Contemporary Challenges in Trefftz Methods, from Theory to Applications
Casa Matemática Oaxaca — 12–17.5.2024

Stable approximation of Helmholtz solutions by evanescent plane waves

Andrea Moiola

https://euler.unipv.it/moiola/

Università di Pavia
Department of Mathematics
"Felice Casorati"

Nicola Galante (INRIA) — Daan Huybrechs (KU Leuven) — Emile Parolin (INRIA)

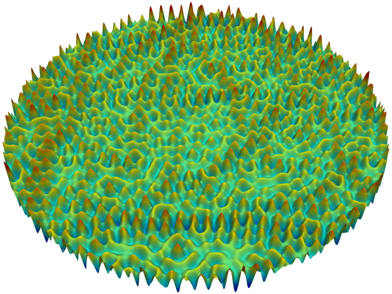
2D: Parolin, Moiola, Huybrechs, M2AN, 2023
3D: Galante, Moiola, Parolin, arXiv:2401.04016, 2024

Helmholtz equation

Homogeneous Helmholtz equation:

$$\Delta u + \kappa^2 u = 0$$

Wavenumber $\kappa = \omega/c > 0$,
$\lambda = \frac{2\pi}{\kappa}$ = wavelength.

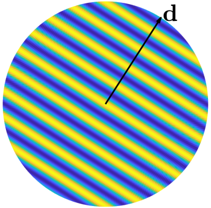


$u(\mathbf{x})$ represents the space dependence of time-harmonic solutions
$U(\mathbf{x}, t) = \Re\{e^{-i\omega t} u(\mathbf{x})\}$ of the wave equation $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

Fundamental PDE in acoustics, electromagnetism, elasticity. . .

- "Easy" PDE for small κ: perturbation of Laplace eq.
- "Difficult" PDE for large κ: high-frequency problems

Propagative plane waves



A difficulty for $\kappa \gg 1$ is the approximation of Helmholtz solutions

One can beat (piecewise) polynomial approximations
using propagative plane waves (PPWs):

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{R}^n \quad \mathbf{d}\cdot\mathbf{d} = 1$$

Some uses of PPWs:

- Trefftz methods: UWVF, TDG, PWDG, DEM, VTCR, WBM, LS, PUM...
- reconstruction of sound fields from point measurements (microphones) in experimental acoustics

PPWs are complex exponentials:
easy & cheap to manipulate, evaluate, differentiate, integrate...
$\rightarrow$ preferred against other Trefftz functions (e.g. circular waves)

Rich PPW approximation theory for Helmholtz solutions:

- Cessenat, Després 1998, Taylor-based, h
- Melenk 1995, Moiola, Hiptmair, Perugia 2011, Vekua theory, hp, κ-explicit

Better rates vs DOFs than polynomials

A negative result

Take $\Omega = B_1 \subset \mathbb{R}^n$ the unit disc/ball, $n \in \{2,3\}$. Choose your favourite

- wavenumber $\kappa > 0$
- norm on Ω (e.g. $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\frac{\|\cdot\|_{H^1(\Omega)}}{\|\mathrm{PW}\|_{H^1(\Omega)}}$)
- target relative accuracy $0 < \delta < 1$ (e.g. 0.5, 1% or 10^{-10})
- finite PPW set $\mathrm{e}^{\mathrm{i}k\mathbf{d}_1\cdot\mathbf{x}} \ldots \mathrm{e}^{\mathrm{i}k\mathbf{d}_P\cdot\mathbf{x}}$ (e.g. equispaced $\mathbf{d}_j$)
- large number M (e.g. 10^{20})

Then we can give you an explicit u such that:

$$u \in C^\infty(\mathbb{R}^n), \qquad \Delta u + \kappa^2 u = 0, \qquad \|u\|_{\text{your favourite}} = 1$$

$$\forall \boldsymbol{\mu} \in \mathbb{C}^P \quad \text{with} \quad \left\| u - \sum_{p=1}^{P} \mu_p \mathrm{e}^{\mathrm{i}\kappa\mathbf{d}_p\cdot\mathbf{x}} \right\|_{\text{your favourite}} \le \delta \qquad \Longrightarrow \qquad \|\boldsymbol{\mu}\|_{\mathbb{C}^P} \ge M$$

Every PPW combination with accuracy δ has huge coefficient vector!
If $M >$ (machine precision)$^{-1}$, we can't represent u in computer arithmetic with PPWs.
Accuracy and stability (bounded coefficients) are mutually exclusive.

Instability

The absence of good approximations to some Helmholtz solutions with coefficient norm proportional to $\|u\|$ is "instability".

This is the source of all notorious troubles with PPW-based Trefftz methods:
ill-conditioning, convergence stagnation, cancellation, high sensitivity to parameters...

Existence of small-coefficient approximations
is a necessary condition for stable floating-point computations.

