One World Numerical Analysis — 21 February 2022

Stable approximation of Helmholtz solutions by evanescent plane waves

Andrea Moiola

https://euler.unipv.it/moiola/

Università di Pavia
Department of Mathematics
"Felice Casorati"

Joint work with:

Daan Huybrechs (KU Leuven) Emile Parolin (Pavia)

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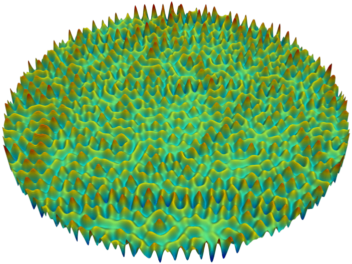
Helmholtz equation

Homogeneous Helmholtz equation:

$$-\Delta u - \kappa^2 u = 0$$



$u(\mathbf{x})$ represents the space dependence of time-harmonic solutions

$$U(\mathbf{x}, t) = \Re\{e^{-i\omega t} u(\mathbf{x})\}$$

of the wave equation $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

Wavenumber $\kappa = \omega/c > 0$,
$\lambda = \frac{2\pi}{\kappa}$ = wavelength.

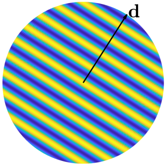
Fundamental PDE in acoustics, electromagnetism, elasticity...

- "Easy" PDE for small κ: perturbation of Laplace eq.
- "Difficult" PDE for large κ: high-frequency problems

Propagative plane waves

A difficulty for $\kappa \gg 1$ is the approximation of Helmholtz solutions.

One can beat (piecewise) polynomial approximations using propagative plane waves (PPWs):



$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{R}^n \quad \mathbf{d}\cdot\mathbf{d} = 1$$

Some uses of PPWs:

- Trefftz methods:
  Galerkin schemes whose basis functions are local PDE solutions.
  E.g.: UWVF, TDG, PWDG, DEM, VTCR, WBM, LS, PUM ...
- reconstruction of sound fields from point measurements (microphones) in experimental acoustics.

PPWs are complex exponentials:
easy & cheap to manipulate, evaluate, differentiate, integrate...
$\rightarrow$ preferred against other Trefftz functions (e.g. circular waves)

Approximation and instability

Rich PPW approximation theory for Helmholtz solutions:

- CESSENAT, DESPRÉS 1998, Taylor-based, h
- MELENK 1995; MOIOLA, HIPTMAIR, PERUGIA 2011, Vekua theory, hp

κ-explicit, better rates vs DOFs than polynomials.

So why isn't everybody using plane waves?

The issue is "instability".
Increasing # of PPWs, at some point convergence stagnates.

Numerical phenomenon: due to computer arithmetic.

PPW instability

Instability already observed in all PPW-based Trefftz methods.

PPW instability usually described as matrix ill-conditioning.

Several solutions have been proposed, e.g.

- HUTTUNEN, GAMALLO, ASTLEY 2009: limit on PPW#
- ANTUNES 2018: change of basis
- CONGREVE, GEDICKE, PERUGIA 2019: basis orthogonalization
- HUYBRECHS, OLTEANU 2019: SVD regularization
- BARUCQ, BENDALI, DIAZ, TORDEAUX 2021: local SVD/QR + precond.
- ...

Adcock–Huybrechs theory

BEN ADCOCK, DAAN HUYBRECHS, SiRev 2019 & JFAA 2020,
"Frames and numerical approximation I & II"

Goal: Approximate some $v \in V$ with linear combination of $\{\phi_m\} \subset V$.

Result: If there exists $\sum_m a_m \phi_m$ with

- good approximation of v, ← OK for PPW
- small coefficients a_m, ← Is it true for PPW?

then the approximation of v in computer arithmetic is stable,
if one uses oversampling and SVD regularization.

Stability does not depend on (LS, Galerkin,...) matrix conditioning.

Importance of small coefficients for Trefftz & Helmholtz (no PPWs) already understood by BARNETT, BETCKE 2008.

Spoiler!

In this talk:

- Show that PPWs can not approximate general Helmholtz solution u with small coefficients.
- Modify PPW space $\rightarrow$ small-coefficient approximation $\rightarrow$ stability. Key idea: use evanescent plane waves.

Here we consider only the approximation in the unit disk $B_1 \subset \mathbb{R}^2$.

