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Boundary element methods for scattering by fractal screens

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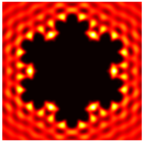

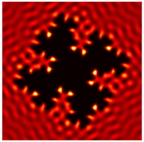

Joint work with
S.N. Chandler-Wilde (Reading), D.P. Hewett (UCL)
A. Caetano (Aveiro)

Acoustic wave scattering by a planar screen

Acoustic waves in free space governed by wave eq. $\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

In time-harmonic regime, assume $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}kt}\}$ and look for u.
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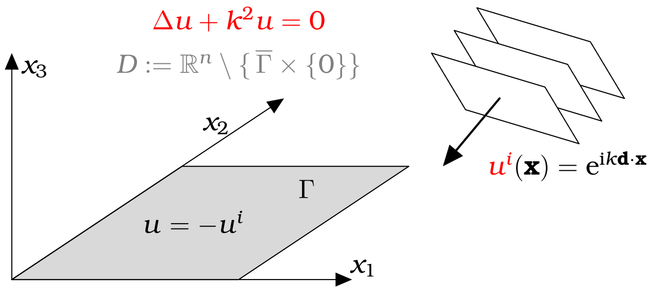
Scattering: incoming wave u^i hits obstacle Γ and generates field u.

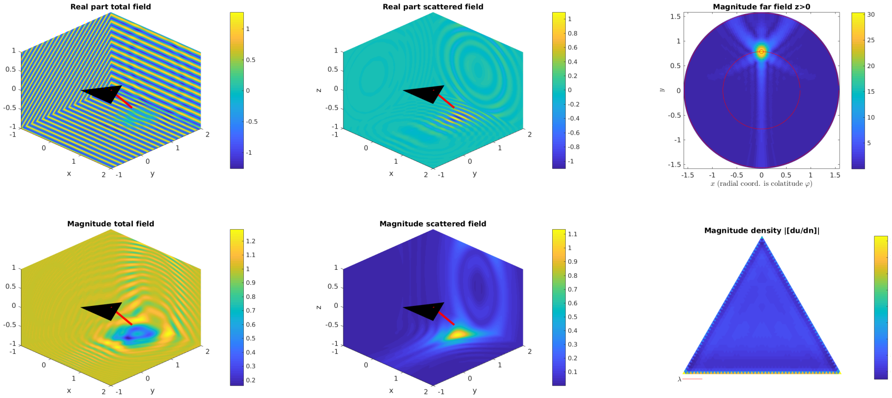
Γ bounded subset of $\Gamma_\infty := \{\mathbf{x} \in \mathbb{R}^n : x_n = 0\} \cong \mathbb{R}^{n-1}$, $n = 2, 3$



u satisfies Sommerfeld radiation condition (SRC) at infinity
(i.e. $\partial_r u - iku = o\left(r^{(1-n)/2}\right)$ uniformly as $r = |\mathbf{x}| \to \infty$).

Scattering by Lipschitz and rough screens

Incident field is plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}$, $|\mathbf{d}| = 1$. $\qquad u^{tot} = u + u^i$



Classical problem when Γ is open and Lipschitz.

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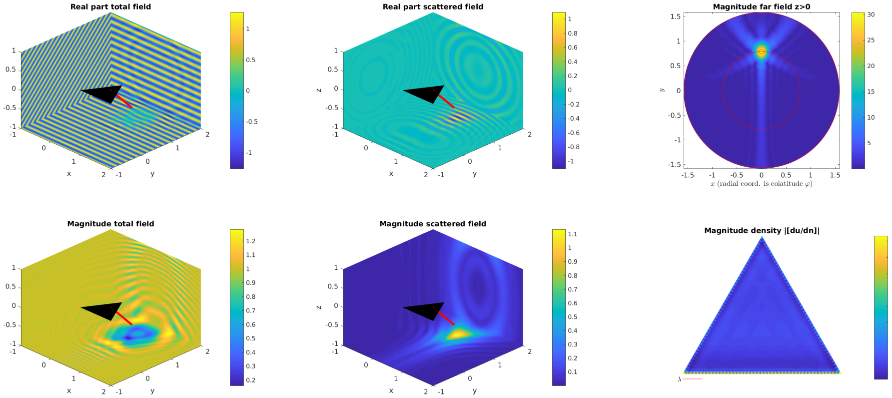
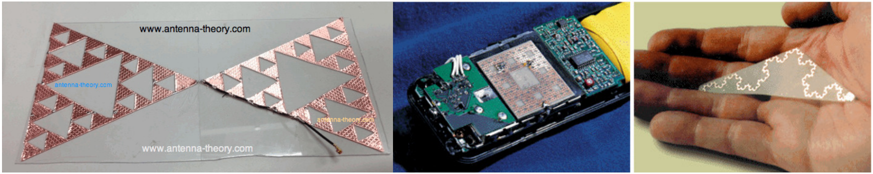


Classical problem when Γ is open and Lipschitz.

What happens for arbitrary (rougher than Lipschitz, e.g. fractal) Γ?