It is also sufficient, according to

ADCOCK, HUYBRECHS, "Frames and numerical approximation I & II", 2019 & 2020

Goal: Approximate some $v \in V$ with linear combination of $\{\phi_p\} \subset V$.

Result: If there exists $\sum_p \mu_p \phi_p$ with

- good approximation of v, ← OK for PPW
- small coefficients μ_p, ← False for PPW

then the approximation of v in computer arithmetic is stable,
if one uses oversampling and SVD regularization.

Stability does not depend on (LS, Galerkin,...) matrix conditioning.

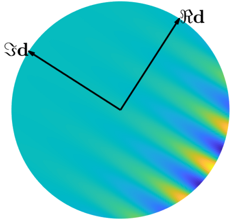
Part I

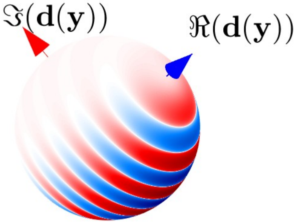
Evanescent plane waves

Evanescent plane waves

Evanescent plane waves (EPW): $\quad e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d}\in\mathbb{C}^n \quad \mathbf{d}\cdot\mathbf{d} = d_1^2+\cdots+d_n^2 = 1$

- Complex $\mathbf{d}$!
- Idea from WBM (wave-based method) by Wim Desmet etc (Leuven)
- Helmholtz solutions
- Complex exponentials: cheap computations, exact quadrature...
- $e^{i\kappa\mathbf{d}\cdot\mathbf{x}} = e^{i\kappa\Re\mathbf{d}\cdot\mathbf{x}}\, e^{-\kappa\Im\mathbf{d}\cdot\mathbf{x}}$, $\qquad \mathbf{d}\cdot\mathbf{d}=1 \Rightarrow \Re\mathbf{d}\cdot\Im\mathbf{d}=0$
  $\Re\mathbf{d}$: propagation direction, $\kappa|\Re\mathbf{d}| \geq \kappa$
  $\Im\mathbf{d}$: evanescence direction
- $|e^{i\kappa\mathbf{d}\cdot\mathbf{x}}| = e^{-\kappa\Im\mathbf{d}\cdot\mathbf{x}}$ essentially localised, need normalisation, easy e.g. in L^∞

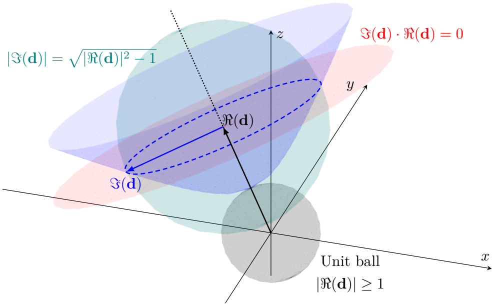




Evanescent plane waves: parametrisation

$e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$ $\mathbf{d} \in \mathbb{C}^n$

$$\mathbf{d}\cdot\mathbf{d} = 1 \iff \begin{cases} |\Re\mathbf{d}|^2 - |\Im\mathbf{d}|^2 = 1 \\ \Re\mathbf{d}\cdot\Im\mathbf{d} = 0 \end{cases}$$



$\mathbf{d}$ parametrised by:

- $\mathbf{p} = \frac{\Re\mathbf{d}}{|\Re\mathbf{d}|} \in \mathbb{S}^{n-1}$: propagation direction
- $\mathbf{e} \in \mathbb{S}^{n-2}$: evanescence direction in the hyperplane $\perp \mathbf{p}$
- $\eta = |\Im\mathbf{d}| \in [0,\infty)$: evanescence strength

$\eta = 0 \iff$ EPW is PPW $\qquad |\Re\mathbf{d}| = \sqrt{1+\eta^2}$

Parameter vector $\mathbf{y} := (\mathbf{p}, \mathbf{e}, \eta) \in Y := \mathbb{S}^{n-1} \times \mathbb{S}^{n-2} \times [0,\infty)$, $\quad \mathrm{EW}_{\mathbf{y}}(\mathbf{x}) := e^{i\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}}$

In 2D: $\mathbf{p} \in \mathbb{S}^1 \sim \theta \in [0, 2\pi)$, $\mathbf{e} = \pm 1$

In 3D: use Euler angles of rotation from reference direction $\mathbf{d}_\uparrow = (i\eta,\ 0,\ \sqrt{1+\eta^2}) \to \mathbf{d}$

Herglotz functions & EPW Herglotz representation

Herglotz functions are continuous superposition of PPWs: (COLTON, KRESS...)

$$u(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} v(\mathbf{d})\ e^{i\kappa\mathbf{d}\cdot\mathbf{x}}\ \mathrm{d}\mathbf{d} \qquad \text{for} \quad v \in L^2(\mathbb{S}^{n-1})$$

Only some Helmholtz solutions $u \in C^\infty(\mathbb{R}^n)$ are Herglotz:
$L^2(\mathbb{S}^{n-1}) \ni v \mapsto u$ has dense image (WECK 2004) but is not surjective.

