\mp}_{i/o})$$

$$P^{\pm}_{i/o}(\phi_1, \phi_2) = \pm\gamma^{\pm}_{C}(\mathcal{K}_{i/o}\phi_1 - \mathcal{V}_{i/o}\phi_2)$$

$K_{i/o}$ = double-layer BIO

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$W_{i/o}$ = hypersingular BIO

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$\mathcal{K}_{i/o}$ = double-layer potential

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for $\Delta + k^2 n_i$ & $\Delta + k^2 n_o$

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$P^{\pm}_{i/o}$ characterise traces of outer/inner Helmholtz solutions:

$$P^{\pm}_{i/o}\phi = \phi \quad\Longleftrightarrow\quad P^{\mp}_{i/o}\phi = \mathbf{0} \quad\Longleftrightarrow\quad \begin{cases} \phi = \gamma^{\pm}_{C} v \\ \Delta v + k^2 n_{i/o} v = 0 \quad \text{in } \Omega^{\pm} \\ \quad (\text{\& radiating on } \Omega^{+}) \end{cases}$$

Boundary integral equations (BIEs)

(Costabel, Stephan 1985, von Petersdorff 1989, Claeys, Hiptmair, Jerez-Hanckes...)

Single-trace I and II-kind BIOs:

$$A_{\mathrm{I}} := P_o^- - P_i^+ = \begin{bmatrix} -(K_i + K_o) & V_i + V_o \\ W_i + W_o & K_i' + K_o' \end{bmatrix} \qquad A_{\mathrm{II}} := P_o^- + P_i^+ = I + \begin{bmatrix} K_i - K_o & -(V_i - V_o) \\ -(W_i - W_o) & -(K_i' - K_o') \end{bmatrix}$$

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If u solves HTP $\implies$ its trace solves the direct BIEs:
$$A_{\mathrm{I}}(\gamma_C^- u) = P_o^- \mathbf{f}$$
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Proof for A_{I} using Calderón projectors:
$$A_{\mathrm{I}}\gamma_C^- u = P_o^- \underbrace{\gamma_C^- u}_{=\gamma_C^+ u + \mathbf{f}} - \underbrace{P_i^+ \gamma_C^- u}_{=0} = \underbrace{P_o^- \gamma_C^+ u}_{=0} + P_o^- \mathbf{f}$$

For scattering BVPs $\mathbf{f} = \gamma_C u^{\mathrm{Inc}} \implies P_o^- \mathbf{f} = \mathbf{f}$ at BIE RHS

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$A_{\mathrm{I}}, A_{\mathrm{II}} : \mathcal{H} \to \mathcal{H}$ bounded and invertible: BIEs are well-posed

T-coercive with T=sign change in one component of $\mathcal{H}$

Also: $A_{\mathrm{II}} : H^1(\Gamma) \times L^2(\Gamma) \circlearrowleft$ Fredholm and invertible

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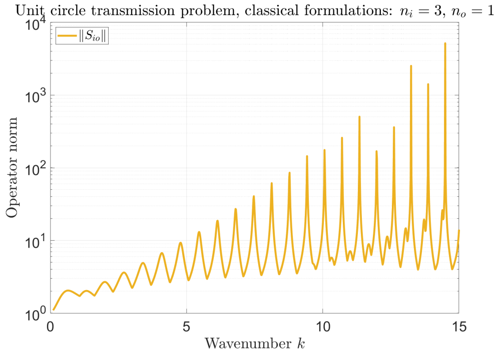
Also: $A_{\rm II} : H^1(\Gamma) \times L^2(\Gamma) \circlearrowleft$ Fredholm and invertible

