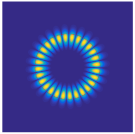
ROMA, 14 MAY 2019

Acoustic and electromagnetic transmission problems: wavenumber-explicit bounds and resonance-free regions

Andrea Moiola



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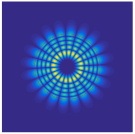

Joint work with E.A. Spence (Bath)

Part I

Helmholtz equation

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Acoustic waves in free space governed by wave eq. $\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

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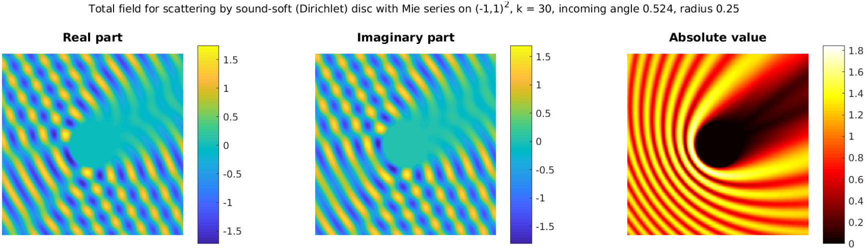
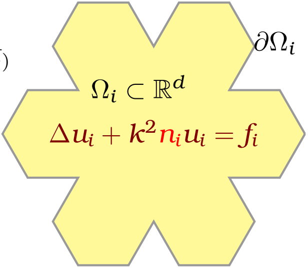
Typical Helmholtz scattering problem:
plane wave $u^{Inc}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ hitting a sound-soft (i.e. Dirichlet) obstacle



Wavelength: $\lambda = \frac{2\pi}{k}$, distance between two crests of a plane wave.

Single penetrable homogeneous obstacle Ω_i:

Sommerfeld radiation condition
$\partial_r u_o - \mathrm{i} k u_o = o(\sqrt{r^{1-d}})$



$$\partial\Omega_i \quad \begin{cases} u_o = u_i + g_D \\ \partial_n u_o = A_N \partial_n u_i + g_N \end{cases}$$

$$\Omega_o = \mathbb{R}^d \setminus \overline{\Omega_i}$$

$$\Delta u_o + k^2 u_o = f_o$$

Helmholtz transmission problem

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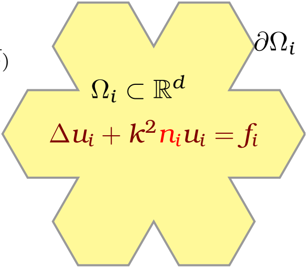

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Data: $f_i \in L^2(\Omega_i)$, $f_o \in L^2_{\text{comp}}(\Omega_o)$, $g_D \in H^1(\partial\Omega_i)$, $g_N \in L^2(\partial\Omega_i)$,
wavenumber $k > 0$, refractive index² $n_i > 0$, $A_N > 0$,
scatterer $\Omega_i \subset \mathbb{R}^d$ (Lipschitz bounded).

What is A_N? E.g. in TE modes $\varepsilon\mu = \begin{cases} 1 & \text{in } \Omega_o, \\ n_i & \text{in } \Omega_i, \end{cases}$ $u = H_z$: $A_N = \frac{\varepsilon_o}{\varepsilon_i}$.
In TM modes, $u = E_z$: $A_N = \frac{\mu_o}{\mu_i}$. In acoustics $A_N = \frac{\rho_o}{\rho_i}$.

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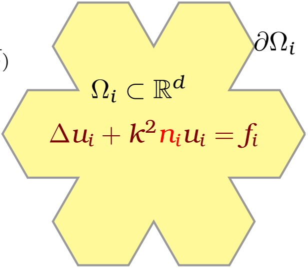

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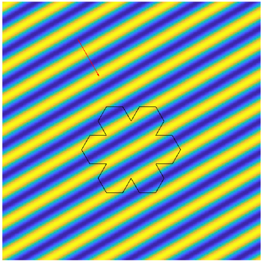
Solution exists and is unique for Ω_i Lipschitz and $k \in \mathbb{C} \setminus \{0\}$, $\Im k \geq 0$
TORRES, WELLAND 1999.