Idea: Define the EPW version of Herglotz functions:

$$u(\mathbf{x}) = (Tv)(\mathbf{x}) := \int_Y v(\mathbf{y})\ \underbrace{e^{i\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}}}_{\mathrm{EW}_{\mathbf{y}}(\mathbf{x})}\ w^2(\mathbf{y})\ \mathrm{d}\mathbf{y} \qquad \text{for} \quad v \in L^2_{w^2}(Y)$$

Weight $w > 0$ is a normalisation, needed since $Y = \mathbb{S}^{n-1} \times \mathbb{S}^{n-2} \times [0,\infty)$ is unbounded in η.

Goal:
For every Helmholtz solution $u \in H^1(\Omega)$ we want v with $Tv = u$ and $\|v\|_{L^2_{w^2}(Y)} \sim \|u\|_{H^1(\Omega)}$.

Herglotz representation on the disc and the sphere

Theorem: Helmholtz solutions on B_1 are EPW superposition

For Ω the disc/ball B_1, $T : \mathcal{A} \subset L^2_{w^2}(Y) \to \mathcal{B} := \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\}$ is invertible.

In particular, for all Helmholtz solutions $u \in H^1(B_1)$, there is a density $v = T^{-1}u$ such that

$$u(\mathbf{x}) = \int_Y v(\mathbf{y})\ \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\ w^2(\mathbf{y})\ \mathrm{d}\mathbf{y}, \qquad \|v\|_{L^2_{w^2}(Y)} \le C\|u\|_{H^1(B_1)}$$

Need appropriate weight $w(\mathbf{y}) = \mathrm{e}^{-\kappa\eta}\eta^{\frac{2n-5}{4}}$

Key tool: expansion of EPWs in circular/spherical wave basis, extending Jacobi–Anger

EPWs are a continuous frame for the Helmholtz solution space $\mathcal{B}$. $T =$ synthesis operator

Numerical recipes from discretisation of integral representation

Conjecture: the same theorem holds for all convex Ω (with the right w)

Circular & spherical waves

Separable Helmholtz solutions in polar and spherical coordinates:

$$2D: \qquad b_\ell(\mathbf{x}) = \beta_\ell\, J_\ell(\kappa r)\, e^{i\ell\vartheta} \qquad \ell \in \mathbb{Z}, \qquad \mathbf{x} = (r, \vartheta) \in B_1$$

$$3D: \qquad b_\ell^m(\mathbf{x}) = \beta_\ell\, j_\ell(\kappa|\mathbf{x}|)\, Y_\ell^m(\mathbf{x}/|\mathbf{x}|) \qquad \ell, m \in \mathbb{Z},\ |m| \le \ell, \qquad \mathbf{x} \in B_1$$

β_ℓ = normalisation in $H^1_\kappa(B_1)$ norm

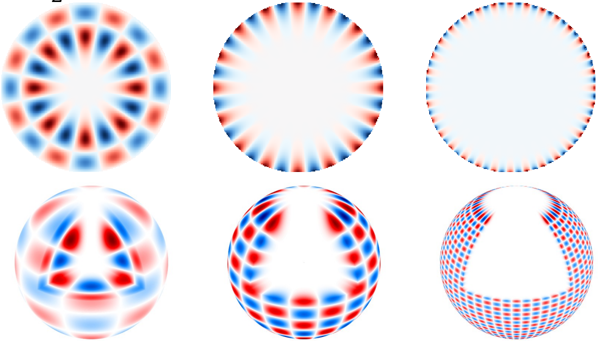
Orthonormal basis of $\mathcal{B} = \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\}$

$\beta_\ell \sim \kappa(\frac{2}{e\kappa})^{|\ell|}|\ell|^{|\ell|+\frac{n-2}{2}}$ for $|\ell| \to \infty$

$\ell = \frac{\kappa}{2}$ "bulk" $\qquad \ell = \kappa \qquad \ell > \kappa$ "evanescent"

2D

3D, $m = \frac{\ell}{2}$



b_ℓ and b_ℓ^m are Herglotz functions with density

$v(\theta) = \beta_\ell \frac{e^{i\ell\theta}}{2\pi i^\ell}$,

$v(\mathbf{d}) = \beta_\ell \frac{Y_\ell^m(\mathbf{d})}{4\pi i^\ell}$:

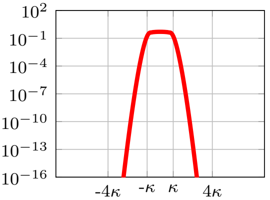
$\|v\|_{L^2(\mathbb{S}^{n-1})} \sim |\ell|^{|\ell|}$

Expansion of PPW in Fourier modes

Jacobi–Anger expansion: $\mathbf{d} \in \mathbb{R}^n, \mathbf{d} \cdot \mathbf{d} = 1$