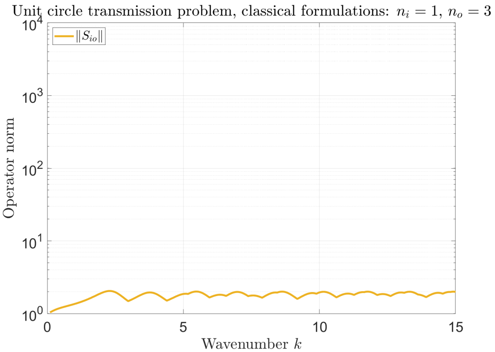
But... swapping $n_i \leftrightarrow n_o$ the k-dependence of S_{io} changes radically, while $A_{\rm I}$ does not change!

$$P_o^- - P_i^+ = P_i^- - P_o^+$$

Spurious quasi-resonances in BIOs

$\Omega^- =$ unit disc in $\mathbb{R}^2$





$n_i = 3 > 1 = n_o$

$\exists k_j \nearrow \infty$ s.t. $\|S_{io}\|_{\mathcal{H}\to\mathcal{H}} \geq C_N k_j^N \quad \forall N > 0$

Quasi-resonances

$n_i = 1 < 3 = n_o$

$\|S_{io}\|_{\mathcal{H}\to\mathcal{H}} \lesssim k \quad \forall k > k_0$

S_{io} is stable uniformly in k

Yellow line: $\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}$

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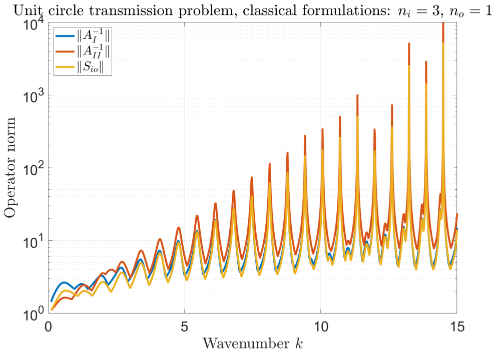

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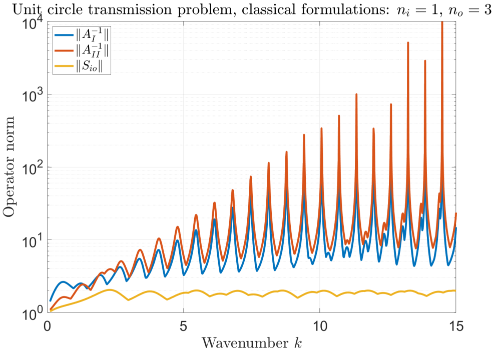

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Blue and **red** lines: $\|A_{\mathrm{I}}^{-1}\|_{\mathcal{H}\to\mathcal{H}}$, $\|A_{\mathrm{II}}^{-1}\|_{\mathcal{H}\to\mathcal{H}}$ ▶ spurious quasi-resonances for $n_i < n_o$

"Fictitious eigenvalues" mentioned in (Misawa, Niino, Nishimura, SIAM J.Appl.Math. 2017)

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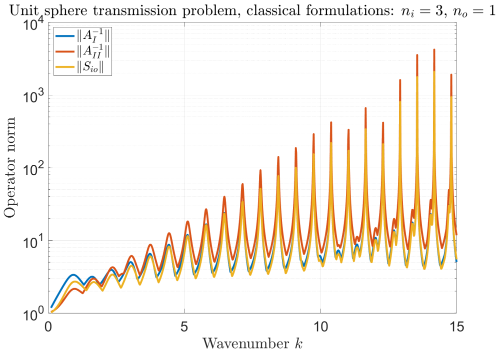

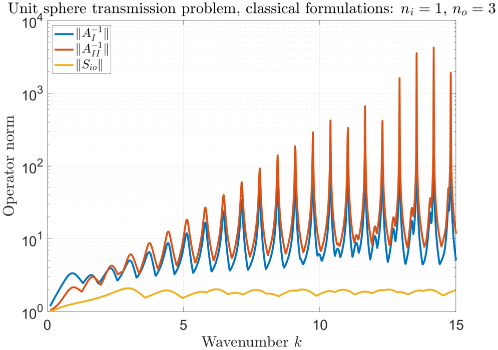

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Classical spurious resonances for exterior Dirichlet problems