Waves and fractals: applications

Wideband fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

Waves and fractals: applications

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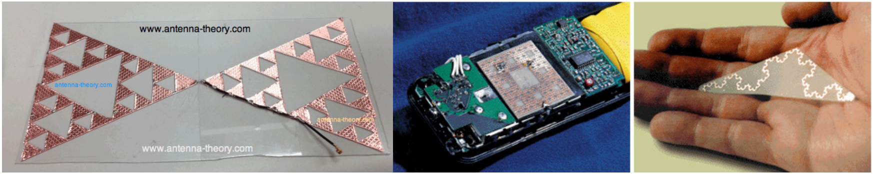

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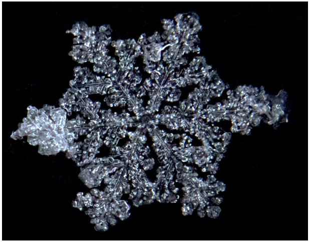
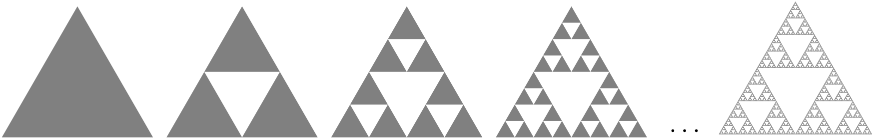
Scattering by ice crystals
in atmospheric physics
e.g. C. Westbrook



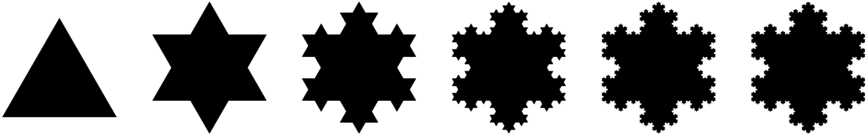
Fractal apertures in laser optics
e.g. J. Christian

Scattering by fractal screens



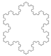
Lots of mathematical challenges:

- How to formulate well-posed BVPs?
  (What is the right function space setting? How to impose BCs?)
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?
- ...



Note: several tools developed here might be used in the (numerical) analysis of different IEs & BVPs involving complicated domains.

Outline

- Sobolev spaces on rough sets
- BVPs and BIEs
  - open screens 
  - compact screens 
- Prefractal to fractal convergence
- BEM and convergence
- Examples & numerics
  - Cantor dust: dependence on Hausdorff dimension
  - Sierpinski triangle: dependence on frequency
  - Snowflakes: inner and outer approximations
  - ...

Sobolev spaces on rough subsets of $\mathbb{R}^{n-1}$

We need fractional (Bessel) Sobolev spaces on $\Gamma \subset \mathbb{R}^{n-1}$. For $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}^{n-1}) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^{n-1}) : \|u\|^2_{H^s(\mathbb{R}^{n-1})} := \int_{\mathbb{R}^{n-1}} (1+|\boldsymbol{\xi}|^2)^s |\hat{u}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} < \infty \right\}$$

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For $\Gamma \subset \mathbb{R}^{n-1}$ open and $F \subset \mathbb{R}^{n-1}$ closed define (McLean)

$H^s(\Gamma) := \{u\|_\Gamma : u \in H^s(\mathbb{R}^{n-1})\}$	restriction
$\widetilde{H}^s(\Gamma) := \overline{C_0^\infty(\Gamma)}^{H^s(\mathbb{R}^{n-1})}$	closure
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When Γ is Lipschitz it holds that

- $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$ with equal norms
- $s \in \mathbb{N} \Rightarrow \|u\|^2_{H^s(\Gamma)} \sim \sum_{|\alpha| \le s} \int_\Gamma |\partial^\alpha u|^2$
- $\widetilde{H}^s(\Gamma) = H^s_\Gamma$ $\quad (\cong H^s_{00}(\Gamma),\ s \ge 0)$
- $H^{\pm 1/2}_{\partial\Gamma} = \{0\}$
- $\{H^s(\Gamma)\}_{s\in\mathbb{R}}$ and $\{\widetilde{H}^s(\Gamma)\}_{s\in\mathbb{R}}$ are interpolation scales.

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When Γ is Lipschitz it holds that	For general open Γ	
▶ $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$ with equal norms	▶ ✓	
▶ $s \in \mathbb{N} \Rightarrow \|u\|^2_{H^s(\Gamma)} \sim \sum_{|\alpha| \le s} \int_\Gamma |\partial^\alpha u|^2$	▶ ×	LIPSCHITZ
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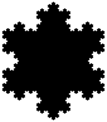
BVPs for open and compact screens

BVP $\mathrm{D}^{op}(\Gamma)$ for open screens

Let $\Gamma \subset \Gamma_\infty$ be bounded & open.
Given $g \in H^{1/2}(\Gamma)$
(for instance, $g = -(\gamma^\pm u^i)|_\Gamma$),
find $u \in C^2(D) \cap W^{1,\mathrm{loc}}(D)$
satisfying

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$
$$(\gamma^\pm u)|_\Gamma = g,$$
Sommerfeld RC.



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BVP D$^{co}(\Gamma)$ for compact scr.