Wave scattering

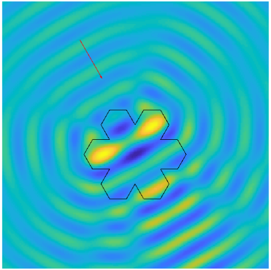
The example we have in mind is scattering of incoming wave u^{Inc}:

$$f_i = k^2(1 - n_i)u^{Inc}, \qquad f_o = 0, \qquad g_D = 0, \qquad g_N = (A_N - 1)\partial_{\mathbf{n}} u^{Inc}.$$

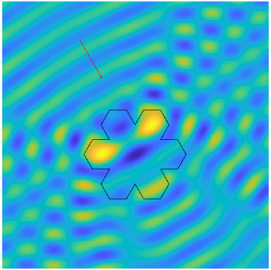
Incoming field
$u^{Inc} = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ (datum)



Scattered field
$u = (u_i, u_o)$



Total field
$u + u^{Inc}$



$n_i = \frac{1}{4}$, $A_N = 1$, $\mathbf{d} = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $k = 20$, $\lambda = 0.314$, 3×3 box,
figures represent real parts of fields.
$\rightarrow U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}kt}\}$

Goal and motivation

From Fredholm theory we have

$$\left\| \begin{pmatrix} u_i \\ u_o \end{pmatrix} \right\|_{\Omega_{i/o}} \leq \mathcal{C}_1 \left\| \begin{pmatrix} f_i \\ f_o \end{pmatrix} \right\|_{\Omega_{i/o}} + \mathcal{C}_2 \left\| \begin{pmatrix} g_D \\ g_N \end{pmatrix} \right\|_{\partial\Omega_i}$$

Goal: find out how $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on k, n_i, A_N, and Ω_i and deduce results about resonances.

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Motivation: NA of Helmholtz problems with variable wavenumber:

- Barucq, Chaumont-Frelet, Gout (2016)
- Ohlberger, Verfürth (2016)
- Brown, Gallistl, Peterseim (2017)
- Sauter, Torres (2017)
- Graham, Pembery, Spence (2019)
- Graham, Sauter (2018)

and with random parameters (from UQ perspective):

- Feng, Lin, Lorton (2015)
- Hiptmair, Scarabosio, Schillings, Schwab (2018)
- Pembery, Spence (2018)...

Who cares?

Lafontaine, Spence, Wunsch, arXiv 2019:

The following is a non-exhaustive list of papers on the frequency-explicit convergence analysis of numerical methods for solving the Helmholtz equation where a central role is played by *either* the non-trapping resolvent estimate (1.5), *or* its analogue (with the same k-dependence) for the commonly-used approximation of the exterior problem where the exterior domain $\mathcal{O}_+$ is truncated and an impedance boundary condition is imposed:

- conforming FEMs (including continuous interior-penalty methods) [72, Proposition 2.1], [74, Proposition 8.1.4], [56, Lemma 2.1], [77, Lemma 3.5], [78, Assumptions 4.8 and 4.18], [45, §2.1], [110, Theorem 3.1], [113, §3.1], [44, §3.2.1], [40, Remark 3.2], [41, Remark 3.1], [29, Assumption 1], [30, Definition 2], [55, Theorem 3.2], [50, Lemma 6.7], [14, Equation 4],
- least squares methods [33, Assumption A1], [10, Remark 1.2], [64, Assumption 1 and equation after Equation 5.37],
- DG methods based on piece-wise polynomials [46, Theorem 2.2], [47, Theorem 2.1], [39, Assumption 3], [48, §3], [62, Assumption A (Equation 4.5)], [76, Equation 4.4], [35, Remark 3.2], [32, Equation 2.4], [84, Equation 4.3], [112, Remark 3.1], [94, Theorem 2.2],
- plane-wave/Trefftz-DG methods [3, Theorem 1], [59, Equation 3.5], [60, Theorem 2.2], [2, Lemma 4.1], [61, Proposition 2.1],
- multiscale finite-element methods [51, Equation 2.3], [13, §1.2], [88, Assumption 5.3], [8, Theorem 1], [87, Assumption 3.8], [31, Assumption 1],
- integral-equation methods [71, Equation 3.24], [75, Equation 4.4], [24, Chapter 5], [53, Theorem 3.2], [111, Remark 7.5], [43, Theorem 2], [49, Theorem 3.2], [52, Assumption 3.2],

In addition, the following papers focus on proving bounds on the solution of Helmholtz boundary-value problems (with these bounds often called "stability estimates") motivated by applications in numerical analysis: [36], [57], [26], [11], [7], [70], [98], [28], [6], [9], [27], [93], [54], [55] [83], [50], Of these papers, all but [70], [6], [27], [11] are in nontrapping situations, [70], [6], [27] are in parabolic trapping scenarios, and [11] proves the exponential growth (1.7) under elliptic trapping.