Part I

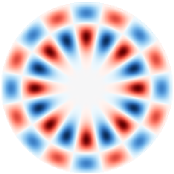
Circular and propagative plane waves

Circular waves — Fourier–Bessel functions

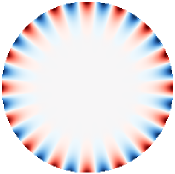
Separable solutions in polar coordinates:

$$b_p(r,\theta) := \beta_p J_p(kr)\mathrm{e}^{\mathrm{i}p\theta} \qquad \forall p \in \mathbb{Z}, \quad (r,\theta) \in B_1$$

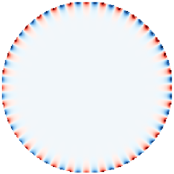
β_p = normalization, e.g. in $H^1(B_1)$ norm. $\qquad \beta_p \sim \kappa\left(\frac{2|p|}{\mathrm{e}\kappa}\right)^{|p|}$ as $p \to \infty$.



$p = 8 = \kappa/2$
Propagative mode



$p = 16 = \kappa$



$p = 32 = 2\kappa$
Evanescent mode

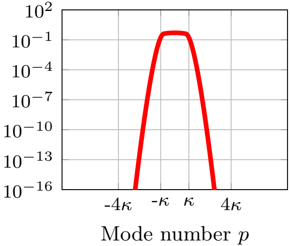
$\{b_p\}_{p\in\mathbb{Z}}$ is orthonormal basis of $\quad \mathcal{B} := \left\{u \in H^1(B_1) : -\Delta u - \kappa^2 u = 0\right\}$

PPW instability

The Jacobi–Anger expansion relates PPWs and circular waves b_p:

$$\mathrm{PW}_\varphi(\mathbf{x}) := e^{i\kappa\mathbf{d}\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} i^p J_p(kr) e^{ip(\theta-\varphi)} = \sum_{p\in\mathbb{Z}} \left(i^p e^{-ip\varphi} \beta_p{}^{-1}\right) b_p(r,\theta)$$

$$\begin{cases} \mathbf{d} = (\cos\varphi,\ \sin\varphi) \\ \mathbf{x} = (r\cos\theta,\ r\sin\theta) \end{cases}$$



Modulus of Fourier coefficient
$|i^p e^{-ip\varphi} \beta_p^{-1}| = |\beta_p^{-1}| \sim |p|^{-|p|}$ indep. of φ.

Approximation of $u = \sum_p \widehat{u}_p b_p \in \mathcal{B}$
requires exponentially large coefficients.

$u \in H^s(B_1), s \geq 1 \iff |\widehat{u}_p| \sim o(|p|^{-s+\frac{1}{2}})$
but $|\beta_p^{-1}| \sim |p|^{-|p|}$ is much smaller!

$$\begin{matrix} \forall p \in \mathbb{Z} \\ \forall M \in \mathbb{N} \\ \forall \boldsymbol{\mu} \in \mathbb{C}^M \\ \forall \eta \in (0,1) \end{matrix} \quad \left\| b_p - \sum_{m=1}^{M} \mu_m \mathrm{PW}_{\frac{2\pi m}{M}} \right\|_{\mathcal{B}} \leq \eta \quad \Longrightarrow \quad \|\boldsymbol{\mu}\|_{\ell^1(\mathbb{C}^M)} \geq (1-\eta) \underbrace{|\beta_p|}_{\sim |p|^{|p|}}$$

Part II

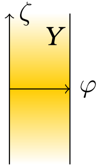
Evanescent plane waves

Evanescent plane waves

Idea from WBM (wave-based method) by Wim Desmet etc (Leuven).

Stability improves using PPWs & evanescent plane waves (EPW):

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d}\in\mathbb{C}^2 \quad \mathbf{d}\cdot\mathbf{d}=1$$

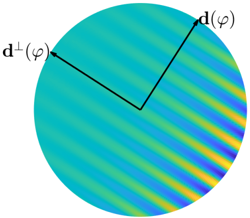


Complex $\mathbf{d}$! Again: exponential Helmholtz solutions.