Let $\Gamma \subset \Gamma_\infty$ be compact.
Given $g \in \widetilde{H}^{1/2}(\Gamma^c)^\perp$
(e.g., $g = -P_\Gamma u^i$),
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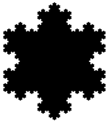

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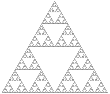

Orthogonal projection
$P_\Gamma : H^{1/2}(\Gamma_\infty) \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$.

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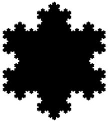

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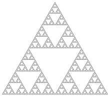

Orthogonal projection
$P_\Gamma : H^{1/2}(\Gamma_\infty) \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$.

If Ω bdd open, $\widetilde{H}^{-1/2}(\Omega) = H^{-1/2}_{\overline{\Omega}}$, then $\mathrm{D}^{op}(\Omega)$&$\mathrm{D}^{co}(\overline{\Omega})$ are equivalent.

Well-posedness & boundary integral equations

Theorem (CW, H, M 2019)

If $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ then problem D$^{op}(\Gamma)$ has a unique solution u.

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Problem D$^{co}(\Gamma)$
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If $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$ then problem D$^{op}(\Gamma)$ has a unique solution u.

Theorem (CW, H, M 2019)

Problem D$^{co}(\Gamma)$ has a unique solution u.

u satisfies the representation formula $u(\mathbf{x}) = -\mathcal{S}_\Gamma\phi(\mathbf{x}), \mathbf{x} \in D$,
where $\phi = [\partial_\mathbf{n} u] := \partial_\mathbf{n}^+ u - \partial_\mathbf{n}^- u$ is the unique solution of BIE $S_\Gamma \phi = -g$.

$\mathcal{S}_\Gamma$ = single-layer potential,
S_Γ = single layer operator: cont. & coercive in $H^{-1/2}(\mathbb{R}^{n-1})$ norm.

$$\mathcal{S}_\Gamma\psi(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\mathrm{d}s(\mathbf{y})$$

$\mathcal{S}_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to C^2(D)\cap W^{1,loc}(\mathbb{R}^n)$	$\mathcal{S}_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to C^2(D) \cap W^{1,loc}(\mathbb{R}^n)$
$S_\Gamma\psi = (\gamma^\pm \mathcal{S}_\Gamma\psi)\vert_\Gamma$	$S_\Gamma = P_\Gamma \gamma^\pm \mathcal{S}_\Gamma$
$S_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$	$S_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$

Φ is the Helmholtz fundamental solution ($\Phi(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$ for $n = 3$)

When is $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$?

The previous theorems extend classical results for Lipschitz domains (STEPHAN & WENDLAND 1984, STEPHAN 1987).

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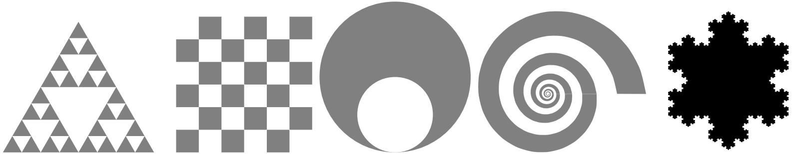
Sufficient conditions for $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$ are that $|\partial\Gamma| = 0$ and either

- Γ is C^0 (e.g. Lipschitz);
- Γ is C^0 except at a set of countably many points $P \subset \partial\Gamma$ such that P has only finitely many limit points;
- Γ is "thick", in the sense of Triebel.



$$(\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2} \iff C_0^\infty(\Gamma) \overset{\text{dense}}{\subset} \{v \in H^{-1/2}(\mathbb{R}^{n-1}) : \operatorname{supp} v \subset \overline{\Gamma}\})$$

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Cases with $\widetilde{H}^{-1/2}(\Gamma) \neq H^{-1/2}_{\overline{\Gamma}}$ constructed using characterisation:

If $s \in \mathbb{R}$, $\mathrm{int}(\overline{\Gamma})$ is C^0 then $\quad \widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}} \iff H^{-s}_{\mathrm{int}(\overline{\Gamma})\setminus\Gamma} = \{0\}$.

Prefractal to fractal convergence of BVPs

BIEs can be written as continuous & coercive variational problems posed in subspaces of $H^{-1/2}(\Gamma_\infty)$: either $\widetilde{H}^{-1/2}(\Gamma)$ or $H_\Gamma^{-1/2}$.

Let Γ_j be a sequence of "prefractals" approximating "fractal" Γ. Denote ϕ_j and ϕ the corresponding BIE solutions.

