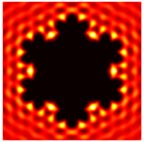
UMI — 4–9 September 2023 — Pisa

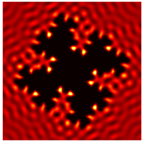
Integral equation methods for acoustic scattering by fractals

Andrea Moiola

http://matematica.unipv.it/moiola/



Università di Pavia
Department of Mathematics "Felice Casorati"



A. Caetano (Aveiro), S.N. Chandler-Wilde (Reading), X. Claeys (LJLL), A. Gibbs (UCL), D.P. Hewett (UCL)

arXiv:2309.02184

Acoustic wave scattering

Time-harmonic acoustic waves:

Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^n$, $n \in \{2, 3\}$, with wavenumber $k > 0$.

Direct scattering: incoming wave $\underbrace{u^i}_{\text{datum}}$ hits obstacle $\underbrace{\Gamma}_{\text{datum}}$ and generates scattered field $\underbrace{u^s}_{\text{unknown}}$.

Consider Dirichlet (sound-soft) boundary conditions on a bounded Γ.

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Time-harmonic acoustic waves:
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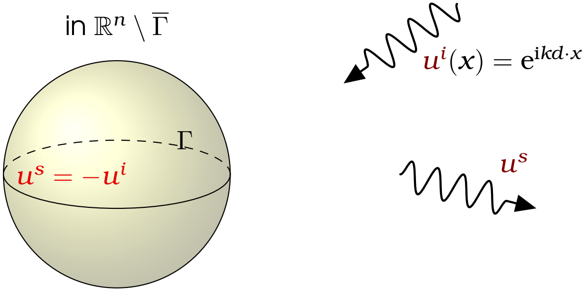
Consider Dirichlet (sound-soft) boundary conditions on a bounded Γ.

$$\Delta u^s + k^2 u^s = 0$$

in $\mathbb{R}^n \setminus \overline{\Gamma}$

$$u^{tot} = u^i + u^s$$

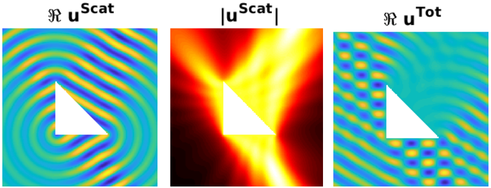


u^s satisfies Sommerfeld radiation condition (SRC) at infinity: $\lim_{r=|x|\to\infty} r^{\frac{n-1}{2}} (\partial_r u^s - iku^s) = 0$

Scattering by Lipschitz domains and screens

Classical problem e.g. when:

1. Γ is the boundary of a Lipschitz domain of $\mathbb{R}^n$



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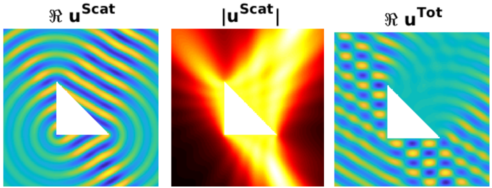

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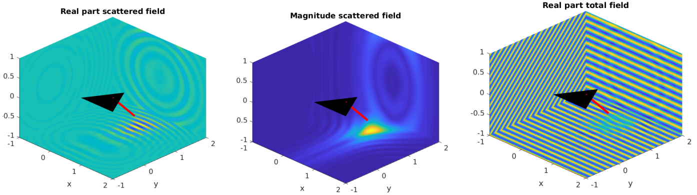
2. Γ is Lipschitz subset of $\{x \in \mathbb{R}^n, x_n = 0\}$ (planar screen)



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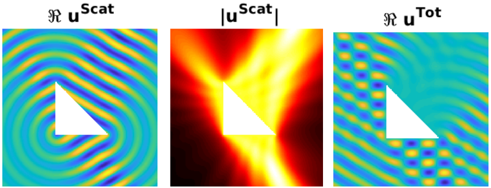

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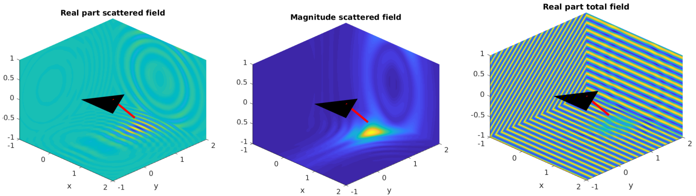

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Neumann trace (jump, in case 2) $\phi = [\partial_n u^s]$ on Γ is solution of single-layer BIE $S\phi = -\gamma u^i$,
scattered field represented with layer potential $u^s = \mathcal{S}\phi$. BIE approximated with BEM.