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = \begin{cases} \sum_{\ell\in\mathbb{Z}} \left(i^\ell e^{-i\ell\theta_{\mathbf{d}}} \beta_\ell^{-1} \right) b_\ell(\mathbf{x}) \\ 4\pi \sum_{\ell=0}^{\infty} i^\ell \beta_\ell^{-1} \sum_{m=-\ell}^{\ell} \overline{Y_\ell^m(\mathbf{d})}\, b_\ell^m(\mathbf{x}) \end{cases}$$

$\mathbf{d} = (\cos\theta_{\mathbf{d}}, \sin\theta_{\mathbf{d}})$



Mode number ℓ (2D)

The modulus of Fourier coefficient decays $\sim \beta_\ell^{-1} \sim |\ell|^{-|\ell|}$

In 2D: $|i^\ell e^{-i\ell\theta_{\mathbf{d}}} \beta_\ell^{-1}| = |\beta_\ell^{-1}| \sim |\ell|^{-|\ell|}$ indep. of $\theta_{\mathbf{d}}$

$\Rightarrow$ the approximation of $u = \sum_\ell \widehat{u}_\ell b_\ell \in \mathcal{B}$
with $\widehat{u}_\ell \neq 0$ for some $|\ell| \gg \kappa$
requires exponentially large coefficients

$$\begin{matrix} \forall \ell \in \mathbb{Z} \quad (|m| \le \ell) \\ \forall P \in \mathbb{N} \\ \forall \mathbf{d}_1 \ldots, \mathbf{d}_P \in \mathbb{S}^{n-1} \\ \forall \boldsymbol{\mu} \in \mathbb{C}^P \\ \forall \delta \in (0,1) \end{matrix} \qquad \left\| b_\ell^{(m)}(\mathbf{x}) - \sum_{p=1}^{P} \mu_p e^{i\kappa \mathbf{d}_p\cdot\mathbf{x}} \right\|_{H^1(B_1)} \le \delta \quad \Longrightarrow \quad \|\boldsymbol{\mu}\|_{l^1(\mathbb{C}^P)} \ge (1-\delta) \underbrace{|\beta_\ell|}_{\sim |\ell|^{|\ell|}}$$

Complex-direction Jacobi–Anger & EPW Fourier expansion

Now we expand EPWs in Fourier modes.

Generalised Jacobi–Anger expansion:

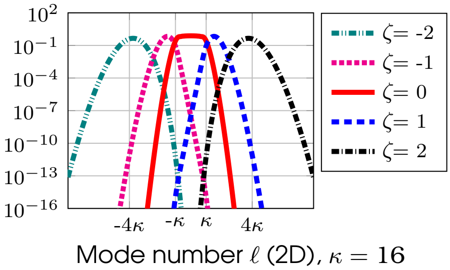
$$e^{i\kappa \mathbf{d}(\mathbf{y})\cdot\mathbf{x}} = \begin{cases} \displaystyle\sum_{\ell\in\mathbb{Z}} \left(i^\ell e^{-i\ell\theta} (\eta + \sqrt{\eta^2+1})^{\pm\ell} \beta_\ell^{-1} \right) b_\ell(\mathbf{x}) & \mathbf{y} = (\theta, \pm, \eta) \in [0, 2\pi) \times \{\pm 1\} \times [0, \infty) \\ \displaystyle 4\pi \sum_{\ell=0}^{\infty} i^\ell \sum_{m=-\ell}^{\ell} \left[\sum_{m'=-\ell}^{\ell} \overline{D_\ell^{m',m}(\boldsymbol{\theta}, \psi)} \gamma_\ell^{m'} i^{-m'} P_\ell^{m'}(\sqrt{\eta^2+1}) \right] \beta_\ell^{-1} b_\ell^m(\mathbf{x}) & \mathbf{y} = (\boldsymbol{\theta}, \psi, \eta) \end{cases}$$

$D_\ell^{m',m}$ = Wigner matrix entry (spherical harmonic rotation)

P_ℓ^m = associated Legendre function (evaluated out of $[-1, 1]$)