Parametrised by φ = direction, ζ = "evanescence".
Parametric cylinder: $\mathbf{y} := (\varphi,\zeta) \in Y := [0,2\pi)\times\mathbb{R}$.

$$\mathbf{d}(\mathbf{y}) := \big(\cos(\varphi+i\zeta),\, \sin(\varphi+i\zeta)\big) \in \mathbb{C}^2$$

$$\begin{aligned}\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) &:= e^{i\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}}\\ &= e^{i\kappa(\cosh\zeta)\mathbf{x}\cdot\mathbf{d}(\varphi)}\ e^{-\kappa(\sinh\zeta)\mathbf{x}\cdot\mathbf{d}^{\perp}(\varphi)},\end{aligned}$$



oscillations along $\mathbf{d}(\varphi) := (\cos\varphi,\, \sin\varphi)$

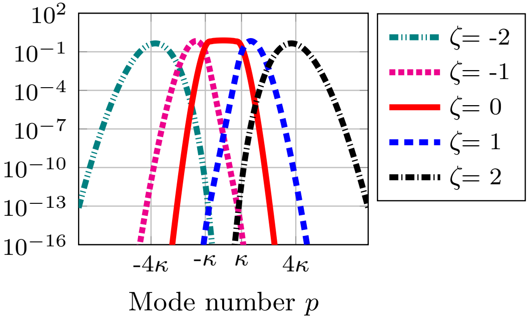
decay along $\mathbf{d}^{\perp}(\varphi) := (-\sin\varphi,\, \cos\varphi)$

EPW modal analysis

Jacobi–Anger expansion holds also for EPWs:

$$e^{i\kappa \mathbf{d}(\mathbf{y})\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} i^p J_p(\kappa r) e^{ip(\theta-[\varphi+i\zeta])} = \sum_{p\in\mathbb{Z}} \left(i^p e^{-ip\varphi} e^{p\zeta} \beta_p^{-1}\right) b_p(\mathbf{x}).$$

Absolute values of Fourier coefficients $|i^p e^{-ip\varphi} e^{p\zeta} \beta_p^{-1}|$, $\kappa = 16$:



Looks promising!

We can hope to approximate large-p Fourier modes with EPWs & small coefficients.

Herglotz representation

How to write general Helmholtz solutions in terms of PWs?

Classical definition from inverse problems:

Helmholtz solutions u that can be written as

$$u(\mathbf{x}) = \int_0^{2\pi} v(\varphi)\mathrm{PW}_\varphi(\mathbf{x})\,\mathrm{d}\varphi \qquad v \in L^2(0, 2\pi)$$

are called "Herglotz functions" with kernel (or density) v.

- **+** Continuous linear combination of PPWs: easily approximated.
- **–** Herglotz functions are a small class (doesn't even include PPWs).

Idea:
Extend Herglotz representation to continuous combinations of EPWs.

Need a weighted L^2 space on cylinder Y.

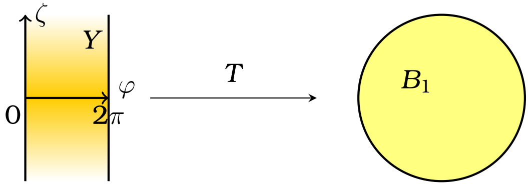
Herglotz representation with EPWs

We want to represent $u \in \mathcal{B}$ as continuous superposition of EPWs:

$$u(\mathbf{x}) = (Tv)(\mathbf{x}) = \int_Y v(\mathbf{y})\,\mathsf{EW}_{\mathbf{y}}(\mathbf{x})\,w^2(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \mathbf{x} \in B_1$$

with density $v \in L^2(Y; w^2)$ and weight w^2.

Parametric space | Physical space



Herglotz density: $v \in \mathcal{A} \subset L^2(Y; w^2)$

Helmholtz solution: $u \in \mathcal{B}$

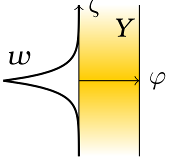
We want T to be invertible: $\|T^{-1}\|$ is a measure of stability.