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If Mosco convergence $V_j \xrightarrow{\mathcal{M}} V$ holds,
then $\phi_j \to \phi$ in $H^{-1/2}(\Gamma_\infty)$ and $\mathcal{S}_{\Gamma_*}\phi_j \to \mathcal{S}_{\Gamma_*}\phi$ in $W^{1,\mathrm{loc}}(\mathbb{R}^n)$,

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Definition of Mosco convergence (1969): $H \supset W_j \xrightarrow{\mathcal{M}} W \subset H$ if

- $\forall v \in W, j \in \mathbb{N}, \exists v_j \in W_j$ s.t. $v_j \to v$ (strong approximability)
- $\forall (j_m)$ subseq. of $\mathbb{N}$, $v_{j_m} \in W_{j_m}$, $v_{j_m} \rightharpoonup v$, then $v \in W$ (weak closure)

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1 open $\Gamma_j \subset \Gamma_{j+1}$

2 compact $\Gamma_j \supset \Gamma_{j+1}$

3 non-nested $\Gamma_j \not\subset\not\supset \Gamma_{j+1}$

Part II

The boundary element method

The boundary element method (BEM)

Partition prefractal Γ_j with mesh $M_j = \{T_{j,1}, \ldots, T_{T_j,N_j}\}$, h_j :=mesh size.
Denote by $V_j^h \subset H^{-1/2}(\Gamma_\infty)$ the space of piecewise constants on M_j.

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Galerkin BEM: solve restriction of (variational form of) BIE to V_j^h:

$$\text{find } \phi_j^h \in V_j^h \text{ s.t. } \forall \psi^h \in V_j^h \qquad \int_{\Gamma_j}\int_{\Gamma_j} \Phi(\mathbf{x},\mathbf{y})\phi_j^h(\mathbf{x})\overline{\psi^h(\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = -\int_{\Gamma_j} g(\mathbf{y})\overline{\psi^h(\mathbf{y})}\mathrm{d}\mathbf{y}$$

ϕ_j^h approximates ϕ on Γ_j, $\quad \mathcal{S}_{\Gamma_j}\phi_j^h$ approximates u in D.

The boundary element method (BEM)

Partition prefractal Γ_j with mesh $M_j = \{T_{j,1}, \ldots, T_{T_j,N_j}\}$, h_j :=mesh size.
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We want to ensure that BEM solution on Γ_j converges to BIE solution on Γ.

$$\phi_j^h \xrightarrow{h_j \to 0} \phi_j \xrightarrow{j \to \infty} \phi, \qquad \phi_j^h \xrightarrow{?} \phi$$

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$$\phi_j^h \xrightarrow{h_j \to 0} \phi_j \xrightarrow{j \to \infty} \phi, \qquad \phi_j^h \overset{?}{\longrightarrow} \phi$$

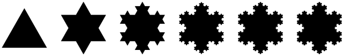
If $V_j^h \xrightarrow{\mathcal{M}} V$, (with either $V = \widetilde{H}^{-1/2}(\Gamma)$ or $V = H_\Gamma^{-1/2}$)
then BEM solution $\phi_j^h \to \phi$ in $H^{-1/2}(\Gamma_\infty)$ and $\mathcal{S}_{\Gamma_*}\phi_j^h \to u$ in $W^{1,\mathrm{loc}}(\mathbb{R}^n)$

Mosco convergence extends Céa argument: Galerkin convergence for discrete spaces not contained in limit space.
Might be useful in very different settings! Non-conforming FEM?

BEM convergence: open screen

Assume all mesh elements have disjoint convex hulls and $|\partial T_{j,l}| = 0$.
(Allow multi-component elements!)

How to choose $(h_j)_{j=0}^{\infty}$ so that $V_j^h \xrightarrow{\mathcal{M}} V$?



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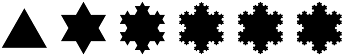

Theorem (CW, H, M 2019)

Let Γ, Γ_j *be bounded open,* $\Gamma_j \subset \Gamma_{j+1}$, $\Gamma = \bigcup_{j=0}^{\infty} \Gamma_j$.
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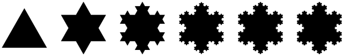

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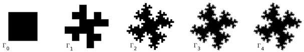
Proof: For $V_j^h \xrightarrow{\mathcal{M}} V = \widetilde{H}^{-1/2}(\Gamma) = \overline{C_0^\infty(\Gamma)}$ we have to show
(i) strong approximability and (ii) weak closedness.
For (i), let $v \in C_0^\infty(\Gamma)$. Then $\exists j_*(v)$ s.t. $v \in C_0^\infty(\Gamma_j)$ for $j \geq j_*(v)$ and

$$\|\Pi_{L^2, V_j^h} v - v\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq (h_j/\pi)^{1/2} \|v\|_{L^2(\Gamma_j)}.$$

For (ii), $V_j^h \subset \widetilde{H}^{-1/2}(\Gamma_j) \xrightarrow{\mathcal{M}} \widetilde{H}^{-1/2}(\Gamma)$. □

Extends to some non-nested $\Gamma_j \overset{\not\subset}{\not\supset} \Gamma_{j+1}$, e.g.



BEM convergence: compact screen

When Γ is compact with empty interior and $\dim_{\mathrm{H}}\Gamma > 1$ this argument fails because $C_0^\infty(\Gamma^\circ) = \{0\}$ is not dense in $V = H_\Gamma^{-1/2} \neq \{0\}$.

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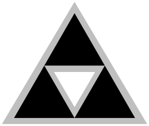
To obtain a smooth approximation we mollify: this enlarges the support.
Currently only results for “thickened prefractals”.