$\gamma_\ell^m = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}$

$\boldsymbol{\theta}, \psi$ = Euler angles



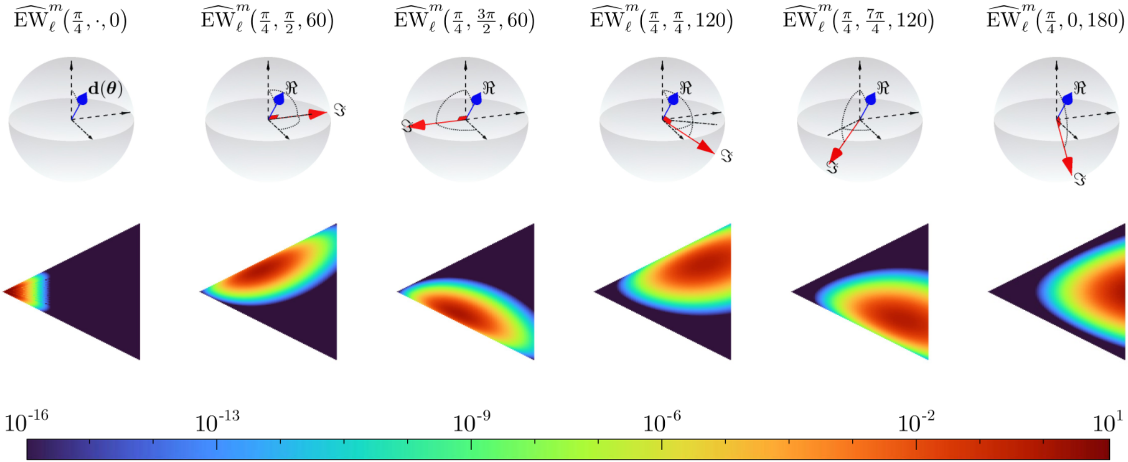
◀ Absolute values of Fourier coefficients (2D)

$(\eta + \sqrt{\eta^2+1})^{\pm\ell}\beta_\ell^{-1} = e^{\ell\zeta}\beta_\ell^{-1}$ $\qquad \zeta = \pm \operatorname{arcsinh} \eta$

Looks promising!

We can hope to approximate large-ℓ Fourier modes with EPWs & small coefficients.

3D EPW modal expansion



Absolute value of Fourier coefficients, plotted against (ℓ, m), $0 \le |m| \le \ell \le 80$

In brackets: 2 Euler angles, $2\kappa(\sqrt{\eta^2+1}-1) \sim 2\kappa\eta$

Invertibility of EPW Herglotz representation

We want to use the EPW Fourier expansion to prove invertibility of

$$\begin{aligned} T: \quad \mathcal{A} \subset L^2_{w^2(Y)} &\to \mathcal{B} := \{u \in H^1(B_1),\ \Delta u + \kappa^2 u = 0\} \\ v &\mapsto u(\mathbf{x}) = \int_Y v(\mathbf{y})\, \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\, w^2(\mathbf{y})\, \mathrm{d}\mathbf{y} \end{aligned}$$

Consider 2D case. $\mathbf{y} = (\theta, \pm, \eta) \in [0, 2\pi) \times \{\pm 1\} \times [0, \infty) = Y$

$w(\mathbf{y}) = \mathrm{e}^{-\kappa\eta}\eta^{-\frac{1}{4}}$, $a_\ell(\mathbf{y}) := \alpha_\ell(\eta + \sqrt{\eta^2+1})^{\pm\ell}\mathrm{e}^{\mathrm{i}\ell\theta} \in L^2_{w^2}(Y)$, α_ℓ = $L^2_{w^2}(Y)$-normalisation

$\{a_\ell, \ell \in \mathbb{Z}\}$ is orthonormal basis of $\mathcal{A} := \mathrm{span}\{a_\ell, \ell \in \mathbb{Z}\} \subsetneq L^2_{w^2}(Y)$

Jacobi Anger: $\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{\ell \in \mathbb{Z}} \tau_\ell \overline{a_\ell(\mathbf{y})} b_\ell(\mathbf{x})$ $\forall \mathbf{x} \in B_1$, $\forall \mathbf{y} \in Y$, $\tau_\ell = \dfrac{\mathrm{i}^\ell}{\alpha_\ell \beta_\ell}$, $0 < \tau_- \le |\tau_\ell| \le \tau_+ < \infty \quad \forall \ell$

The operator $T : \mathcal{A} \to \mathcal{B}$ is diagonal in ONB $\{a_\ell\}, \{b_\ell\}$, bounded and invertible:

$$T : a_\ell \mapsto \sum_{\ell'} \tau_{\ell'} b_{\ell'} \int_Y a_\ell \overline{a_{\ell'}} w^2 = \tau_\ell b_\ell, \qquad \tau_- \|v\|_{\mathcal{A}} \le \|Tv\|_{\mathcal{B}} \le \tau_+ \|v\|_{\mathcal{A}} \qquad \forall v \in \mathcal{A}$$

Every Helmholtz solution is EPW superposition with small coefficients: $\|v\|_{\mathcal{A}} \le \tau_-^{-1} \|u\|_{\mathcal{B}}$

Parameter sampling in Y

How to choose points $\{\mathbf{y}_p\}_p \in Y$ and discrete EPW set $\{e^{i\kappa \mathbf{d}(\mathbf{y}_p)\cdot \mathbf{x}}\}_p$?
Construct quadrature rule, using technique from COHEN, MIGLIORATI, 2017.

Fix Fourier truncation $L \in \mathbb{N}$, probability density $\rho(\mathbf{y}) := \frac{w^2}{2L+1} \sum_{|\ell|\le L} |a_\ell(\mathbf{y})|^2$ on Y

and generate $P \in \mathbb{N}$ nodes $\{\mathbf{y}_p\}_{p=1,\dots,P} \subset Y$ distributed according to ρ.