Exterior Dirichlet BVP:
$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega^+ \\ \gamma^+ u = g & \text{on } \Gamma \\ \text{SRC} & \end{cases}$$

Well-posed $\forall\, k > 0$

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Direct BIEs:
$\varphi = \gamma_N^+ u$

$$V\varphi = \Big(K - \frac{1}{2}\Big) g \qquad \text{I kind}$$

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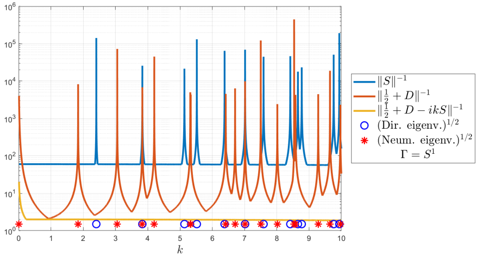

Spurious resonances

If k^2 is Dirichlet / Neumann eigenvalue of $-\Delta$ on Ω^- $\Longrightarrow$ $\ker V \neq \{0\}$ / $\ker\left(\frac{1}{2} + K'\right) \neq \{0\}$

Singular BIEs for well-posed BVPs. Interior resonances pollute BIEs for exterior problem.

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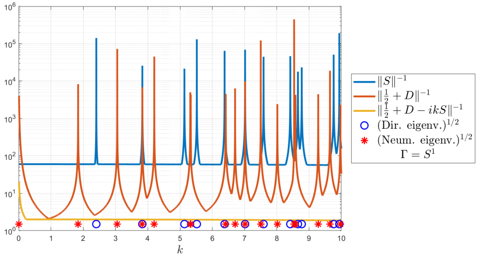

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Singular BIEs for well-posed BVPs. Interior resonances pollute BIEs for exterior problem.

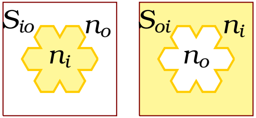
Instead, in HTPs we have spurious quasi-resonances because the operator is not singular.
Resonant BVP pollutes BIEs for stable BVP.

k_j are real part of complex resonances.

BIO inverses involve 2 HTPs

Recall $S_{io} : \mathcal{H} \to \mathcal{H}, \ \mathbf{f} \mapsto \gamma_C^- u,$ is the HTP solution operator

Denote S_{oi} the "unphysical" HTP solution operator with swapped $n_i \leftrightarrow n_o$



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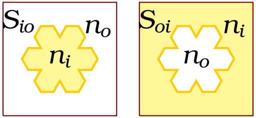

Theorem: BIO inverse formulas

$$A_{\mathrm{I}}^{-1} = S_{io} + S_{oi} - I \qquad A_{\mathrm{II}}^{-1} = I - S_{io} - S_{oi} + 2S_{io}S_{oi}$$

Both HTPs S_{io} ("physical") and S_{oi} ("unphysical") enter the inverse of the BIOs

Even for stable HTPs with $n_i < n_o$, $\exists k_j \nearrow \infty$ s.t. $\|A_{\mathrm{I}}^{-1}\|_{\mathcal{H}\to\mathcal{H}} \geq C_N k_j^N \quad \forall N > 0$
for C^∞ strictly convex Ω^-

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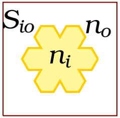
Denote S_{oi} the "unphysical" HTP solution operator with swapped $n_i \leftrightarrow n_o$



Theorem: BIO inverse formulas

$$A_{\mathrm{I}}^{-1} = S_{io} + S_{oi} - I \qquad A_{\mathrm{II}}^{-1} = I - S_{io} - S_{oi} + 2S_{io}S_{oi}$$