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LAFONTAINE, SPENCE, WUNSCH, arXiv 2019:

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Allow to control:

- Quasi-optimality & pollution effect
- Gmres iteration #
- Matrix compression
- *hp*-FEM&BEM (Melenk–Sauter)
- ...

“Cut-off resolvent”: $R_\chi(k)$

Assume $g_D = g_N = 0$ (no jumps/boundary data).

Solution operator: $R(k) = R(k, n_i, A_N, \Omega_i)$: $\begin{pmatrix} f_i \\ f_o \end{pmatrix} \mapsto \begin{pmatrix} u_i \\ u_o \end{pmatrix}$.

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Resonances: poles of meromorphic continuation of $R_\chi(k)$ to $\Im k < 0$.

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We want to bound the norm of $R_\chi(k)$, $k \in \mathbb{R}$.
Consider separately cases $n_i < 1$ and $n_i > 1$: very different!

Resolvent bounds for $n_i < 1$

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CARDOSO, POPOV, VODEV 1999:

- using microlocal analysis
- Ω_i smooth (C^∞), convex, curvature > 0 
- C_0, C_1 not explicit in n_i, A_N
- $k > k_0$ for some $k_0 > 0$
- $n_i < 1$, $A_N > 0$ TE/TM: $\varepsilon_i\mu_i \le \varepsilon_o\mu_o$

M., SPENCE:

- elementary proof
- Ω_i Lipschitz, star-shaped ($\mathbf{x}\cdot\mathbf{n} \ge 0$) 
- C_0, C_1 explicit in n_i, A_N and geometry
- any $k > 0$
- $n_i \le \frac{1}{A_N} \le 1$ TE/TM: $\varepsilon_i \le \varepsilon_o$, $\mu_i \le \mu_o$

(Related results in PERTHAME, VEGA 1999.)

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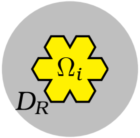
Using VODEV 1999, under either set of assumptions, we have strip of holomorphicity underneath real axis:

$$R_\chi(k) \text{ is holomorphic in } \{k \in \mathbb{C} : \Re k > k_0,\ \Im k > -\delta\} \qquad (\delta > 0)$$

(One of) our bounds



$\Omega_i \subset \mathbb{R}^d$ is star-shaped, $g_N = g_D = 0$, $k > 0$, and

$$0 < n_i \leq \frac{1}{A_N} \leq 1.$$

Given $R > 0$ such that $\operatorname{supp} f_o \subset B_R$, let $D_R := B_R \setminus \overline{\Omega_i}$.

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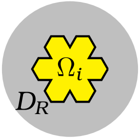

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$$\|\nabla u_i\|^2_{L^2(\Omega_i)} + k^2 n_i \|u_i\|^2_{L^2(\Omega_i)} + \frac{1}{A_N}\left(\|\nabla u_o\|^2_{L^2(D_R)} + k^2 \|u_o\|^2_{L^2(D_R)}\right)$$
$$\le \left[4\,\mathrm{diam}(\Omega_i)^2 + \frac{1}{n_i}\left(2R + \frac{d-1}{k}\right)^2\right]\|f_i\|^2_{L^2(\Omega_i)}$$
$$+ \frac{1}{A_N}\left[4R^2 + \left(2R + \frac{d-1}{k}\right)^2\right]\|f_o\|^2_{L^2(D_R)}.$$

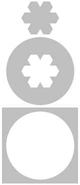
Fully explicit, shape-robust estimate.

(Extended to $g_D, g_N \neq 0$ under strict inequalities and star-shapedness.)

How our bound was obtained

Multiply the PDE by the "test functions" (multipliers, $\mathcal{M}u$)

$$\begin{aligned} &\mathbf{x}\cdot\nabla u - \mathrm{i}kRu + \frac{d-1}{2}u && \text{in } \Omega_i,\\ &\frac{1}{A_N}\Big(\mathbf{x}\cdot\nabla u - \mathrm{i}kRu + \frac{d-1}{2}u\Big) && \text{in } D_R,\\ &\frac{1}{A_N}\Big(\mathbf{x}\cdot\nabla u - \mathrm{i}k|\mathbf{x}|u + \frac{d-1}{2}u\Big) && \text{in } \mathbb{R}^d\setminus D_R, \end{aligned}$$