From Cohen–Migliorati, expect that any $u \in \operatorname{span}\{b_\ell\}_{|\ell|\le L}$ can be approximated by EPWs

$$\left\{ \mathbf{x} \mapsto \frac{1}{\sqrt{P\sum_{|\ell|\le L} |a_\ell(\mathbf{y}_p)|^2}} \mathrm{EW}_{\mathbf{y}_p}(\mathbf{x}) \right\}_{p=1,\dots,P} \subset \mathcal{B}$$

with small coefficients.

$\to$ Stable approximation in computer arithmetic using SVD & oversampling.
Confirmed by numerics!

The P-dimensional EPW space depends on truncation parameter L:
the space is tuned to approximate the Fourier modes b_ℓ with $|\ell| \le L$.

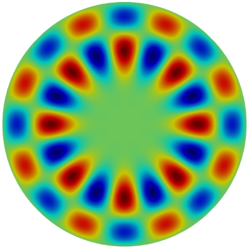
Part II

Numerical results

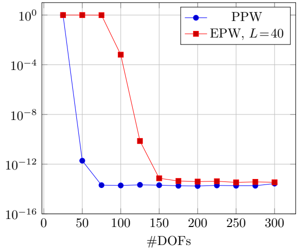
Approximation by PPWs and by EPWs — 2D

$\kappa = 16$, SVD truncation parameter $\epsilon = 10^{-14}$, # boundary nodes $S = \max\{2P, 2|\ell|\}$

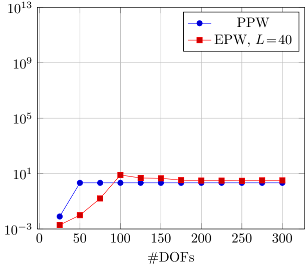
$\ell = 8$



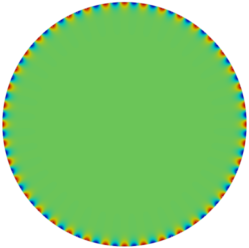
b_ℓ, $\ell = 8$, residual $\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|/\|\mathbf{c}\|$



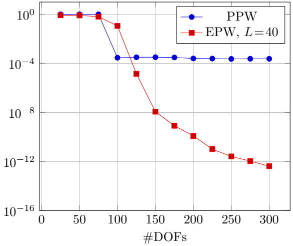
b_ℓ, $\ell = 8$, coefficient norm $\|\boldsymbol{\xi}_\epsilon\|$



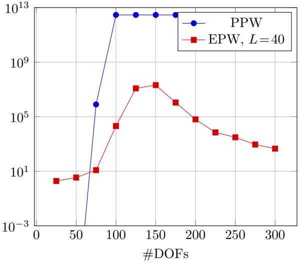
$\ell = 40$



b_ℓ, $\ell = 40$, residual $\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|/\|\mathbf{c}\|$



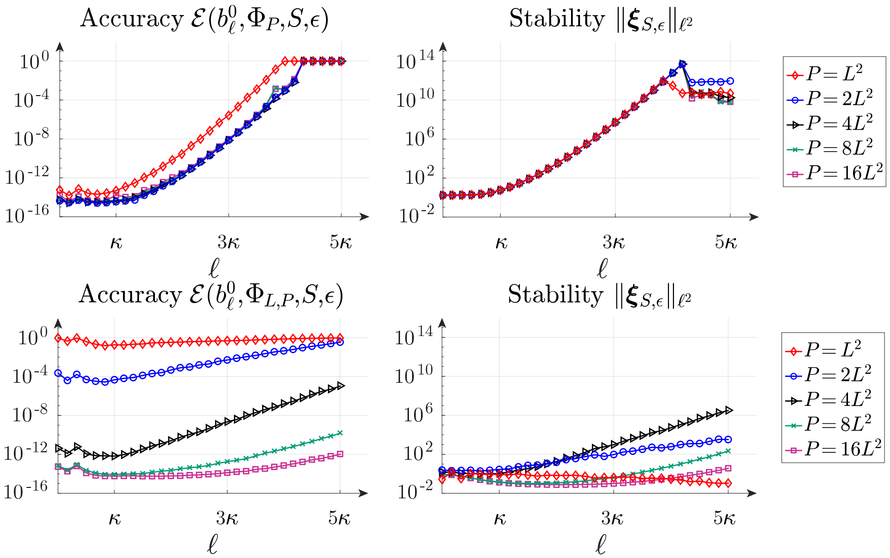
b_ℓ, $\ell = 40$, coefficient norm $\|\boldsymbol{\xi}_\epsilon\|$



Approximation of spherical waves b_ℓ^0 by PPWs and EPWs — 3D

PPWs

EPWs



$L^2 = 576$

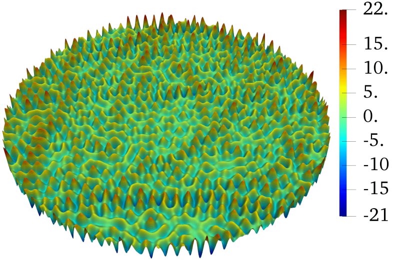
$\kappa = 6$, Fourier truncation $L = 4\kappa$, SVD truncation $\epsilon = 10^{-14}$, oversampling $S \approx 2P$

Increasing #DOFs does not improve PPW error. Condition number is irrelevant.