Both HTPs S_{io} ("physical") and S_{oi} ("unphysical") enter the inverse of the BIOs

Even for stable HTPs with $n_i < n_o$, $\exists k_j \nearrow \infty$ s.t. $\|A_{\mathrm{I}}^{-1}\|_{\mathcal{H}\to\mathcal{H}} \geq C_N k_j^N$ $\forall N > 0$ for C^∞ strictly convex Ω^-

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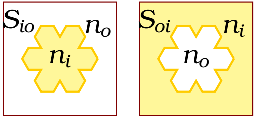

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Theorem: On $\mathrm{range}(P_o^-)$ the unphysical operator does not enter $A_\bullet^{-1}$

$$A_{\mathrm{I}}^{-1} P_o^- = A_{\mathrm{II}}^{-1} P_o^- = S_{io} P_o^-, \qquad A_{\mathrm{I}}^{-1}|_{\mathrm{range}(P_o^-)\to\mathrm{range}(P_i^-)} = A_{\mathrm{II}}^{-1}|_{\mathrm{range}(P_o^-)\to\mathrm{range}(P_i^-)} = S_{io}$$

Proof: $A_{\mathrm{I}}^{-1} = (P_i^- - P_o^+)^{-1} = S_{io} + S_{oi} - I$

Use only Calderón projector identities and characterisation of Helmholtz solution traces

Let $\psi, \mathbf{g} \in \mathcal{H}$, $A_{\mathrm{I}}\psi = \mathbf{g}$

1 Premultiply P_i^- to BIE :

$$\begin{aligned}\mathbf{0} &= A_{\mathrm{I}}\psi - \mathbf{g}\\ &= (P_i^- - P_o^+)\psi - \mathbf{g}\\ &= P_i^-\big((P_i^- - P_o^+)\psi - \mathbf{g}\big)\\ &= P_i^-\big((I - P_o^+)\psi - \mathbf{g}\big)\\ &= P_i^-\big(P_o^-\psi - \mathbf{g}\big)\end{aligned}$$

$$\left.\begin{aligned}\Rightarrow P_o^-\psi - \mathbf{g} &= \gamma_C^+(\text{sol. for } n_i)\\ P_o^-\psi &= \gamma_C^-(\text{sol. for } n_o)\end{aligned}\right\} \Rightarrow P_o^-\psi = S_{oi}\mathbf{g}$$

2 Similarly, swapping $i \leftrightarrow o$:

$$P_i^-\psi = S_{io}\mathbf{g}$$

3

$$\begin{aligned}P_o^+\psi &= (P_o^+ - P_i^- + P_i^-)\psi\\ &= (-A_{\mathrm{I}} + P_i^-)\psi\\ &= -\mathbf{g} + P_i^-\psi\end{aligned}$$

4

$$\begin{aligned}\psi &= (P_o^- + P_o^+)\psi\\ &= S_{oi}\mathbf{g} + P_i^-\psi - \mathbf{g}\\ &= (S_{oi} + S_{io} - I)\mathbf{g}\end{aligned}$$

Augmented BIEs

Motivation:

- Given "unstable" linear system $A\mathbf{x} = \mathbf{y}$, $A \in \mathbb{C}^{n,n}$, $\sigma_n(A) \ll 1$
- Assume to know $B \in \mathbb{C}^{m,n}$ s.t. $B\mathbf{x} = \mathbf{0}$
- Augmented system $M\mathbf{x} = \begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \implies \sigma_n(M) \geq \sigma_n(A)$

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BIE: $A_\bullet \phi = \mathbf{g}$

HTP solution $\phi = \gamma_C^- u$ satisfies: $P_i^+ \phi = \mathbf{0}$

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$\bullet \in \{\text{I,II}\}$

If $\mathbf{f} = \gamma_C u^{\text{Inc}} \implies \mathbf{g} = \mathbf{f} = P_o^- \mathbf{f} = S_{oi}\mathbf{f}$, $\exists$ solution ϕ, $\phi = S_{io}\mathbf{g}$: no S_{oi} involved