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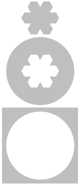
$$\frac{1}{A_N}\left(\mathbf{x}\cdot\nabla u - \mathrm{i}kRu + \frac{d-1}{2}u\right) \qquad \text{in } D_R,$$

$$\frac{1}{A_N}\left(\mathbf{x}\cdot\nabla u - \mathrm{i}k|\mathbf{x}|u + \frac{d-1}{2}u\right) \qquad \text{in } \mathbb{R}^d \setminus D_R,$$



integrate by parts and sum 3 contributions. E.g. on Ω_i we obtain

$$\int_{\Omega_i} |\nabla u_i|^2 + n_i k^2 |u_i|^2$$

$$= -2\Re \int_{\Omega_i} f_i\, \overline{\mathcal{M}u_i} + \int_{\partial\Omega_i} (\mathbf{x}\cdot\mathbf{n}) \left(|\partial_{\mathbf{n}} u_i|^2 - |\nabla_T u_i|^2 + k^2 n_i |u_i|^2\right)$$

$$+ 2\Re\left\{\left(\mathbf{x}\cdot\overline{\nabla_T u_i} + \mathrm{i}kR\overline{u_i} + \frac{d-1}{2}\overline{u_i}\right)\partial_{\mathbf{n}} u_i\right\}.$$

Manipulation of terms on $\partial\Omega_i$ & ∂B_R from 2 sides gives negative value.
First for smooth fields, then proceed by density.

These types of test functions introduced by Morawetz in 1960s/1970s.

Proof for smooth u_i, u_o

$$\int_{\Omega_i}(|\nabla u_i|^2 + k^2 n_i |u_i|^2) + \frac{1}{A_N}\int_{D_R}(|\nabla u_o|^2 + k^2|u_o|^2)$$

$$\overset{\text{IBP!}}{=} -2\Re\int_{\Omega_i}\Big(\mathbf{x}\cdot\nabla\overline{u}_i - \mathrm{i}kR\overline{u}_i + \frac{d-1}{2}\overline{u}_i\Big)f_i - \frac{2}{A_N}\Re\int_{D_R}\Big(\mathbf{x}\cdot\nabla\overline{u}_o - \mathrm{i}kR\overline{u}_o + \frac{d-1}{2}\overline{u}_o\Big)f_o$$

$$+\int_\Gamma(\mathbf{x}\cdot\mathbf{n})\Big(|\partial_\mathbf{n}u_i|^2 - |\nabla_T u_i|^2 + k^2 n_i|u_i|^2\Big) + 2\Re\Big\{\Big(\mathbf{x}\cdot\overline{\nabla_T u_i} + \mathrm{i}kR\overline{u_i} + \frac{d-1}{2}\overline{u_i}\Big)\partial_\mathbf{n}u_i\Big\}$$

$$-\frac{1}{A_N}\int_\Gamma(\mathbf{x}\cdot\mathbf{n})\Big(|\partial_\mathbf{n}u_o|^2 - |\nabla_T u_o|^2 + k^2|u_o|^2\Big) + 2\Re\Big\{\Big(\mathbf{x}\cdot\overline{\nabla_T u_o} + \mathrm{i}kR\overline{u_o} + \frac{d-1}{2}\overline{u_o}\Big)\partial_\mathbf{n}u_o\Big\}$$

$$\underbrace{+\frac{1}{A_N}\int_{\partial B_R}\Big(R(|\partial_r u_o|^2 - |\nabla_T u_o|^2 + k^2|u_o|^2) - 2kR\Im\{\overline{u_o}\partial_r u_o\} + (d-1)\Re\{\overline{u_o}\partial_r u_o\}\Big)}_{=0,\text{ from SRC and Morawetz–Ludwig IBP identity}}$$

$$\le \|f_i\|_{\Omega_i}\Big(2\,\mathrm{diam}(\Omega_i)\,\|\nabla u_i\|_{\Omega_i} + (2kR+d-1)\,\|u_i\|_{\Omega_i}\Big) \qquad \leftarrow \text{Cauchy–Schwarz}$$

$$+\frac{\|f_o\|_{D_R}}{A_N}\left(2R\,\|\nabla u_o\|_{D_R} + (2kR+d-1)\,\|u_o\|_{\Omega_o}\right)$$

$$+\int_\Gamma \underbrace{\mathbf{x}\cdot\mathbf{n}}_{\ge 0,\ \star\text{-shape}}\Big(\underbrace{|\partial_\mathbf{n}u_i|^2 - \frac{1}{A_N}|\partial_\mathbf{n}u_o|^2}_{\le 0,\text{ from jump rel.s and } A_N\ge 1}\ \underbrace{-|\nabla_T u_i|^2 + \frac{1}{A_N}|\nabla_T u_o|^2}_{\le 0,\text{ from jump rel.s and } A_N\ge 1} + \underbrace{k^2 n_i|u_i|^2 - \frac{1}{A_N}k^2|u_o|^2}_{\le 0,\text{ from jump rel.s and } n_i\le\frac{1}{A_N}}\Big)$$