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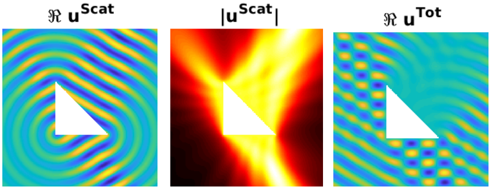

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What happens when Γ is much rougher than this, e.g. fractal?

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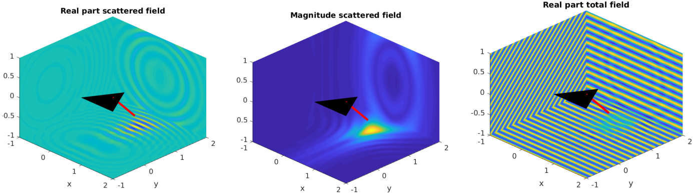

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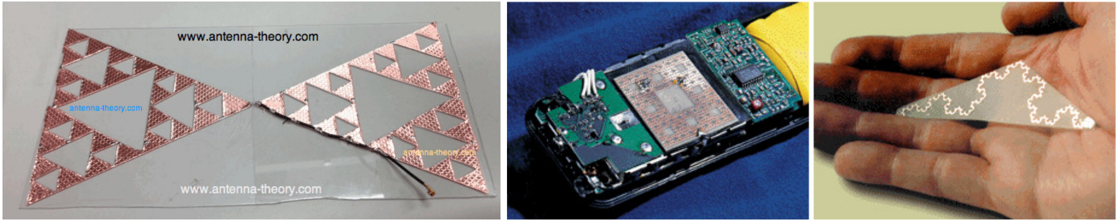
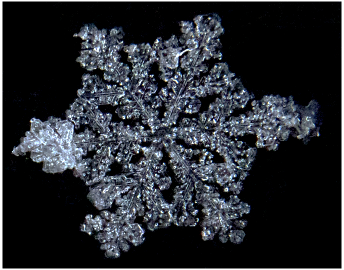
Waves and fractals: applications

Fractals model roughness at multiple scales, in natural and man-made objects:



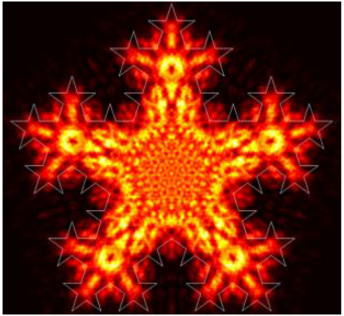
Wideband fractal antennas

▲ http://www.antenna-theory.com/antennas/fractal.php



◀ Scattering by ice crystals in atmospheric physics (C. Westbrook)

Fractal apertures in laser optics (J. Christian) ▶

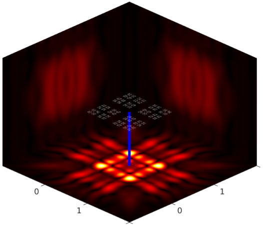


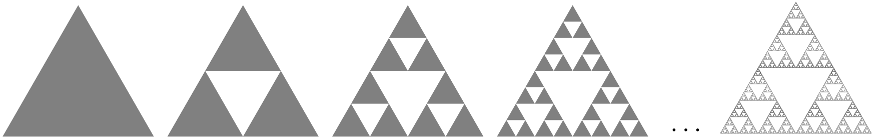
M.V. Berry 1979, "Diffractals": *a new regime in wave physics*

Scattering by fractals

Plenty of mathematical challenges:

- How to formulate well-posed BVPs?
  What is the right function space setting?
  How to write BVP as integral equation?
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?
- How to exploit self-similarity?
- ...





Tools developed here (hopefully!) relevant to (numerical) analysis of
other IEs, ΨDOs, BVPs, numerical integration on rough, complicated, fractal sets.

Our main contributions

This talk: AC, SCW, XC, AG, DH, AM, arXiv:2309.02184
Integral equation methods for acoustic scattering by fractals

BVPs, integral equations, function spaces

- SCW, DH, IEOT, 2015
  Wavenumber-explicit continuity & coercivity est. in acoustic scattering by planar scr.
- SCW, DH, AM, IEOT, 2017
  Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to BIEs on fractal scr.
- SCW, DH, SIAM J. Math. Anal., 2018
  Well-posed PDE and integral equation formulations for scattering by fractal screens,
- AC, DH, AM, JFA, 2021
  Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets

