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Explicit bounds for electromagnetic transmission problems

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Maxwell equations in heterogeneous media

Given:

- wavenumber $k > 0$
- sources $\mathbf{J}, \mathbf{K} \in H(\mathrm{div}^0; \mathbb{R}^3)$, compactly supported
- $\epsilon_0, \mu_0 > 0$
- $\epsilon, \mu \in L^\infty(\mathbb{R}^3; SPD)$ such that
  $\Omega_i := \mathrm{int}\big(\mathrm{supp}(\epsilon - \epsilon_0 \underline{\underline{\mathbf{I}}}) \cup \mathrm{supp}(\mu - \mu_0 \underline{\underline{\mathbf{I}}})\big)$ is bounded and Lipschitz

Find $\mathbf{E}, \mathbf{H} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ such that

$$i k \epsilon \mathbf{E} + \nabla \times \mathbf{H} = \mathbf{J} \qquad \text{in } \mathbb{R}^3,$$

$$-i k \mu \mathbf{H} + \nabla \times \mathbf{E} = \mathbf{K} \qquad \text{in } \mathbb{R}^3,$$

$(\mathbf{E}, \mathbf{H})$ satisfy Silver–Müller radiation condition

$$\left|\sqrt{\epsilon_0}\mathbf{E} - \sqrt{\mu_0}\mathbf{H} \times \tfrac{\mathbf{x}}{|\mathbf{x}|}\right| = \mathcal{O}_{|\mathbf{x}| \to \infty}(|\mathbf{x}|^{-2}).$$

$\epsilon = \epsilon_0$
$\mu = \mu_0$

Special case: "transmission problem", i.e. homogeneous scatterer

$$\epsilon = \begin{cases} \epsilon_i \\ \epsilon_0 \end{cases} \quad \mu = \begin{cases} \mu_i & \text{in } \Omega_i \\ \mu_0 & \text{in } \Omega_o := \mathbb{R}^3 \setminus \overline{\Omega_i} \end{cases} \quad 0 < \epsilon_i, \epsilon_0, \mu_i, \mu_0 \text{ constant.}$$

Wave scattering

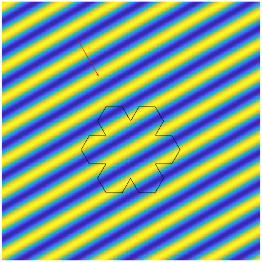
The example we have in mind is incident wave $\mathbf{E}^{Inc}, \mathbf{H}^{Inc}$ hitting Ω_i:

$\to$ BVP with data supported on Ω_i:

$$\mathbf{J} = ik^2(\epsilon_0 - \epsilon)\mathbf{E}^{Inc},$$
$$\mathbf{K} = ik^2(\mu - \mu_0)\mathbf{H}^{Inc}.$$

Incoming field

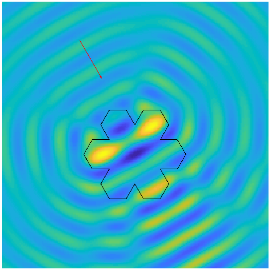
$$\mathbf{E}^{Inc} = \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{A}e^{ik\sqrt{\epsilon_0\mu_0}\mathbf{x}\cdot\mathbf{d}}$$
$$\mathbf{H}^{Inc} = \mathbf{d} \times \mathbf{A}e^{ik\sqrt{\epsilon_0\mu_0}\mathbf{x}\cdot\mathbf{d}}$$



datum

Scattered field

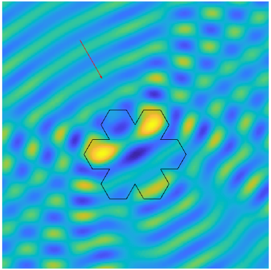
$$\mathbf{E}$$
$$\mathbf{H}$$



BVP solution

Total field

$$\mathbf{E} + \mathbf{E}^{Inc}$$
$$\mathbf{H} + \mathbf{H}^{Inc}$$



physical field

Goal and motivation

If ϵ, μ are sufficiently regular then the problem is well-posed.
From Fredholm theory we have

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\Omega_{i/o}} \leq C \left\| \begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix} \right\|_{\Omega_{i/o}}$$

Goal: find out how $C = C(k, \epsilon, \mu)$ depends on k, ϵ and μ.

Why? In FEM & BEM analysis and in UQ for time-harmonic problems, explicit parameter dependence allows to control:

- Quasi-optimality & pollution effect
- Gmres iteration numbers
- Matrix compression
- hp-FEM & BEM (Melenk–Sauter)
- Shape differentiation & uncertainty quantification
- …

Who cares?