$$+2\Re\int_\Gamma\underbrace{\Big(\mathbf{x}\cdot\overline{\nabla_T u_i} + \mathrm{i}kR\overline{u_i} + \frac{d-1}{2}\overline{u_i}\Big)\partial_\mathbf{n}u_i - \frac{1}{A_N}\Big(\mathbf{x}\cdot\overline{\nabla_T u_o} + \mathrm{i}kR\overline{u_o} + \frac{d-1}{2}\overline{u_o}\Big)\partial_\mathbf{n}u_o}_{=0,\text{ from jump rel.s } u_o=u_i\ \&\ \partial_\mathbf{n}u_o=A_N\partial_\mathbf{n}u_i}$$

$$\le\frac{\text{left-hand side}}{2} + \left[2\,\mathrm{diam}(\Omega_i)^2 + \frac{1}{2n_i}\Big(2R+\frac{d-1}{k}\Big)^2\right]\|f_i\|^2_{\Omega_i} + \frac{1}{A_N}\left[2R^2 + \frac{1}{2}\Big(2R+\frac{d-1}{k}\Big)^2\right]\|f_o\|^2_{D_R}.$$

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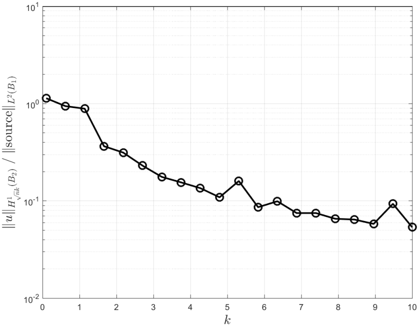

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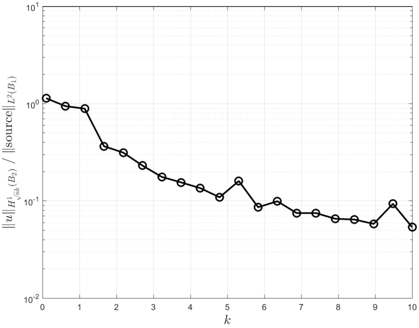
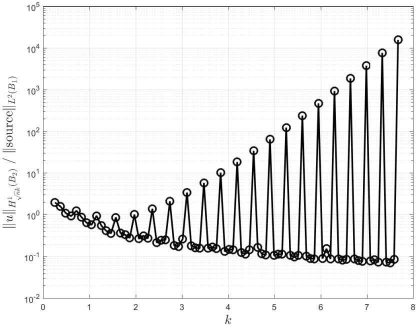




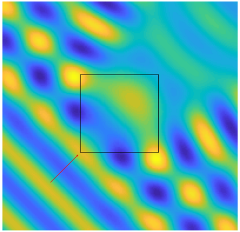
Try many cases and they seem to suggest stability hold.
However...if we choose some special ks $\|u\|_{L^2(B_R)}$ & $\|u\|_{H^1(B_R)}$ blow up!

$n_i < 1$ vs $n_i > 1$ $\quad \left(\lambda_o = \frac{2\pi}{k},\ \lambda_i = \frac{2\pi}{k\sqrt{n_i}},\ \ n_i = \frac{\lambda_o^2}{\lambda_i^2}\right)$

$n_i < 1 \Rightarrow \lambda_i > \lambda_o$

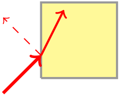
inside Ω_i wavelength is longer

E.g. air bubble in water.



$(n_i = 1/3)$

Snell's law:

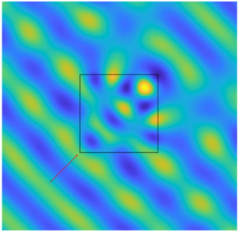


All rays eventually leave Ω_i:
stability for all $k > 0$.

$n_i > 1 \Rightarrow \lambda_i < \lambda_o$

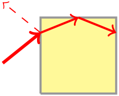
inside Ω_i wavelength is shorter

E.g. glass in air: lenses.



$(n_i = 3)$

Snell's law:



Total internal reflection,
creeping waves, ray trapping:
quasi-resonances.