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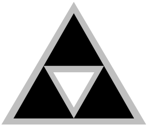

Theorem (CW, H, M 2019)

Let Γ compact & Γ_j open satisfy $\Gamma \subset \Gamma(\epsilon_j) \subset \Gamma_j \subset \Gamma(\eta_j)$, $0 < \epsilon_j < \eta_j \to 0$.
Then BEM convergence holds if $h_j = o(\epsilon_j)$ as $j \to \infty$.
If H_Γ^t is dense in $H_\Gamma^{-1/2}$ for $t \in (-1/2, 0)$ then $h_j = o(\epsilon_j^{-2t})$ suffices.

If Γ is d-set (e.g. IFS attractor), $h_j = o(\epsilon_j^\mu)$, $\mu > n - 1 - \dim_H \Gamma$ is enough.

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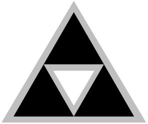

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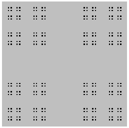
Proof of (i) (strong approx.): Let $v \in H_\Gamma^t$ and set $v_j := (\psi_{\varepsilon_j/2} * v)$, then

$$\|\Pi_{L^2, V_j^h} v_j - v_j\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq (h_j/\pi)^{1/2} \|v_j\|_{L^2(\Gamma_j)} \leq (h_j/\pi)^{1/2} (\varepsilon_j/2)^t \|v\|_{H_\Gamma^t}.$$

Attractors of iterated function systems



Let $s_1, \dots, s_m : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be contracting similarities,
$s(U) := \bigcup_{m=1}^{\nu} s_m(U)$, for $U \subset \mathbb{R}^{n-1}$,
$\Gamma = s(\Gamma)$ the unique attractor (the fractal).

(Open set condition.)
Assume $O \neq \emptyset$ is open, convex, $s(O) \subset O$ and $s_m(O) \cap s_{m'}(O) = \emptyset$.
Define open prefractal sequence: $\Gamma_0 := O$, $\Gamma_{j+1} := s(\Gamma_j)$

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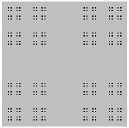

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Let $M_0 = \{T_{0,1}, ..., T_{0,N_0}\}$ be any convex mesh on Γ_0,
then define a convex mesh on Γ_j as

$$M_j := \left\{ s_{m_1} \circ \cdots \circ s_{m_j}(T_{0,l}) : 1 \le m_{j'} \le \nu \text{ for } j' = 1, ..., j \text{ and } 1 \le l \le N_0 \right\}.$$

Then Γ is a d-set, BVP convergence and BEM convergence hold.

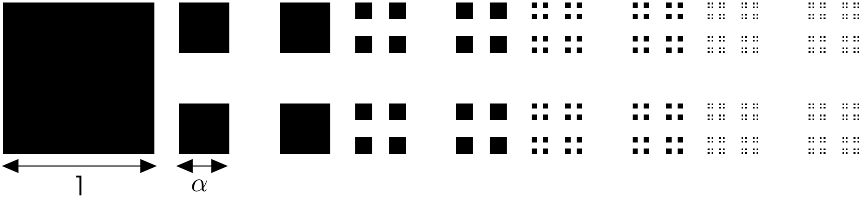
The prefractals Γ_j are not the natural ones, but thickened.
Also extends to "pre-convex" meshes.

Part III

Examples and numerics

Cantor dust

Cantor dust is Cartesian product of 2 copies of Cantor set with parameter $0 < \alpha < 1/2$. Prefractals $\Gamma_0, \ldots, \Gamma_4$:



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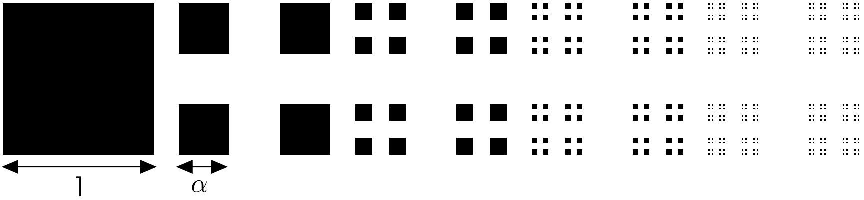
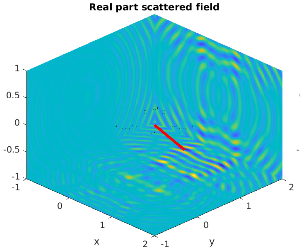


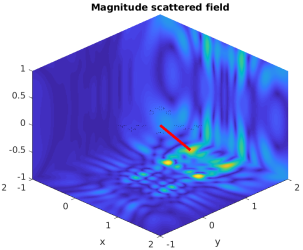
- Γ "audible" ($\phi \neq 0$) $\iff \alpha > \frac{1}{4} \iff \mathrm{dim_H}(\Gamma) > 1$.
  ($\phi \neq 0 \iff \mathrm{dim_H}(\Gamma) > 1$ holds for all d-sets!)
- $H^{-1/2}_{\Gamma_j} \xrightarrow{\mathcal{M}} H^{-1/2}_{\Gamma}$, prefractal solutions ϕ_j converge to ϕ.
- BEM on thickened prefractals converge,
  1 DOF / prefractal component is enough.