LAFONTAINE, SPENCE, WUNSCH, arXiv 2019: (Helmholtz)

The following is a non-exhaustive list of papers on the frequency-explicit convergence analysis of numerical methods for solving the Helmholtz equation where a central role is played by *either* the non-trapping resolvent estimate (1.5), *or* its analogue (with the same k-dependence) for the commonly-used approximation of the exterior problem where the exterior domain $\mathcal{O}_+$ is truncated and an impedance boundary condition is imposed:

- conforming FEMs (including continuous interior-penalty methods) [72, Proposition 2.1], [74, Proposition 8.1.4], [56, Lemma 2.1], [77, Lemma 3.5], [78, Assumptions 4.8 and 4.18], [45, §2.1], [110, Theorem 3.1], [113, §3.1], [44, §3.2.1], [40, Remark 3.2], [41, Remark 3.1], [29, Assumption 1], [30, Definition 2], [55, Theorem 3.2], [50, Lemma 6.7], [14, Equation 4],
- least squares methods [33, Assumption A1], [10, Remark 1.2], [64, Assumption 1 and equation after Equation 5.37],
- DG methods based on piece-wise polynomials [46, Theorem 2.2], [47, Theorem 2.1], [39, Assumption 3], [48, §3], [62, Assumption A (Equation 4.5)], [76, Equation 4.4], [35, Remark 3.2], [32, Equation 2.4], [84, Equation 4.3], [112, Remark 3.1], [94, Theorem 2.2],
- plane-wave/Trefftz-DG methods [3, Theorem 1], [59, Equation 3.5], [60, Theorem 2.2], [2, Lemma 4.1], [61, Proposition 2.1],
- multiscale finite-element methods [51, Equation 2.3], [13, §1.2], [88, Assumption 5.3], [8, Theorem 1], [87, Assumption 3.8], [31, Assumption 1],
- integral-equation methods [71, Equation 3.24], [75, Equation 4.4], [24, Chapter 5], [53, Theorem 3.2], [111, Remark 7.5], [43, Theorem 2], [49, Theorem 3.2], [52, Assumption 3.2],

In addition, the following papers focus on proving bounds on the solution of Helmholtz boundary-value problems (with these bounds often called "stability estimates") motivated by applications in numerical analysis: [36], [57], [26], [11], [7], [70], [98], [28], [6], [9], [27], [93], [54], [55] [83], [50], Of these papers, all but [70], [6], [27], [11] are in nontrapping situations, [70], [6], [27] are in parabolic trapping scenarios, and [11] proves the exponential growth (1.7) under elliptic trapping.

What about Helmholtz? (M. & S. 2019)

Simplest heterogeneous Helmholtz problem: find $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ s.t.

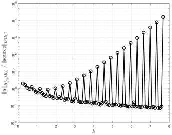
$\Delta u + k^2\, n\, u = f$ in $\mathbb{R}^d$
+Sommerfeld radiation c.

$$f \in L^2(\mathbb{R}^d), \qquad n = \begin{cases} n_i & \text{constant in } \Omega_i, \\ 1 & \text{in } \Omega_o. \end{cases}$$

- If $0 < n_i < 1$, Ω_i star-shaped $\qquad \Omega_i \cup \operatorname{supp} f \subset B_R$

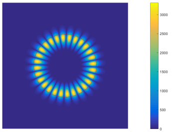
$$\|\nabla u\|^2_{L^2(B_R)} + k^2 \left\|\sqrt{n}\, u\right\|^2_{L^2(B_R)} \le \left[4R^2 + \frac{1}{n_i}\left(2R + \frac{d-1}{k}\right)^2\right] \|f\|^2_{L^2(B_R)}$$

Fully explicit, k-independent, shape-robust estimate.
(For $d = 2$ it implies bounds for Maxwell TE/TM modes.)

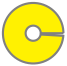


- If $n_i > 1$, Ω_i strictly convex & C^∞:

superalgebraic blow up in k, quasi-resonances,
ray trapping, creeping waves...



Dependence on parameters is complicated!
Monotonicity of n & shape of Ω_i are crucial.



Wavenumber-explicit bounds: a bit of history

- MORAWETZ 1960S/70S: introduced main tools (multipliers)
- MELENK 1995: 1st k-explicit bound for Helmholtz, bdd dom.
- CHANDLER-WILDE, MONK 2008: unbounded domains
- HIPTMAIR, MOIOLA, PERUGIA 2011: Maxwell, bdd dom.

homogeneous coeff.

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heterogeneous coeff.