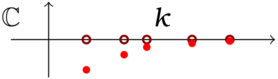
"Quasi-modes" for $n_i > 1$

- POPOV, VODEV 1999:

Ω_i smooth, convex, strictly positive curvature, $n_i > 1$, $A_N > 0$,

$\exists$ complex sequence $(k_j)_{j=1}^{\infty}$, with $|k_j| \to \infty$, $\Re k_j \geq 1$, and $0 > \Im k_j = \mathcal{O}(|k_j|^{-\infty})$ s.t.

$\|R_\chi(k_j)\|_{L^2 \to L^2}$ blows up super-algebraically



We show that $\{\Re k_j\}$ gives the same blow up:

"quasi-modes" with real wavenumber.

These are the peaks in the previous plot.

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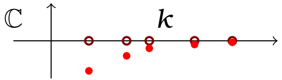

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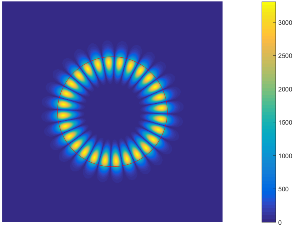
- BELLASSOUED 2003: (blow up is at most exponential in k)

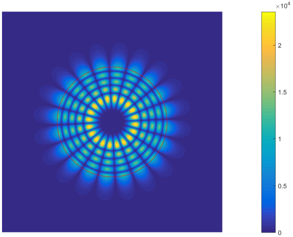
Ω_i smooth, $n_i > 0$, $A_N > 0$, $\exists C_1, C_2, k_0 > 0$, s.t.

$$\|R_\chi(k)\|_{L^2\to L^2} \leq C_1 \exp(C_2 k) \quad \text{for all } k \geq k_0$$

Quasi-resonances and perturbations

Ω_i=unit disc in $\mathbb{R}^2$, $n_i = 100$.





$$k_1 = 1.77945199481921 \approx \Re k_{14,1}, \;\; k_2 = 2.75679178324354 \approx \Re k_{10,5}$$

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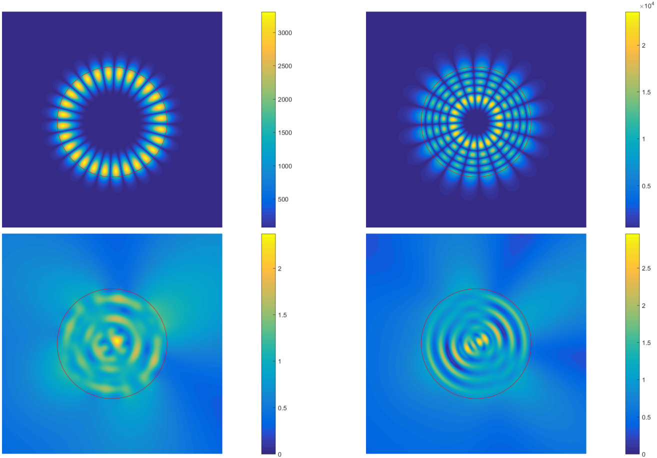

$k_1 = 1.779451994819\textcolor{red}{21} \approx \Re k_{14,1}$, $k_2 = 2.75679178324354 \approx \Re k_{10,5}$

$k_3 = 1.77945199481\textcolor{red}{5}$, $k_4 = 2.757$

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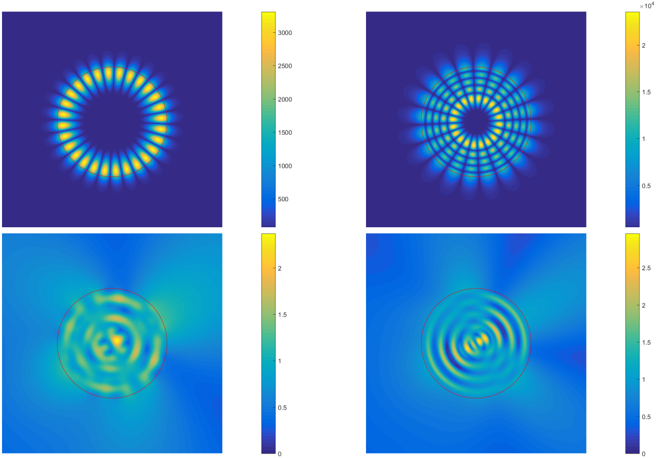

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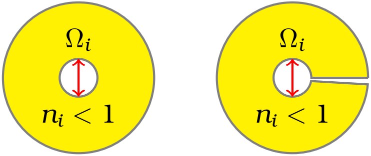
LAFONTAINE, SPENCE, WUNSCH 2019:

$\forall \delta > 0\ \exists J \subset \mathbb{R},\ |J| < \delta$ s.t. $\|R_\chi(k)\|_{L^2 \to L^2} \leq C k^{\frac{5}{2}d+\epsilon}$ $\forall k \in [k_0, \infty) \setminus J$.