Numerical methods

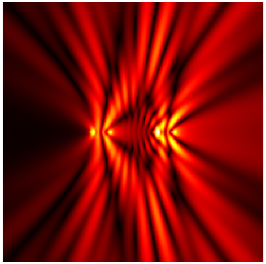
- SCW, DH, AM, J.Besson, Numer. Math., 2021
  Boundary element methods for acoustic scattering by fractal screens
- J.Bannister, AG, DH, M3AS, 2022
  Acoustic scattering by impedance screens/cracks with fractal boundary. . .
- AG, DH, AM, Numer. Algorithms, 2022
  Numerical quadrature for singular integrals on fractals
- AC, SCW, AG, DH, AM, arXiv:2212.06594, 2022
  A Hausdorff-measure BEM for acoustic scattering by fractal screens
- AG, DH, B.Major arXiv:2303.13141, 2023
  Numerical evaluation of singular integrals on non-disjoint self-similar fractal sets

What do we do?

3 levels of generality for Γ

- Arbitrary compact $\Gamma \subset \mathbb{R}^n$:
  BVP, Newton potential & op., variational form
  THEOREM: BVP and IE well-posedness
- d-sets:
  "intrinsic" function spaces, trace operators
  integral operators, piecewise-constant Galerkin
  THEOREM: Galerkin convergence
- IFS attractors:
  tree structure, wavelets, quadrature rule
  THEOREM: Galerkin convergence rates

\+ Numerical results



Arbitrary compact $\Gamma \subset \mathbb{R}^n$

BVP: $\Delta u^s + k^2 u^s = 0$ in $\Omega := \mathbb{R}^n \setminus \Gamma$, Sommerfeld r.c., $u^s + u^i \in W_0^{1,\mathrm{loc}}(\Omega)$

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Standard acoustic Newton potential:

$$\mathcal{A}\psi(x) := \int_{\mathbb{R}^n} \Phi(x,y)\psi(y)\mathrm{d}y, \qquad x \in \mathbb{R}^n, \qquad \Phi(x,y) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) & n=2 \\ \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} & n=3 \end{cases}$$

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Spaces:

$$H_\Gamma^{-1} := \{v \in H^{-1}(\mathbb{R}^n) : \operatorname{supp} v \subset \Gamma\}, \qquad \left(H_\Gamma^{-1}\right)^* = \widetilde{H}^1(\Omega)^\perp$$

$$\widetilde{H}^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\mathbb{R}^n)} \qquad P : H^1(\mathbb{R}^n) \to \widetilde{H}^1(\Omega)^\perp \text{ projection}$$

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"Integral operator": $A := H_\Gamma^{-1} \to \widetilde{H}^1(\Omega)^\perp$, $\quad A\phi := P(\sigma\mathcal{A}\phi)$, $\quad \sigma \in C_0^\infty(\mathbb{R}^n), \sigma|_{\Gamma+B_\epsilon} = 1$

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$a(\phi,\psi) := \langle A\phi, \psi\rangle_{H^1(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)}$ is continuous & compactly-perturb. coercive in $H_\Gamma^{-1} \times H_\Gamma^{-1}$

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THEOREM. Except for possibly countably many k, ($\forall k > 0$ if Ω connected)

- $A := H_\Gamma^{-1} \to \widetilde{H}^1(\Omega)^\perp$ is invertible
- the BVP has unique solution $u^s \in H^{1,\mathrm{loc}}(\mathbb{R}^n)$
- $u^s = \mathcal{A}\phi$ where $\phi \in H_\Gamma^{-1}$ is the unique solution of the IE $A\phi = g$ with $g := -P(\sigma u^i)$

Part I

IE and Galerkin on d-sets

Hausdorff measure and d-sets

Hausdorff measure and dimension of $E \subset \mathbb{R}^n$, $\quad 0 \le d \le n$:

$$\mathcal{H}^d(E) := \lim_{\delta \searrow 0} \inf_{\{U_i\}} \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supset E, \operatorname{diam} U_i < \delta \right\}, \quad \dim_{\mathrm{H}}(E) := \inf\{d : \mathcal{H}^d(E) = 0\}$$

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A compact set $\Gamma \subset \mathbb{R}^n$ is a d-set if $\quad c_1 r^d \le \mathcal{H}^d\big(\Gamma \cap B_r(x)\big) \le c_2 r^d \quad \forall x \in \Gamma,\ 0 < r \le 1$

"Uniformly locally d-dimensional sets".

FALCONER, TRIEBEL, JONSSON&WALLIN, ...