Weighted $L^2(Y)$ space $\mathcal{A}$

Weighted L^2 space on parametric cylinder & orthonormal basis:

$$w(\mathbf{y}) := \mathrm{e}^{-\kappa \sinh|\zeta| + \frac{1}{4}|\zeta|} \qquad \mathbf{y} = (\varphi, \zeta) \in Y$$

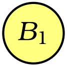
$$\|v\|_{\mathcal{A}}^2 := \|v\|_{L^2(Y;w^2)}^2 = \int_Y |v(\mathbf{y})|^2 w^2(\mathbf{y})\,\mathrm{d}\mathbf{y}$$



$$a_p(\mathbf{y}) := \alpha_p\, \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}$$

$\alpha_p > 0$ normalization in $\|\cdot\|_{\mathcal{A}}$, $p \in \mathbb{Z}$

$$\mathcal{A} := \overline{\mathrm{span}\{a_p\}_{p\in\mathbb{Z}}}^{\|\cdot\|_{\mathcal{A}}} \subsetneq L^2(Y;w^2)$$

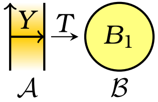
Jacobi–Anger: $\mathbf{x} \in$  $\mathbf{y} \in$ 

$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{p\in\mathbb{Z}} \mathrm{i}^p J_p(\kappa r) \mathrm{e}^{\mathrm{i}p(\theta - [\varphi + \mathrm{i}\zeta])} = \sum_{p\in\mathbb{Z}} \tau_p \overline{a_p(\mathbf{y})}\, b_p(\mathbf{x}), \qquad \tau_p := \frac{\mathrm{i}^p}{\alpha_p \beta_p}.$$

From asymptotics & choice of w: $\quad 0 < \tau_- \leq |\tau_p| \leq \tau_+ < \infty \quad \forall p \in \mathbb{Z}.$

Herglotz transform T

Define Herglotz transform: (synthesis operator)



$$(Tv)(\mathbf{x}) := \int_Y \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\ v(\mathbf{y})\ w^2(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \begin{matrix} T : \mathcal{A} \to \mathcal{B} \\ v \mapsto u \end{matrix}$$

Jacobi–Anger $\Rightarrow$ T is diagonal in ONB's $\{a_p\}, \{b_p\}$: $\forall v \in \mathcal{A},\ \mathbf{x} \in B_1$

$$\begin{aligned}(Tv)(\mathbf{x}) :&= \int_Y \Big(\sum_{p\in\mathbb{Z}} \tau_p b_p(\mathbf{x})\overline{a_p(\mathbf{y})}\Big) v(\mathbf{y}) w^2(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \sum_{p\in\mathbb{Z}} \tau_p \Big(\int_Y \overline{a_p(\mathbf{y})} v(\mathbf{y}) w^2(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big) b_p(\mathbf{x}) \quad = \sum_{p\in\mathbb{Z}} \tau_p (v,\ a_p)_{\mathcal{A}} b_p(\mathbf{x}).\end{aligned}$$

The operator $T : \mathcal{A} \to \mathcal{B}$ is bounded and invertible:

$$Ta_p = \tau_p b_p, \qquad \tau_- \|v\|_{\mathcal{A}} \le \|Tv\|_{\mathcal{B}} \le \tau_+ \|v\|_{\mathcal{A}} \qquad \forall v \in \mathcal{A}$$

Every Helmholtz solution is (continuous) linear combination of EPW!

Part III

Discrete EPW spaces

Frames, RKHS, sampling

All good at continuous level, but what about finite sums of EPWs?

The evanescent plane waves $\{\mathrm{EW}_{\mathbf{y}}\}_{\mathbf{y}\in Y}$ form a continuous frame.
Optimal frame bounds: $A = \tau_-^2$ and $B = \tau_+^2$.

Let $K_{\mathbf{y}} \in \mathcal{A}$ be Riesz representation of the evaluation functional at $\mathbf{y}$:

$$v(\mathbf{y}) = (v, K_{\mathbf{y}})_{\mathcal{A}} \qquad \forall v \in \mathcal{A}, \qquad \mathbf{y} \in Y.$$

$\mathcal{A}$ is reproducing-kernel Hilbert space, kernel: $K_{\mathbf{y}}(\mathbf{z}) = \sum_{p\in\mathbb{Z}} \overline{a_p(\mathbf{y})} a_p(\mathbf{z})$

T maps evaluation functionals into evanescent plane waves:

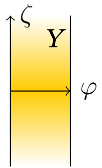
$$T : K_{\mathbf{y}} \mapsto \mathrm{EW}_{\mathbf{y}} \qquad \forall \mathbf{y} \in Y.$$

Approximation of u by EPWs "maps" to
reconstruction of $v = T^{-1}u$ by point sampling:

$$\mathcal{A} \ni \qquad v \approx \sum_{m=1}^{M} \mu_m K_{\mathbf{y}_m} \qquad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \qquad u \approx \sum_{m=1}^{M} \mu_m \mathrm{EW}_{\mathbf{y}_m} \qquad \in \mathcal{B}$$

Parameter sampling in Y

How to do sampling in $\mathcal{A} = \mathrm{span}\{\alpha_p \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}\} \subset L^2(Y; w^2)$?
How to choose points $\{\mathbf{y}_m\}_m \in Y$?