Non star-shaped scatterers ($n_i < 1$)

For $n_i < 1$, need of star-shaped scatterer Ω_i is now clear:
general Ω_i can contain cavities, trap waves, support quasi-modes.



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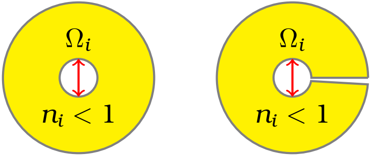
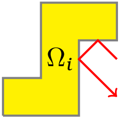


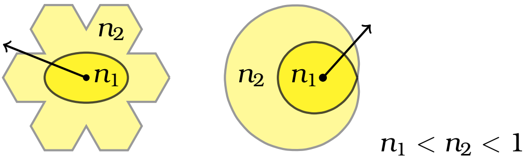
We expect that k-uniform bounds hold for more general obstacles: non-trapping domains.



Morawetz techniques are not useful in this case.

What if n_i takes more than two values?

For piecewise-constant n_i, i.e. several materials, similar bounds hold if n_i increases radially:



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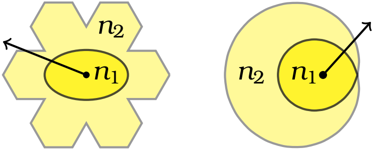

$n_1 < n_2 < 1$

More general case: $n \in C^{0,1}$

GRAHAM, PEMBERY, SPENCE 2019

If $2n(\mathbf{x}) + \mathbf{x} \cdot \nabla n(\mathbf{x}) \geq \bigstar > 0$,

$1 - n$ compactly supported

$\Rightarrow$ the solution of $\Delta u + nk^2 u = f$ satisfies $\|u\|_{H^1_k(B_R)} \leq \frac{C}{\bigstar} \|f\|_{L^2(B_R)}$.

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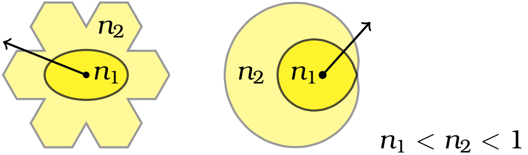

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Extensions:

- $\operatorname{div}(A\nabla u) + nk^2 u = f$
- $n \in L^\infty(\mathbb{R}^d)$ radially non-decreasing,
  $A \in L^\infty(\mathbb{R}^d; SPD)$ radially non-increasing
- Star-shaped Dirichlet scatterer
- Truncated domain and impedance BCs

Helmholtz equation: summary

(M3AS 2019) Moiola, Spence, Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions.

- $n_i < 1$: explicit bounds on $\|u\|_{H^1(B_R)}$ from Morawetz multipliers,
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Open question for $n_i > 1$:

Does non-smooth Ω_i support quasi-modes? What's blow up in k?

Think: Ω_i polygon/polyhedron.

PDE guess: Yes, what's bad for smooth is worse for rough.
Wave guess: No, corners diffract energy and stop creeping waves.

Interesting numerical project!

Part II

Maxwell equations

Maxwell "transmission" problem

Given:

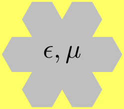
- $k > 0$
- $\mathbf{J}, \mathbf{K} \in H(\mathrm{div}^0, \mathbb{R}^3)$, compactly supported
- $\epsilon_0, \mu_0 > 0$
- $\epsilon, \mu \in L^\infty(\mathbb{R}^3, SPD)$ such that
  $\Omega_i := \mathrm{int}\big(\mathrm{supp}(\epsilon - \epsilon_0 \underline{\underline{\mathbf{I}}}) \cup \mathrm{supp}(\mu - \mu_0 \underline{\underline{\mathbf{I}}})\big)$ is bounded and Lipschitz

Find $\mathbf{E}, \mathbf{H} \in H_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$ such that

$$\mathrm{i}k\epsilon\mathbf{E} + \nabla \times \mathbf{H} = \mathbf{J} \qquad \text{in } \mathbb{R}^3,$$

$$-\mathrm{i}k\mu\mathbf{H} + \nabla \times \mathbf{E} = \mathbf{K} \qquad \text{in } \mathbb{R}^3,$$

$(\mathbf{E}, \mathbf{H})$ satisfy Silver–Müller radiation condition.