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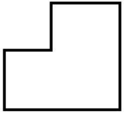
Examples of d-sets in $\mathbb{R}^2$:

(a) Closure of a bounded Lipschitz open set

$d = 2$,

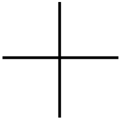


(b) Boundary of a bounded Lipschitz open set

$d = 1$,

(c) Line segment screen

$d = 1$,



(d) Multi-screen

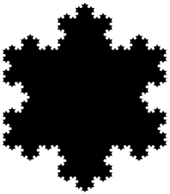
$d = 1$,

(e) Cantor set screen

$d = \frac{\log 2}{\log 3}$,

(f) Koch curve

$d = \frac{\log 4}{\log 3}$,



(g) Koch snowflake

$d = 2$

d-sets: function spaces and integral operator

On d-set Γ, define $\mathbb{L}_2(\Gamma)$ as the space of square-integrable function wrt measure $\mathcal{H}^d|_\Gamma$.

Can define "intrinsic" Sobolev spaces $\mathbb{H}^t(\Gamma)$. $\mathbb{H}^t(\Gamma) \subset \mathbb{L}_2(\Gamma) \subset \mathbb{H}^{-t}(\Gamma) = \mathbb{H}^t(\Gamma)^*$, $t > 0$.

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tr_Γ and its adjoint tr_Γ^* are unitary isomorphisms in: $(n - 2 < d \le n)$

$$
\begin{array}{ccccc}
\mathbb{H}^{1-\frac{n-d}{2}}(\Gamma) & \subset & \mathbb{L}_2(\Gamma) & \subset & \mathbb{H}^{-1+\frac{n-d}{2}}(\Gamma) \\
\mathrm{tr}_\Gamma \uparrow & & & & \downarrow \mathrm{tr}_\Gamma^* \\
\widetilde{H}^1(\mathbb{R}^n \setminus \Gamma)^\perp & & & & H_\Gamma^{-1} \\
\cap & & & & \cap \\
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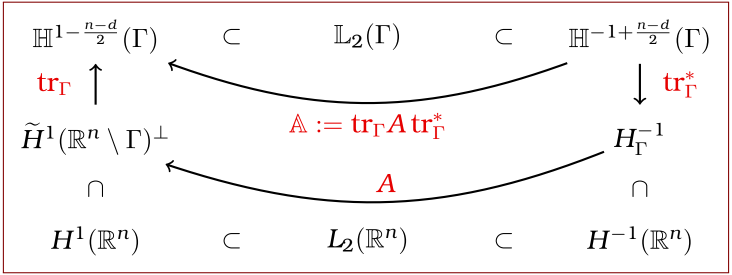

$\mathbb{A}$ is a single-layer operator between $\mathbb{H}^t(\Gamma)$ spaces

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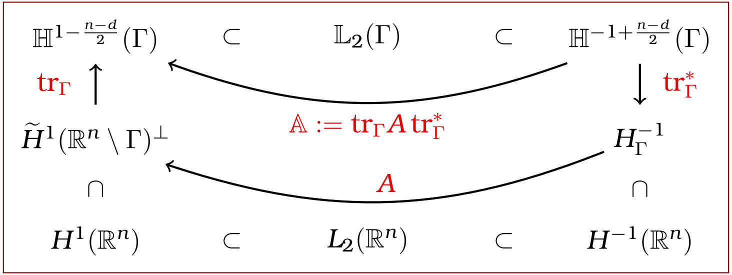
For $s > \frac{n-d}{2}$, it extends to $\mathrm{tr}_\Gamma : H^s(\mathbb{R}^n) \to \mathbb{L}_2(\Gamma)$ (continuous linear op. with dense image)

tr_Γ and its adjoint tr_Γ^* are unitary isomorphisms in: $(n-2 < d \le n)$



$\mathbb{A}$ is a single-layer operator between $\mathbb{H}^t(\Gamma)$ spaces

Theorem. $\mathbb{A}$ is an integral operator in Hausdorff measure:

$$\forall \Psi \in L_\infty(\Gamma), \qquad \mathbb{A}\Psi(x) = \int_\Gamma \Phi(x,y)\Psi(y)\, \mathrm{d}\mathcal{H}^d(y) \qquad \mathcal{H}^d\text{-a.e. } x \in \Gamma$$

The Hausdorff-measure integral equation method

Re-write IE and (coercive+compact) variational problem for $\widetilde{\phi} \in \mathbb{H}^{-t_d}(\Gamma)$, $t_d := 1 - \frac{n-d}{2}$:

$$\mathbb{A}\widetilde{\phi} = \mathbf{tr}_\Gamma g \qquad \Longleftrightarrow \qquad \langle \mathbb{A}\widetilde{\phi}, \widetilde{\psi} \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} = \langle \mathbf{tr}_\Gamma g, \widetilde{\psi} \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} \quad \forall \widetilde{\psi} \in \mathbb{H}^{-t_d}(\Gamma)$$

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We can apply Galerkin method with any N-dimensional $\mathbb{V}_N \subset \mathbb{L}_2(\Gamma) \overset{\text{dense}}{\subset} \mathbb{H}^{-t_d}(\Gamma)$.