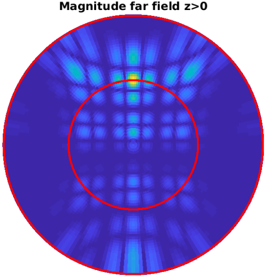
Actually BEM converges with even less than 1 DOF/component:
m_j components/element on Γ_j for $1 \leq m_j < 4^{(\frac{\log 4}{\log 1/\alpha} - 1)j}$.

Cantor dust: field plots

Prefractal level $j = 6$, $N_j = 4^6 = 4\,096$ DOFs, $k = 50$, $\alpha = 1/3$.







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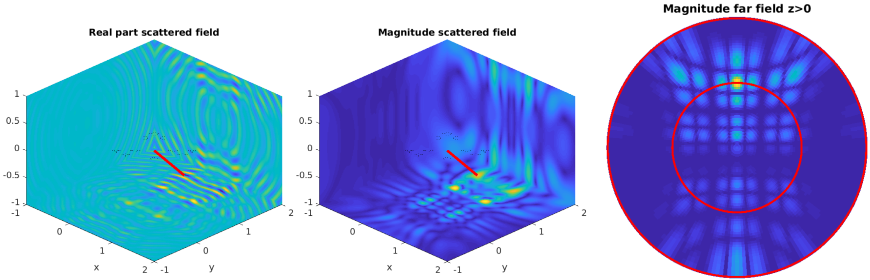
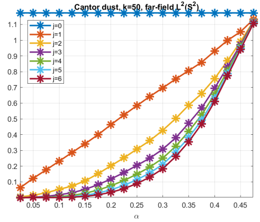
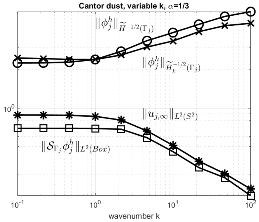




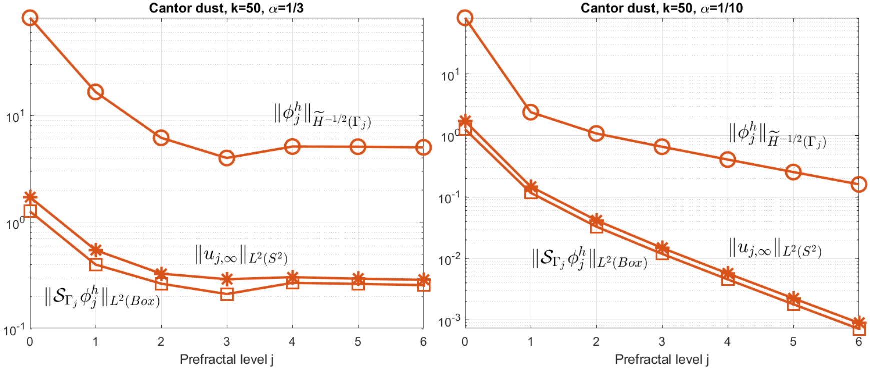
◀ L^2 norms of far-field, $\alpha \in (0.025, 0.475)$, prefractal levels $j = 0, \ldots, 6$.

Solution norms for $\alpha = \frac{1}{3}$ ▶ wavenumber $k \in [0.1, 100]$.



Cantor dust, solution norms

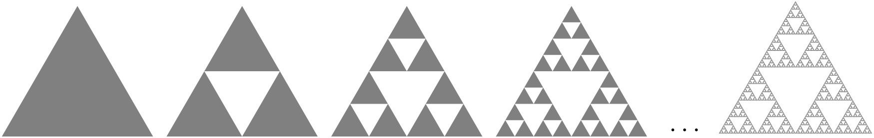
Norm of ◯ Neumann jumps (BIE solution), □ near-field, $*$ far-field:



Norms of the solution on the prefractals converge:

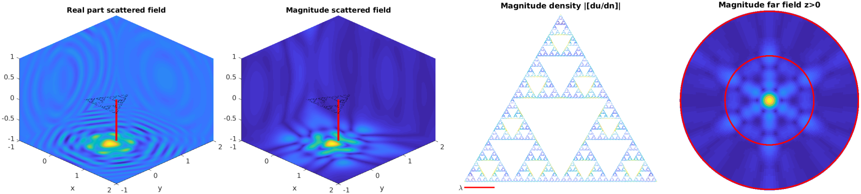
- to positive constant values for $\alpha = 1/3$ (left),
- to 0 for $\alpha = 1/10$ (right).