$\epsilon = \epsilon_0$
$\mu = \mu_0$

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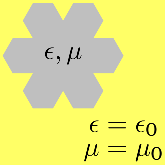

The Morawetz multipliers for this problem are

$$(\epsilon\overline{\mathbf{E}} \times \mathbf{x} + R\sqrt{\epsilon\mu}\,\overline{\mathbf{H}}) \quad \& \quad (\mu\overline{\mathbf{H}} \times \mathbf{x} - R\sqrt{\epsilon\mu}\,\overline{\mathbf{E}}) \qquad \text{in } B_R \supset \Omega_i,$$

$$(\epsilon_0\overline{\mathbf{E}} \times \mathbf{x} + r\sqrt{\epsilon_0\mu_0}\,\overline{\mathbf{H}}) \quad \& \quad (\mu_0\overline{\mathbf{H}} \times \mathbf{x} - r\sqrt{\epsilon_0\mu_0}\,\overline{\mathbf{E}}) \qquad \text{in } \mathbb{R}^3 \setminus B_R.$$

Single homogeneous scatterer

The analogous of the Helmholtz problem seen earlier is

$$\epsilon = \begin{cases} \epsilon_i & \text{in } \Omega_i \\ \epsilon_0 & \text{in } \Omega_o \end{cases}, \quad \mu = \begin{cases} \mu_i & \text{in } \Omega_i \\ \mu_0 & \text{in } \Omega_o \end{cases} \quad 0 < \epsilon_i, \epsilon_0, \mu_i, \mu_0 \text{ constant.}$$

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If $\boxed{\epsilon_i \le \epsilon_0}$, $\boxed{\mu_i \le \mu_0}$, $\boxed{\Omega_i \text{ star-shaped}}$, $\Omega_i \cup \operatorname{supp} \mathbf{J} \cup \operatorname{supp} \mathbf{K} \subset B_R$, then

$$\epsilon_i \|\mathbf{E}\|^2_{B_R} + \mu_i \|\mathbf{H}\|^2_{B_R} \quad \le \quad 4R^2 \left(\frac{\epsilon_0}{\epsilon_i} + \frac{\mu_0}{\mu_i} \right) \left(\epsilon_0 \|\mathbf{K}\|^2_{B_R} + \mu_0 \|\mathbf{J}\|^2_{B_R} \right).$$

Equivalent to wavenumber-independent $H(\mathrm{curl}; B_R)$ bound for $\mathbf{E}$.

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Equivalent to wavenumber-independent $H(\mathbf{curl}; B_R)$ bound for $\mathbf{E}$.

- If ϵ_i is (constant) SPD matrix, same holds if $\max \mathit{eig}(\epsilon_i) \le \epsilon_0$ and with ϵ_i substituted by $\min \mathit{eig}(\epsilon_i)$ in the bound. Same for μ_i.
- Similar results when $\mathbb{R}^3$ is truncated with impedance BCs.

What about more general ϵ, μ?

Assume:

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- $\epsilon_* := \operatorname{ess\,inf}_{\mathbf{x}\in\Omega_i} \big(\epsilon + (\mathbf{x}\cdot\nabla)\epsilon\big) > 0$, $\mu_* := \operatorname{ess\,inf}_{\mathbf{x}\in\Omega_i} \big(\mu + (\mathbf{x}\cdot\nabla)\mu\big) > 0$
  "weak monotonicity" in radial direction, avoid trapping of rays

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  "weak monotonicity" in radial direction, avoid trapping of rays
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Then we have explicit, wavenumber-indep., bound:

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To get rid of "extra regularity" assumption, need density of $C^\infty(\overline{D})^3$ in

$$\Big\{ \mathbf{v} \in H(\mathrm{curl}; D) : \nabla\cdot[\alpha\mathbf{v}] \in L^2(D), \alpha\mathbf{v}\cdot\hat{\mathbf{n}} \in L^2(\partial D), \mathbf{v}_T \in L^2_T(\partial D) \Big\}, \alpha \in \{\epsilon, \mu\}$$

For $\epsilon = \mu =$identity: density proved in COSTABEL, DAUGE 1998.

Summary

Helmholtz equation in $\mathbb{R}^d$, homogeneous inclusion:

- $n_i < 1$: explicit bounds on $\|u\|_{H^1(B_R)}$ from Morawetz multipliers,
  resolvent bounded uniformly in k, holomorphicity strip
- $n_i > 1$: exponential growth of stability constant through $(k_j)_{j=1}^{\infty}$,
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Maxwell equations in $\mathbb{R}^3$, inhomogeneous inclusion:

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Thank you!