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If Γ is boundary of bdd Lipschitz domain, screen or multi-screen (Claeys, Hiptmair 2013), then this coincides with classical single-layer BIE and BEM.

If Γ is planar d-set, it coincides with (AC, SCW, AG, DH, AM 2022).

Part II

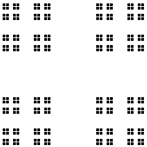
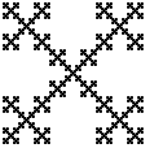
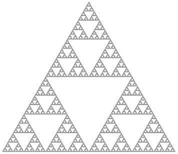
IEM on IFS attractors

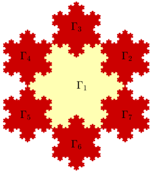
Iterated function systems (IFS)

IFS is a family of M contracting similarities: (FALCONER, HUTCHINSON, TRIEBEL,...)

$$s_m : \mathbb{R}^n \to \mathbb{R}^n, \qquad |s_m(x) - s_m(y)| = \rho_m |x - y|, \qquad 0 < \rho_m < 1, \qquad m = 1, \ldots, M.$$

There exists a unique non-empty compact Γ with $\Gamma = s(\Gamma)$, where $s(E) := \bigcup_{m=1}^{M} s_m(E)$.

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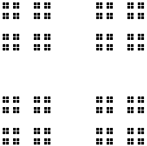
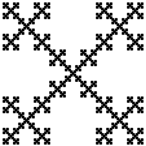
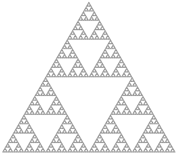


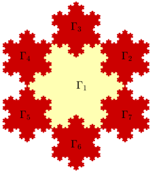

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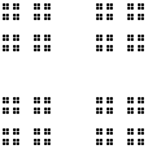
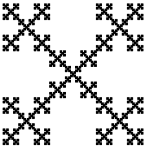
IFS is homogeneous if $\rho_m = \rho \ \forall m$ (then $d = \frac{\log M}{\log 1/\rho}$).

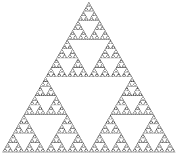
Γ is disjoint if $\Gamma_m := s_m(\Gamma)$ are all disjoint.
Disjoint implies OSC and $d < n$.



$M=4$, D, H



$M=5$, ND, H



$M=3$, ND, H

$M=4$, D, NH

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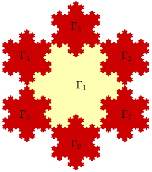
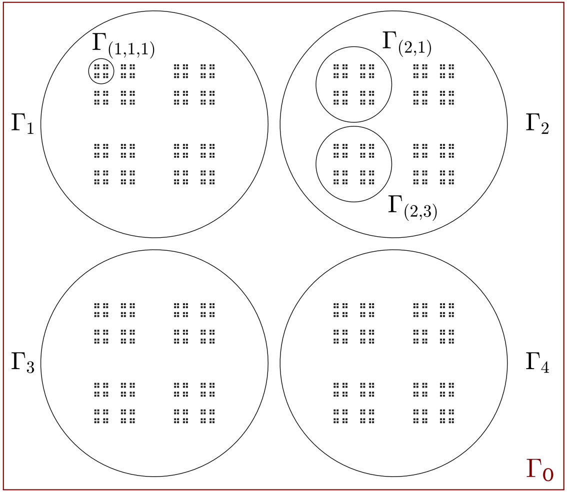


$M=7$, ND, NH

IFS tree structure and wavelets

Disjoint IFS attractor Γ have natural decompositions in elements $\Gamma_{\mathbf{m}} = s_{m_1} \circ \cdots \circ s_{m_\ell}(\Gamma)$, $\mathbf{m} = (m_1, \ldots, m_\ell) \in \{1, \ldots, M\}^\ell$, $\ell \in \mathbb{N}$, that are similar copies of Γ itself.