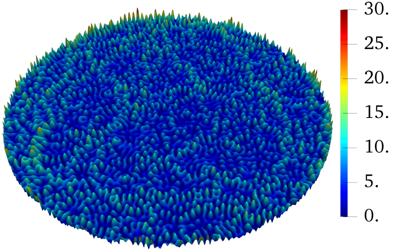
EPW approximation of random circular wave combination

$u = \sum_{|\ell|\leq L} \hat{u}_\ell b_\ell, \qquad \widehat{u}_\ell \sim (\max\{1, |\ell| - \kappa\})^{-1/2}, \qquad \kappa = 100, \qquad L = 2\kappa, \qquad \#DOFs = P = 802$

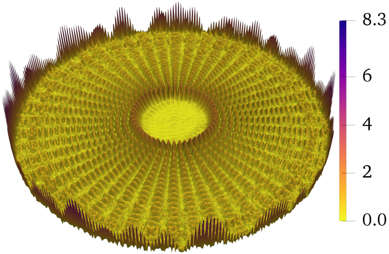
$\Re\{u\}$



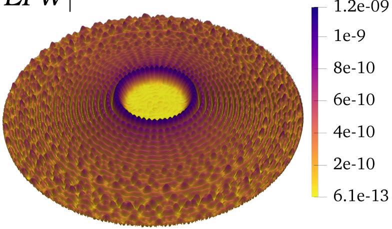
$|u|$



$|u - PPW|$



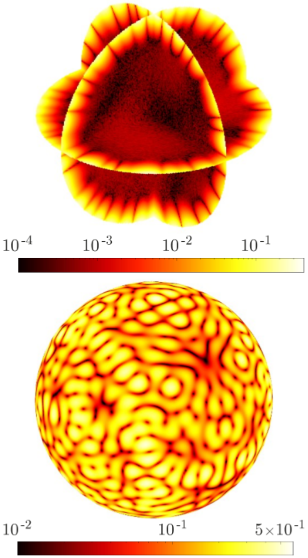
$|u - EPW|$



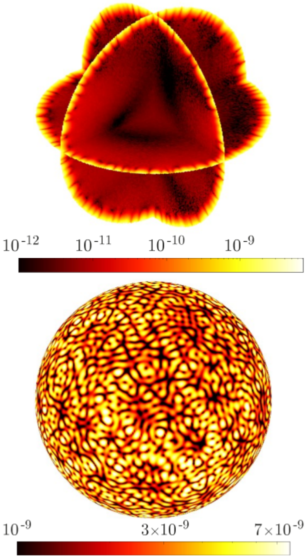
$\|u - PPW\|_{L^\infty} \gtrsim 7 \cdot 10^9 \, \|u - EPW\|_{L^\infty}$

#DOFs/wavelength $= \lambda\sqrt{P/|B_1|} \approx 1$

EPW approximation of random spherical wave combination



▲ PPW error



▲ EPW error

$\kappa = 5$

$L = 25$

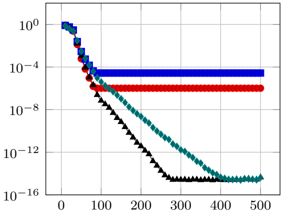
#random params = $\dim \mathcal{A}_L = 676$

$\#DOFs = P = 2704$

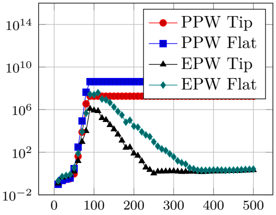
Polygonal domain: discrete space for circumscribed circle

$\kappa = 16$, $\#DOFs = P = 200$, $u =$ fundamental solution at distance 0.25, L^∞ normalisation

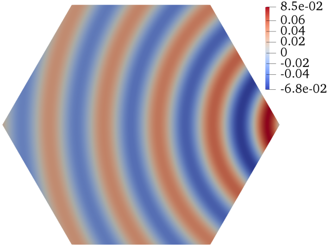
Accuracy $\mathcal{E}$



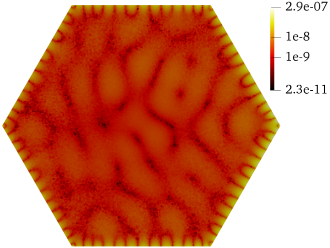
Stability $\|\boldsymbol{\xi}_{S,\epsilon}\|_{\ell^2}$