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Theorem: Inf-sup stability of the augmented BIEs $\bullet \in \{\mathrm{I,II}\}$

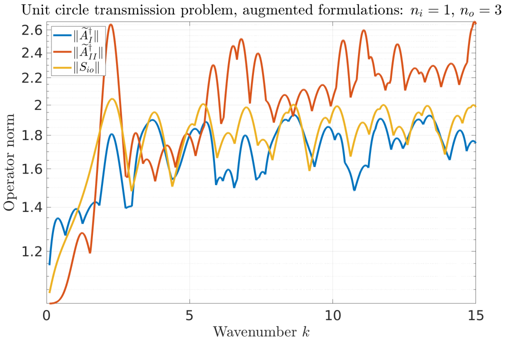
$$\inf_{\phi \in \mathcal{H}\setminus\{\mathbf{0}\}} \sup_{(\psi_1,\psi_2) \in \mathcal{H}\times\mathcal{H}\setminus\{(\mathbf{0},\mathbf{0})\}} \frac{\left| \left(\begin{bmatrix} A_\bullet \\ P_i^+ \end{bmatrix} \phi, \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right)_{\mathcal{H}\times\mathcal{H}} \right|}{\|\phi\|_{\mathcal{H}} \, \|(\psi_1,\psi_2)\|_{\mathcal{H}\times\mathcal{H}}} \geq \frac{1}{4 \max\left\{ \|S_{io}\|_{\mathcal{H}\to\mathcal{H}}, 1 \right\}}$$

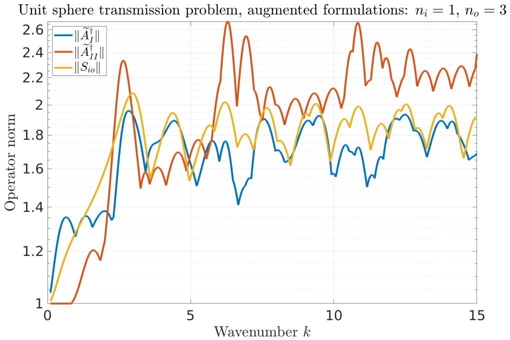
Proof relies on test fields $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} S_{io}^* P_i^- \phi \\ P_i^+ \phi \end{bmatrix}$ for A_{I} and $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} S_{io}^* P_i^- \phi \\ P_i^+ \phi - 2 S_{io}^* P_i^- \phi \end{bmatrix}$ for A_{II}

Augmented BIO pseudoinverses

$n_i = 1 < 3 = n_o$

Norms of the augmented BIO pseudoinverses $\left\| \begin{bmatrix} A_\bullet \\ P_i^+ \end{bmatrix}^\dagger \right\|_{\mathcal{H}\to\mathcal{H}}$ follow $\|S_{io}\|_{\mathcal{H}\to\mathcal{H}}$:





unit disc in $\mathbb{R}^2$ ▲

▲ unit ball in $\mathbb{R}^3$

Augmentation removes spurious quasi resonances!

Summary

Helmholtz transmission problem with $n_i < n_o$, Ω^- star-shaped:

- boundary value problem is stable for all k
- classical direct BIEs are unstable for a sequence of frequencies: spurious quasi-resonances

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Remedy: augmented BIEs ► inf-sup stability

Open problem: stabilisation of Galerkin BEM

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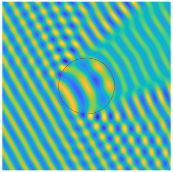

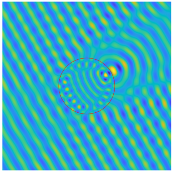

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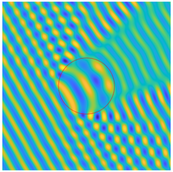

Thank you!