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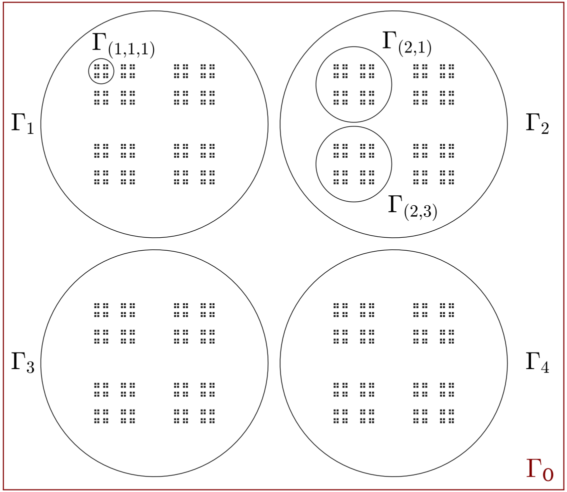

Linear combinations of characteristic functions $\chi_{\mathbf{m}}$ of $\Gamma_{\mathbf{m}}$ give hierarchical orthonormal wavelet basis of $\mathbb{L}_2(\Gamma)$.

Collecting $\Gamma_{\mathbf{m}}$s according to diameter, wavelet basis gives characterisation of $\mathbb{H}^t(\Gamma)$ and its norm. (JONSSON 1998)

We use $\mathrm{span}\{\chi_{\mathbf{m}}\}$ for a suitable partition with $\mathrm{diam}(\Gamma_{\mathbf{m}}) \le h$ as Galerkin space $\mathbb{V}_N$

Piecewise-constant IEM convergence for disjoint IFS attractors

Using Fredholm, relation Galerkin space/wavelets, coefficient decay in $\mathbb{H}^t(\Gamma)$:

Theorem (AC, SCW, XC, AG, DH, AM 2023)

Γ disjoint IFS attractor, $n-2 < d = \dim_{\mathrm{H}}(\Gamma) < n$.
$\mathbb{V}_N$ piecewise constants on self-similar partition $\{\Gamma_{\mathbf{m}}\}$ of Γ, $\mathrm{diam}(\Gamma_{\mathbf{m}}) \le h$.
Assume IE solution $\phi \in H^s_\Gamma$ for some $-1 < s < -\frac{n-d}{2}$.

Then $$\left\|\widetilde{\phi} - \widetilde{\phi}_N\right\|_{\mathbb{H}^{-1+\frac{n-d}{2}}(\Gamma)} = \|\phi - \phi_N\|_{H^{-1}_\Gamma} \le c\, h^{s+1} \|\phi\|_{H^s_\Gamma}$$

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- h^{2s+2} super-convergence of linear functionals, e.g.: point value $u^s(x)$ and far-field
- No higher regularity (and rate) can be expected: $H_\Gamma^{-\frac{n-d}{2}} = \{0\}$
- For homogeneous IFS ($\rho_m = \rho$), if maximal regularity is achieved, rates are

$$M^{-\ell/2} \quad \text{for } n = 2, \qquad (\rho M)^{-\ell/2} \quad \text{for } n = 3$$

  with ℓ the “level” of the pw-constant space $(h = \rho^\ell \mathrm{diam}(\Gamma),\ N = M^\ell)$
- For $d = n-1$, we recover classical results for Lipschitz screens and boundaries
  For $\Gamma \subset \{x_n = 0\}$, we recover (AC, SCW, AG, DH, AM 2022)

Part III

Numerics

Numerical quadrature on IFS attractors

Galerkin integral equation method for general class of IFS implemented in julia
`https://github.com/AndrewGibbs/IFSintegrals`

Linear system requires quadrature rule to approximate

$$A_{j,j'} = \int_{\Gamma_{\mathbf{m}(j)}} \int_{\Gamma_{\mathbf{m}(j')}} \Phi(x,y)\, \mathrm{d}\mathcal{H}^d(y)\mathrm{d}\mathcal{H}^d(x), \qquad b_j = -\int_{\Gamma_{\mathbf{m}(j)}} u^i(x)\, \mathrm{d}\mathcal{H}^d(x)$$

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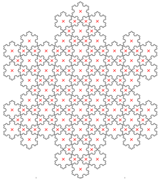
Recipe based on:

- decomposing $\Gamma_{\mathbf{m}}$ in similar sub-components, using IFS structure
- splitting Helmholtz kernel in Laplace + smoother terms
- exploiting Laplace kernel homogeneity and IFS self-similarity to reduce singular integral to a smooth one
- treating smooth integrands with composite barycentre rule, using IFS
- expressing all singular integrals in terms of a few "fundamental" ones

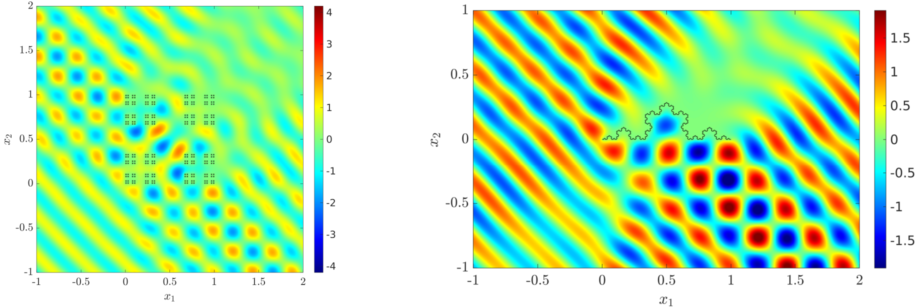


Convergence analysis of quadrature error and of fully discrete Galerkin system.