Sierpinski triangle

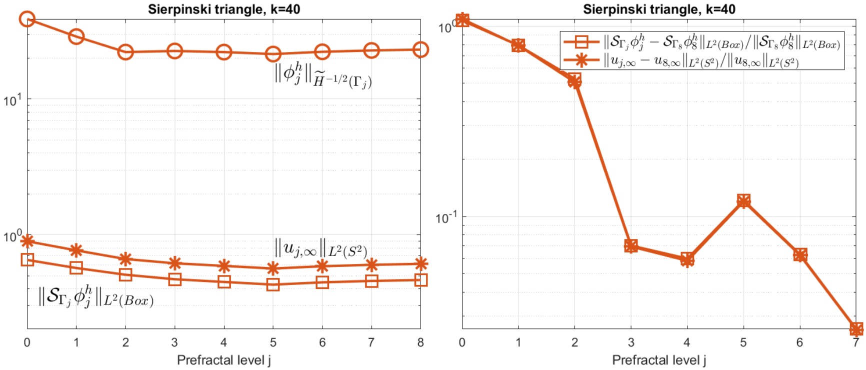


$H^{-1/2}_{\Gamma_j} \xrightarrow{\mathcal{M}} H^{-1/2}_{\Gamma}$, prefractal solutions ϕ_j converge to ϕ.
BEM on thickened prefractals converges if $h_j = o((\frac{3}{4} - \epsilon)^j)$.

Prefractal level $j = 8$, $N_j = 3^8 = 6\,561$ DOFs, $k = 40$:



Sierpinski triangle, solution norms



Right plot near- & far-field:

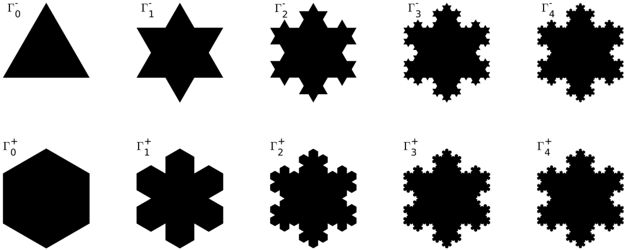
$$\square = \frac{\|\mathcal{S}_{\Gamma_j}\phi_j - \mathcal{S}_{\Gamma_8}\phi_8\|_{L^2(BOX)}}{\|\mathcal{S}_{\Gamma_8}\phi_8\|_{L^2(BOX)}}, \qquad * = \frac{\|u_{j,\infty} - u_{8,\infty}\|_{L^2(\mathbb{S}^2)}}{\|u_{8,\infty}\|_{L^2(\mathbb{S}^2)}}.$$

Prefractal level 3 is where density maxima are located and all wavelength-size prefractal features are resolved: big error reduction!

Koch snowflake

We can approximate Γ from inside and outside with polygons $\Gamma_j^{\pm}$:

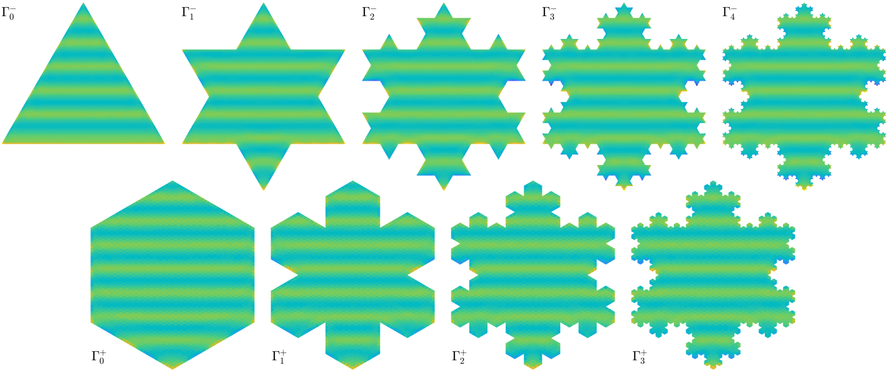
$$\underbrace{\Gamma_1^- \subset \Gamma_2^- \subset \Gamma_3^-}_{\text{open}} \subset \cdots \subset \bigcup_{j\in\mathbb{N}} \Gamma_j^- = \Gamma \subset \overline{\Gamma} = \bigcap_{j\in\mathbb{N}} \Gamma_j^+ \subset \cdots \subset \underbrace{\Gamma_3^+ \subset \Gamma_2^+ \subset \Gamma_1^+}_{\text{closed}}.$$



For a scattering BVP, since Γ is "thick", $\widetilde{H}^{\pm 1/2}(\Gamma) = H^{\pm 1/2}_{\overline{\Gamma}}$
and both sequences $u_j^{\pm}$ converge to the same limit.

(Caetano + H + M, 2018)

Real part of fields on inner and outer prefractals

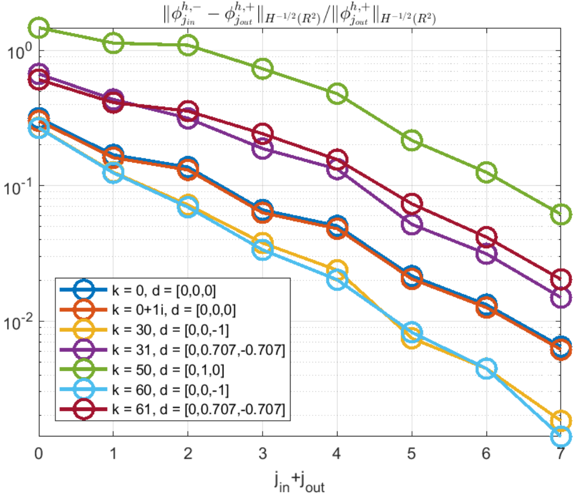


$k = 61, \ \mathbf{d} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^\top,$ 3576 to 10344 DOFs.