- MOIOLA, SPENCE 2019: Helmholtz & piecewise-constant n
- GRAHAM, PEMBERY, SPENCE 2019: Helmholtz & general coeff.
- VERFÜRTH 2019: Maxwell & impedance

Plenty of other related contributions exist!
BARUCQ, CHAUMONT-FRELET, FENG, HETMANIUK, LORTON, PETERSEIM, SAUTER, TORRES, WIENERS&WOHLMUTH, (your name here), . . .

Our goal: extend (GRAHAM, PEMBERY, SPENCE 2019) to Maxwell eq.s.

Bound #1: transmission problem

Single homogeneous scatterer:

$$\epsilon = \begin{cases} \epsilon_i & \text{in } \Omega_i \\ \epsilon_0 & \text{in } \Omega_o \end{cases}, \qquad \mu = \begin{cases} \mu_i & \text{in } \Omega_i \\ \mu_0 & \text{in } \Omega_o \end{cases} \qquad 0 < \epsilon_i, \epsilon_0, \mu_i, \mu_0 \text{ constant.}$$

If $\boxed{\epsilon_i \le \epsilon_0}$, $\boxed{\mu_i \le \mu_0}$, $\boxed{\Omega_i \text{ star-shaped}}$, $\Omega_i \cup \operatorname{supp} \mathbf{J} \cup \operatorname{supp} \mathbf{K} \subset B_R$, then

$$\epsilon_i \|\mathbf{E}\|_{B_R}^2 + \mu_i \|\mathbf{H}\|_{B_R}^2 \quad \le \quad 4R^2 \left(\frac{\epsilon_0}{\epsilon_i} + \frac{\mu_0}{\mu_i} \right) \left(\epsilon_0 \|\mathbf{K}\|_{B_R}^2 + \mu_0 \|\mathbf{J}\|_{B_R}^2 \right).$$

$\|\cdot\|_{B_R} = \|\cdot\|_{L^2(B_R)}$

Equivalent to wavenumber-independent $H(\mathrm{curl}; B_R)$ bound for $\mathbf{E}$.

If ϵ_i is (constant) SPD matrix, same holds if $\max \mathit{eig}(\epsilon_i) \le \epsilon_0$ and with ϵ_i substituted by $\min \mathit{eig}(\epsilon_i)$ in the bound. Same for μ_i.

Bound #2: more general ϵ, μ

Assume $\epsilon, \mu \in W^{1,\infty}(\Omega_i; SPD)$, $\quad \Omega_i$ Lipschitz,

- Ω_i star-shaped
- $\|\epsilon_i\|_{L^\infty(\partial\Omega_i)} \le \epsilon_0$, $\|\mu_i\|_{L^\infty(\partial\Omega_i)} \le \mu_0$, i.e. jumps are "upwards" on $\partial\Omega_i$
- $\epsilon_* := \operatorname{ess\,inf}_{\mathbf{x}\in\Omega_i} \big(\epsilon + (\mathbf{x}\cdot\nabla)\epsilon\big) > 0$, $\mu_* := \operatorname{ess\,inf}_{\mathbf{x}\in\Omega_i} \big(\mu + (\mathbf{x}\cdot\nabla)\mu\big) > 0$
  "weak monotonicity" in radial direction, avoid trapping of rays
- "extra regularity" ($\mathbf{E}, \mathbf{H} \in H^1(\Omega_i \cup \Omega_o)^3$ or $\epsilon, \mu \in C^1(\Omega_i)$ or $W^{1,\infty}(\mathbb{R}^3)$)

Then we have explicit wavenumber-independent bound:

$$\epsilon_* \|\mathbf{E}\|^2_{B_R} + \mu_* \|\mathbf{H}\|^2_{B_R}$$
$$\le 4R^2 \left(\frac{\|\epsilon\|^2_{L^\infty(B_R)}}{\epsilon_*} + \frac{\epsilon_0\mu_0}{\mu_*} \right) \|\mathbf{K}\|^2_{B_R} + 4R^2 \left(\frac{\|\mu\|^2_{L^\infty(B_R)}}{\mu_*} + \frac{\epsilon_0\mu_0}{\epsilon_*} \right) \|\mathbf{J}\|^2_{B_R}.$$

Expect (from Helmholtz analogy) superalgebraic blow up in k
if any of the first 3 assumptions is lifted.

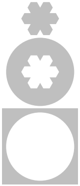
Similar results when $\mathbb{R}^3$ is truncated with impedance BCs.

How our bound was obtained

First consider smooth case $\mathbf{E},\mathbf{H}\in C^1(\mathbb{R}^3;\mathbb{C}^3)$.