Disjoint case: (AG, DH, AM 2022). Non-disjoint case: (AG, DH, B. Major 2023).

$n = 2$

Total field for scattering by Cantor dust and Koch curve. $M = 4, \rho = \frac{1}{3}, d = \frac{\log 4}{\log 3}, k = 20.$



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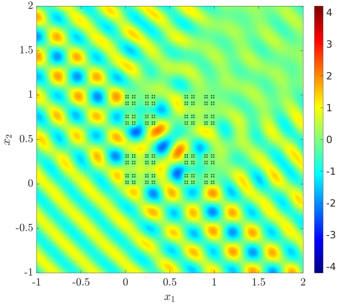
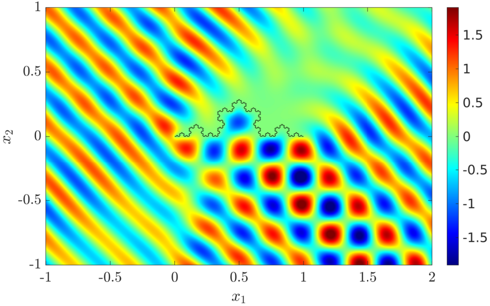
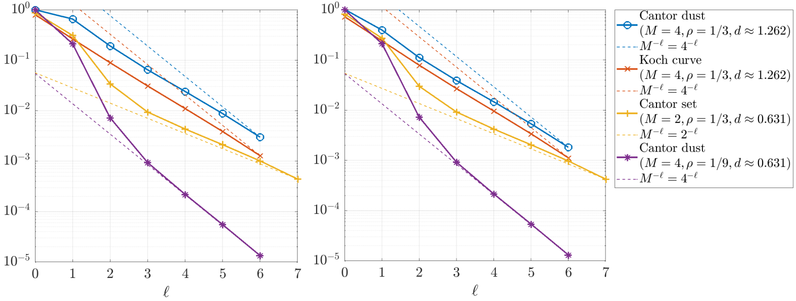


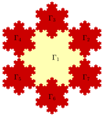


Near- & far-field relative L_∞ error for different shapes, $k = 5$.

Dashed lines $= M^{-\ell}$ conv. rates under maximal regularity: achieved for $d \leq 1$

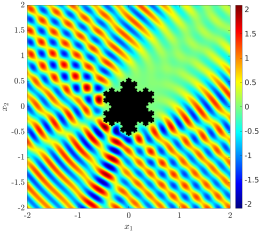


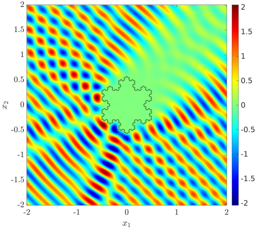
Koch snowflake

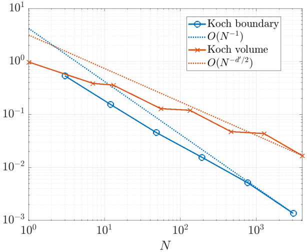


Two ways of approximating the scattering by a Koch snowflake Γ:

1. Γ = closure of open set: non-homog. IFS with $M = 7$, $d = 2$, $\rho_1 = \frac{1}{\sqrt{3}}$, $\rho_{2:7} = \frac{1}{3}$
2. $\partial\Gamma$ = union of 3 Koch curves: 3 IFSs with $M = 4$ each, $d = \frac{\log 4}{\log 3}$, $\rho = \frac{1}{3}$







◀ Far-field L_∞ relative error

2 requires that k^2 is not eigenvalue of Γ

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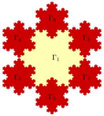

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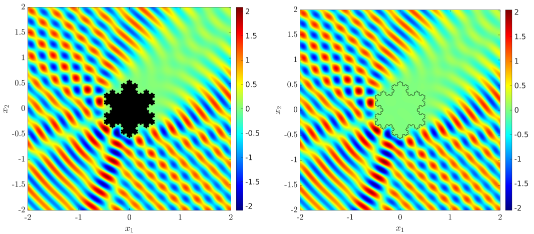