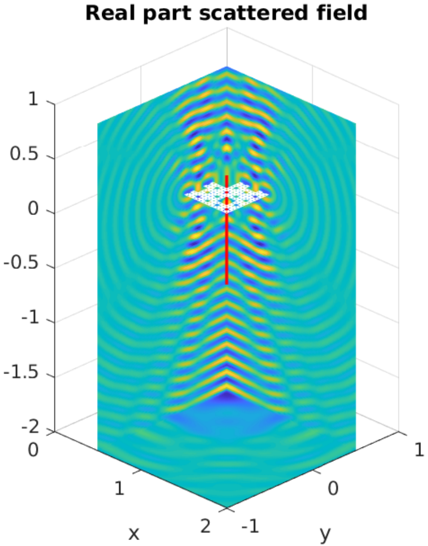
Now I compare $\phi_j^{h,-}$ against $\phi_{j-1}^{h,+}$ and $\phi_j^{h,+}$.

Inner and outer snowflake approximations

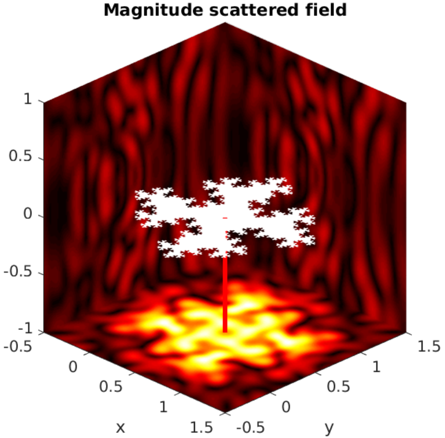
$$\frac{\|\phi^{h,-}_{j_{in}} - \phi^{h,+}_{j_{out}}\|_{H^{-1/2}(\mathbb{R}^2)}}{\|\phi^{h,+}_{j_{out}}\|_{H^{-1/2}(\mathbb{R}^2)}}$$



Other shapes



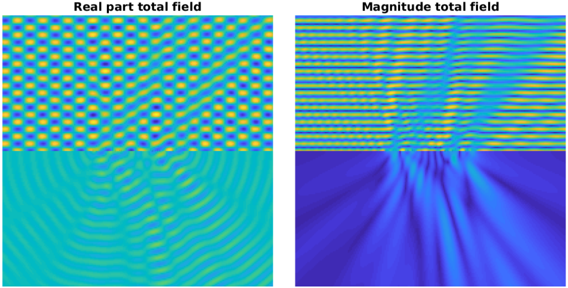
◁ Sierpinski carpet.



△ "Square snowflake",
limit of non-monotonic prefractals.

Apertures

Field through bounded apertures in unbounded Neumann screens computed via Babinet's principle.



$n = 1$, Cantor set $\alpha = 1/3$, prefractal level 12:
field through 0-measure holes!

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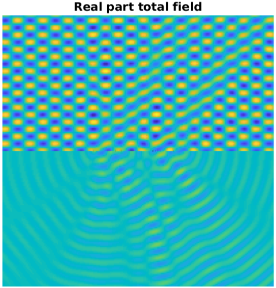

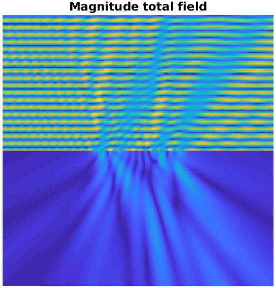

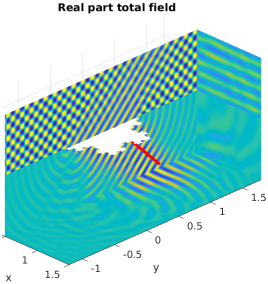

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Koch snowflake-shaped aperture △

Bibliography

- SNCW, DPH, *Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens,* IEOT, 2015.
- SNCW, DPH, AM, *Interpolation of Hilbert and Sobolev spaces: quantitative estimates and counterexamples,* Mathematika, 2015.
- DPH, AM, *On the maximal Sobolev regularity of distributions supported by subsets of Euclidean space,* An. and Appl., 2017.
- SNCW, DPH, AM, *Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to BIEs on fractal screens,* IEOT, 2017.
- DPH, AM, *A note on properties of the restriction operator on Sobolev spaces,* JAA 2017.
- SNCW, DPH, *Well-posed PDE and integral equation formulations for scattering by fractal screens,* SIAM J. Math. Anal., 2018.
- A. Caetano, DPH, AM, *Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets* arXiv 2019.
- SNCW, DPH, AM, *Boundary element methods for acoustic scattering by fractal screens* coming soon!

. . .

Open questions

- ✓ Impedance (Robin) bc's: **see Dave Hewett's talk!**
- Regularity theory for the fractal solution
- Rates of convergence
- Approximation on fractals
- Fast BEM implementation
- What about curved screens?
  More general rough scatterers?
- What about the Maxwell case?
  Other PDEs? (Laplace, reaction–diffusion already covered.)
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