We follow Cohen, Migliorati, 2017
"Optimal weighted least-squares methods"

Fix $P \in \mathbb{N}$, set $\mathcal{A}_P := \mathrm{span}\{a_p\}_{|p| \le P} \subset \mathcal{A}$. Define probability density

$$\rho(\mathbf{y}) := \frac{w^2}{2P+1} \sum_{|p| \le P} |a_p(\mathbf{y})|^2 \qquad \text{on } Y$$

$\rho^{-1} =$ "Christoffel function"

and generate $M \in \mathbb{N}$ nodes $\{\mathbf{y}_m\}_{m=1,\dots,M}$ distributed according to ρ.

We expect the span of the normalised sampling functionals !

$$\left\{ \mathbf{y} \mapsto \frac{1}{\sqrt{\sum_{|p| \le P} |a_p(\mathbf{y}_m)|^2}} K_{\mathbf{y}_m}(\mathbf{y}) \right\}_{m=1,\dots,M} \subset \mathcal{A}$$

to approximate any $v_P \in \mathcal{A}_P$ with small coefficients.

Helmholtz solution approximation by EPWs

Then any $u \in \mathrm{span}\{b_p\}_{|p|\le P}$ can be approximated by EPWs

$$\left\{\mathbf{x} \mapsto \frac{1}{\sqrt{M\sum_{|p|\le P}|a_p(\mathbf{y}_m)|^2}}\mathrm{EW}_{\mathbf{y}_m}(\mathbf{x})\right\}_{m=1,\dots,M} \subset \mathcal{B}$$

with small coefficients.

Then u can be stably approximated in computer arithmetic using SVD and oversampling.

The M-dimensional EPW space depends on truncation parameter P: space is tuned to approximate Fourier modes b_p with $|p| \le P$.

(EPW choice is very different from WBM!)

Part IV

Numerical results

Boundary sampling method

Given (PPW, EPW,...) approximation set $\mathrm{span}\{\phi_m\}_{m=1,\dots,M}$,
how do we approximate $u \in \mathcal{B}$ in practice?

We use boundary sampling on $\left\{\mathbf{x}_s = \left(\begin{smallmatrix} r=1 \\ \theta_s=\frac{2\pi s}{S}\end{smallmatrix}\right)\right\}_{s=1,\dots,S} \subset \partial B_1$:

$$A\boldsymbol{\xi} = \mathbf{c} \qquad \text{with} \qquad \begin{matrix} A_{s,m} := \phi_m(\mathbf{x}_s), \\ c_s := u(\mathbf{x}_s) \end{matrix} \quad \begin{matrix} s=1,\dots,S \\ m=1,\dots,M \end{matrix} \quad \rightarrow \quad u_M = \sum_m \xi_m \phi_m \approx u.$$

Choose $\kappa^2 \neq$ Laplace–Dirichlet eigenvalue on B_1.

Could use instead:
- sampling in the bulk of B_1,
- impedance trace,
- $\mathcal{B}$ / $L^2(B_1)$ / $L^2(\partial B_1)$ projection...