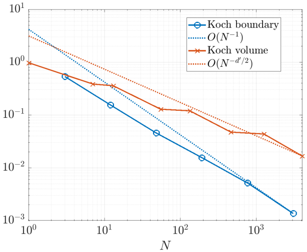

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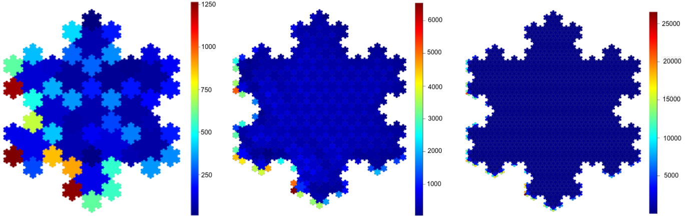
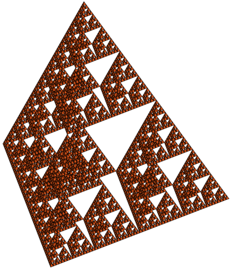
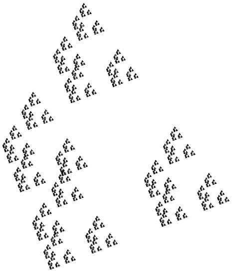
We show that the solution of IE 1 satisfies $\phi \in H^{-1}_{\partial\Gamma} \subset H^{-1}_{\Gamma}$

Refining the mesh, ϕ_N localises on boundary: plot of $|\phi_N|$ ►



$n = 3$

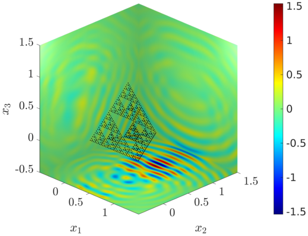


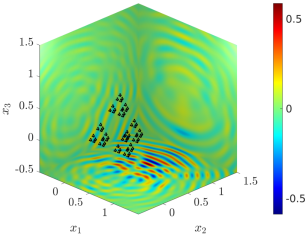


◀ Sierpinski tetrahedron, $M = 4$.

Left: $\rho = \frac{1}{2}$, $d = 2$, connected

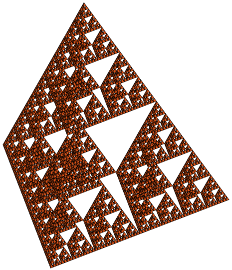
Right: $\rho = \frac{3}{8}$, $d = \frac{\log 4}{\log(8/3)}$, disjoint

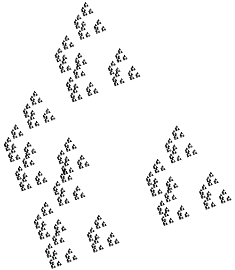




▲ scattered field, $k = 50$, $\ell = 7$, $N = 16\,384$

$n = 3$

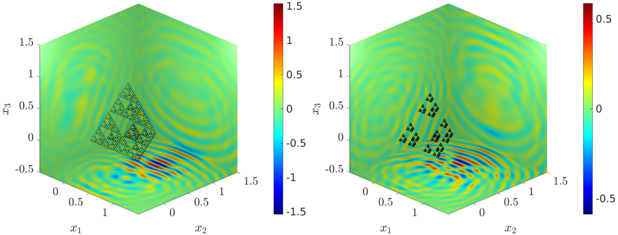


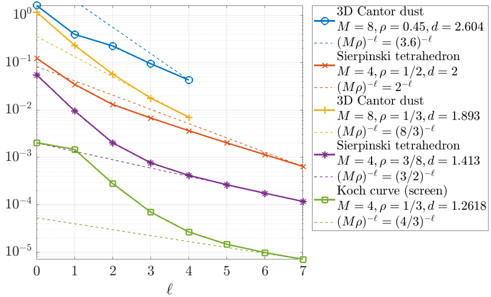


- Sierpinski tetrahedron, $M = 4$.

Left: $\rho = \frac{1}{2}$, $d = 2$, connected

Right: $\rho = \frac{3}{8}$, $d = \frac{\log 4}{\log(8/3)}$, disjoint





▲ scattered field, $k = 50$, $\ell = 7$, $N = 16\,384$

far-field L_∞ error (increments), $k = 2$ ▲

Summary and outlook

Scattering of time-harmonic acoustic waves by sound-soft obstacle Γ:

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Γ d-set:	IE in Hausdorff measure, convergence of piecewise-constant Galerkin
Γ disjoint IFS:	concrete recipe for Galerkin space & quadrature, convergence rates

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Open questions and ongoing work:

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