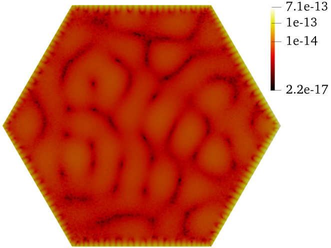
$\Re\{u\}$



$|u - PPW|$



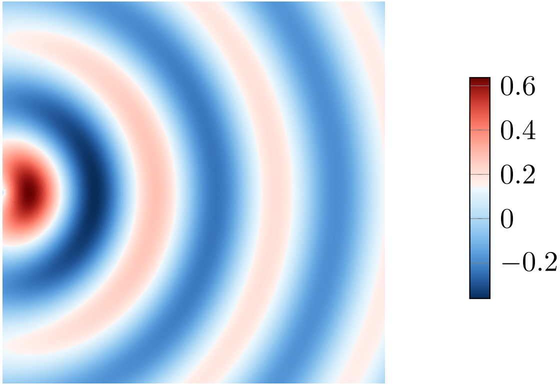
$|u - EPW|$

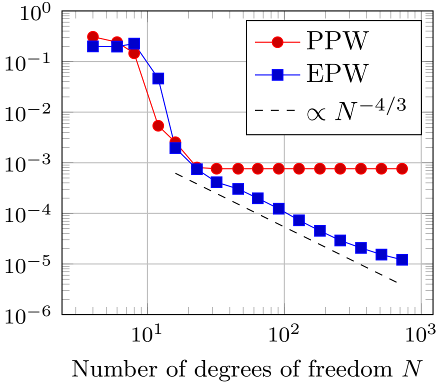


Trefftz discontinuous Galerkin

Corner singularity: $u(\mathbf{x}) = J_\nu(kr)\mathrm{e}^{\mathrm{i}\nu\vartheta} \in H^{1+\nu-\epsilon}((0,1)\times(-\frac{1}{2},\frac{1}{2}))$, $\nu = \frac{2}{3}$, $\kappa = 10$

TDG on 8-triangle mesh





PPW error stalls at 10^{-3} while EPW error keeps decreasing.

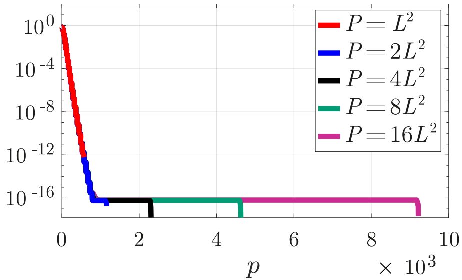
We observe that EPWs gives better results than PPWs also for smooth $J_1(kr)\mathrm{e}^{\mathrm{i}\vartheta} \in C^\infty(\mathbb{R}^2)$.

Collocation matrix singular values — 3D

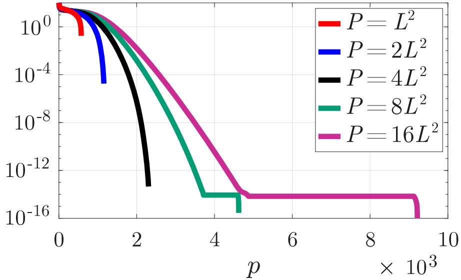
Singular values σ_p of $\mathbf{A} \in \mathbb{C}^{2P\times P}$, $A_{s,p} = \phi_p(\mathbf{x}_s)$ $\kappa = 6, L = 4\kappa$

Collocation nodes: $\mathbf{x}_s$ = Sloan–Womersley extremal points on $\partial B_1 \subset \mathbb{R}^3$

▼ ϕ_p = PPWs



▼ ϕ_p = normalised EPWs



EPWs do not reduce condition number for large P.

Higher ϵ-rank ($\#\{\sigma_p \geq \epsilon\sigma_{\max}\}$) gives larger numerically achievable approximation space.

Summary

- Approximation by PPWs is unstable: accuracy requires large coefficients
- Approximation by evanescent PWs seems to be stable
- Key new result is stable Herglotz transform $u = \int_Y v \,\mathrm{EW}$
- EPWs parameters chosen with sampling in Y
- Ill-conditioning is not the issue: the key is small-coefficient representation

Ongoing:

- General convex geometries
- Proof of discrete EPW stability
- Simpler computational recipes
- Faster linear algebra
- Use in Trefftz-DG
- Presence of evanescent modes in BVPs
- ...

Thank you!

2D: E. Parolin, D. Huybrechs, A. Moiola — M2AN 2023

julia code: https://github.com/EmileParolin/evanescent-plane-wave-approx

3D: N. Galante, A. Moiola, E. Parolin — arXiv:2401.04016

Matlab: https://github.com/Nicola-Galante/evanescent-plane-wave-approximation

See also talks by Nicola and Emile at Waves 2024.