- Oversampling: $S > M$
- SVD regularization, threshold ϵ:

} required by Adcock–Huybrechs

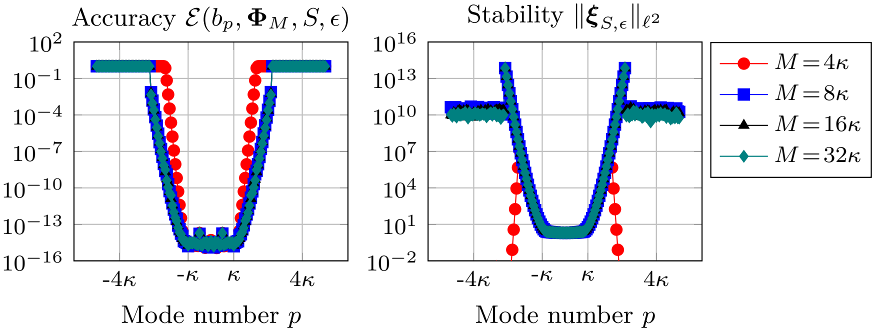
$$A = U\,\mathrm{diag}(\sigma_1,\dots,\sigma_M)\,V^*, \qquad \Sigma_\epsilon := \mathrm{diag}(\{\sigma_m > \epsilon \max_{m'} \sigma_{m'}\}),$$

$$\boldsymbol{\xi}_\epsilon = V\Sigma_\epsilon^\dagger U^* \mathbf{c}$$

Approximation by PPWs

Approximation of circular waves $\{b_p\}_p$ by equispaced PPWs

$\kappa = 16$, $\epsilon = 10^{-14}$, $S = \max\{2M, 2|p|\}$, residual $\mathcal{E} = \frac{\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|}{\|\mathbf{c}\|}$

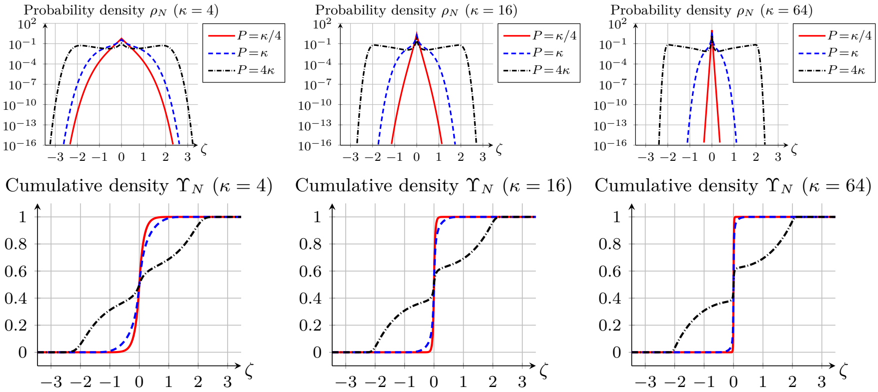


- Propagative modes $|p| \lesssim \kappa$: $\mathcal{O}(\epsilon)$ error $\forall M$, $\mathcal{O}(1)$ coeff.'s
- Evanescent modes $|p| \gtrsim 3\kappa$: $\mathcal{O}(1)$ error $\forall M$, large coeff.'s

Condition number is irrelevant!

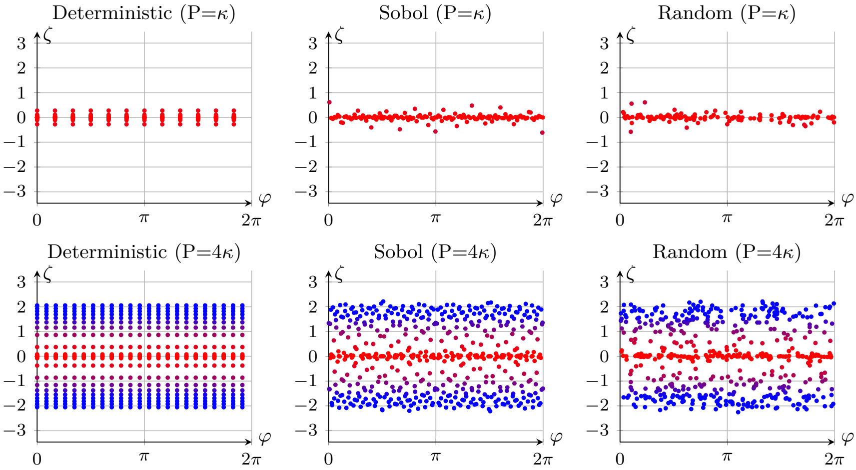
EPW approximation: probability measure on Y

Probability density ρ & cumulative d.f. as functions of evanescence ζ:



They depend on P: target functions in $\mathrm{span}\{b_p\}_{|p|\leq P}$.
Modes at $\zeta \approx \pm\log(2P/\kappa)$.

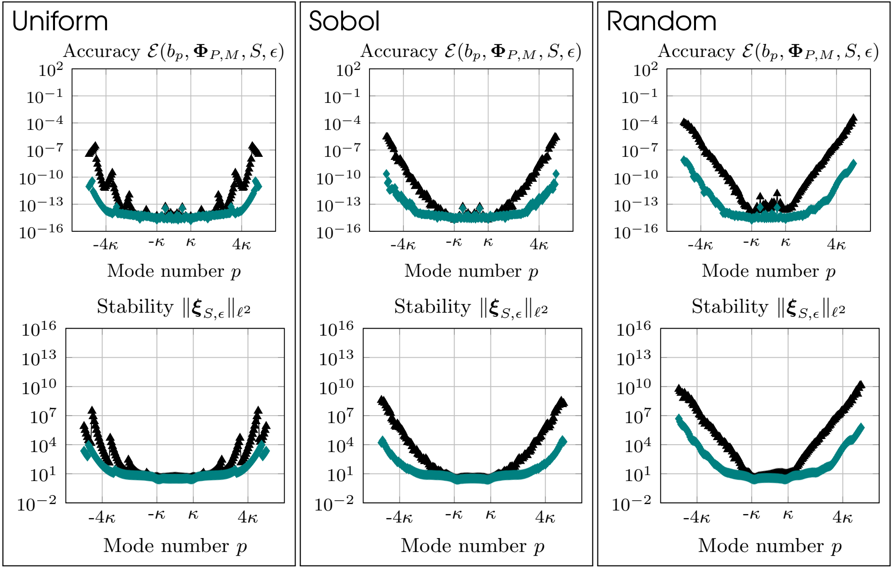
Parameter samples in the cylinder Y



Samples computed on $(0, 1)^2$ & uniform prob., mapped to Y by Υ^{-1}.

Approximation by EPWs

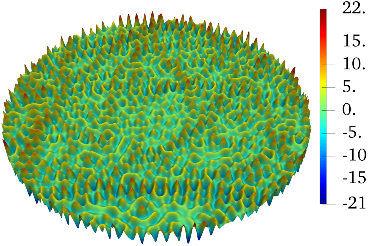
Approximation of $\{b_p\}$, ▲ $M = 4P$, ◆ $M = 8P$ | $P = 4\kappa,\ \kappa = 16$



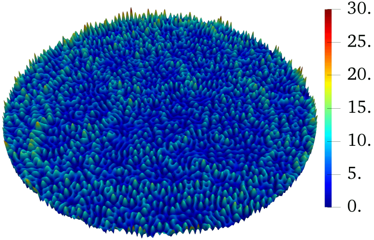
Solution and error plots

$u = \sum_{|p|\leq P} \hat{u}_p b_p, \quad \hat{u}_p \sim (\max\{1, |p| - \kappa\})^{-1/2}, \quad \kappa = 100, \quad P = 2\kappa, \quad M = 802$

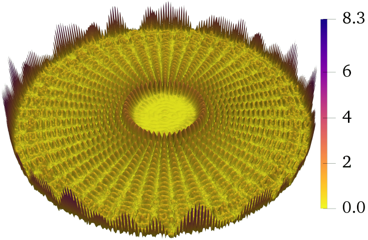
$\Re\{u\}$



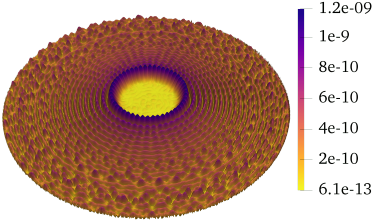
$|u|$



$|u - PPW|$



$|u - EPW|$



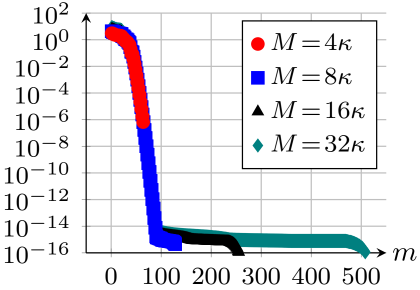
$\|u - PPW\|_{L^\infty} \gtrsim 7 \cdot 10^9 \, \|u - EPW\|_{L^\infty}$

DOFs/wavelength $= \lambda\sqrt{M/|B_1|} \approx 1$

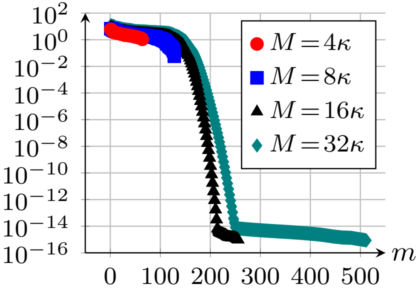
Singular values of the matrix A

$\kappa = 16$

PPWs



EPWs (Sobol, $P = 4\kappa$)



Comparable condition numbers, larger ϵ-rank for EPWs.
Can further increase ϵ-rank by raising P.

Summary

- Approximation of Helmholtz solutions by PPWs is unstable: accuracy only with large coefficients.
- Approximation by evanescent PWs seems to be stable.
- EPWs parameters chosen with sampling in Y.
- Key new result is stable Herglotz transform $u = Tv$.

Next steps:

- General geometries
- 3D
- Maxwell & elasticity
- Complete proof of EPW stability
- Use in Trefftz and in sampling
- ...

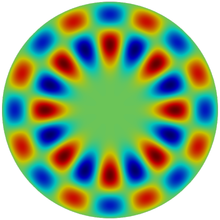
Thank you!

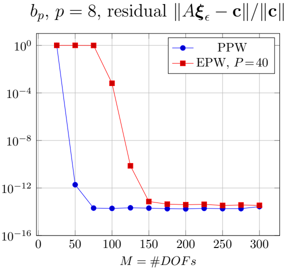
E. Parolin, D. Huybrechs, A. Moiola arXiv:2202.05658
Stable approximation of Helmholtz solutions by evanescent plane waves
Julia code on:
`https://github.com/EmileParolin/evanescent-plane-wave-approx`

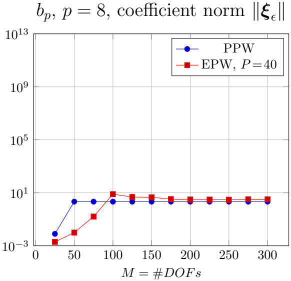
Approximation by PPWs and by EPWs

$\kappa = 16, \qquad \epsilon = 10^{-14}, \qquad S = \max\{2M, 2|p|\}$

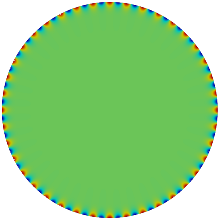
$p = 8$

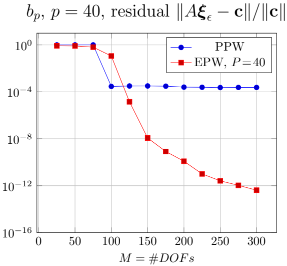


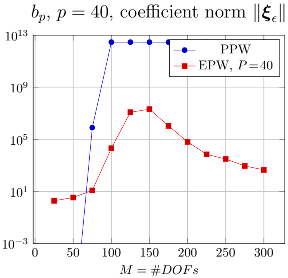




$p = 40$







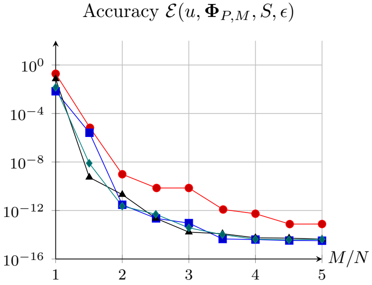
Approximation of general (truncated) u

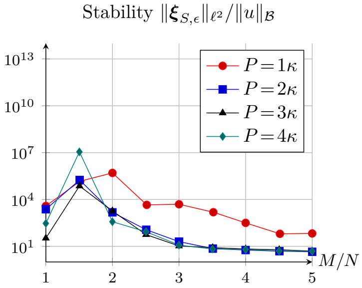
Evanescent PW approximation of rough u: $(S = 2M, \kappa = 16)$

$$u = \sum_{|p| \le P} \hat{u}_p b_p, \qquad \hat{u}_p \sim (\max\{1, |p| - \kappa\})^{-1/2}$$

EPWs constructed assuming that P is known. Deterministic sampling.

Convergence for $M \nearrow$ plotted against $\frac{M}{2P+1} = \frac{\dim(\text{approx. space})}{\dim(\text{solution space})}$:





Error is P-independent.