(i) Multiply the 2 PDEs by the "test fields" (Morawetz multipliers)

$$\begin{aligned}
&(\epsilon\overline{\mathbf{E}}\times\mathbf{x}+R\sqrt{\epsilon\mu}\overline{\mathbf{H}}) \quad \& \quad (\mu\overline{\mathbf{H}}\times\mathbf{x}-R\sqrt{\epsilon\mu}\overline{\mathbf{E}}) \qquad \text{in } B_R\supset\Omega_i,\\
&(\epsilon_0\overline{\mathbf{E}}\times\mathbf{x}+r\sqrt{\epsilon_0\mu_0}\overline{\mathbf{H}}) \quad \& \quad (\mu_0\overline{\mathbf{H}}\times\mathbf{x}-r\sqrt{\epsilon_0\mu_0}\overline{\mathbf{E}}) \qquad \text{in } \mathbb{R}^3\setminus B_R,
\end{aligned}$$



(ii) integrate by parts in Ω_i, $B_R\setminus\overline{\Omega_i}$ and $\mathbb{R}^3\setminus B_R$,
(iii) sum 3 contributions, (iv) take $\Re$eal part, (v) have fun!

$$\int_{B_R}\overline{\mathbf{E}}\cdot\underbrace{\big(\epsilon+(\mathbf{x}\cdot\nabla)\epsilon\big)}_{\geq\epsilon_*\text{ by assumpt.}}\mathbf{E}+\overline{\mathbf{H}}\cdot\underbrace{\big(\mu+(\mathbf{x}\cdot\nabla)\mu\big)}_{\geq\mu_*\text{ by assumpt.}}\mathbf{H}$$

Using PDEs & $\nabla\cdot[\epsilon\mathbf{E}]=\nabla\cdot[\mu\mathbf{H}]=0$

$$=2\int_{B_R}\Re\Big\{\mathbf{K}\cdot(\epsilon\overline{\mathbf{E}}\times\mathbf{x}+\sqrt{\epsilon_0\mu_0}R\overline{\mathbf{H}})+\mathbf{J}\cdot(\mu\overline{\mathbf{H}}\times\mathbf{x}-\sqrt{\epsilon_0\mu_0}R\overline{\mathbf{E}})\Big\}$$

$$+\int_{\partial\Omega_i}\underbrace{[\text{terms from IBP}]}_{\substack{\leq 0\text{ by } \epsilon_i\preceq\epsilon_0,\mu_i\preceq\mu_0,\quad \mathbf{n}\cdot\mathbf{x}\geq0,\\ [\![\mathbf{E}_T,\mathbf{H}_T,(\epsilon\mathbf{E})_N,(\mu\mathbf{H})_N]\!]=0}}+\int_{\partial B_R}\underbrace{[\text{terms from IBP}]}_{\leq0\text{ by S–M radiation c.}}$$

Conclude by Cauchy–Schwarz.

Rough coefficients, regularity and density

Proof in previous slide only uses elementary results if $\mathbf{E}, \mathbf{H} \in C^1(\mathbb{R}^3; \mathbb{C}^3)$.

For general case we need density of inclusion

$$C^\infty(\overline{D})^3 \subset \left\{ \mathbf{v} \in H(\mathrm{curl}; D), \nabla\cdot[\mathsf{A}\mathbf{v}] \in L^2(D), \mathsf{A}\mathbf{v}\cdot\hat{\mathbf{n}} \in L^2(\partial D), \mathbf{v}_T \in L^2_T(\partial D) \right\}$$

for $\mathsf{A} = \epsilon$ & $\mathsf{A} = \mu$, D Lipschitz bdd.

If $\mathsf{A} \in C^1(\Omega_i; SPD)$, this density is non-trivial but follows from regularity results for layer potentials on manifolds (MITREA, TAYLOR 1999).

- Equivalent step for Helmholtz was much simpler.
- Constant scalar ϵ & μ: density proved in COSTABEL, DAUGE 1998.
- If $\mathbf{E}, \mathbf{H} \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^3)$ then no density is needed.
  E.g. ensured if $\epsilon, \mu \in W^{1,\infty}(\mathbb{R}^3; SPD)$ (no jumps).
- What about $\mathsf{A} \in W^{1,\infty}(\Omega_i; SPD)$?

Summary

Time-harmonic Maxwell eq.s in $\mathbb{R}^3$ with heterogeneous inclusion:

- fully explicit bounds on $\|\mathbf{E}\|_{H(\mathrm{curl},B_R)}$ if ϵ, μ "radially growing"
- also for impedance BVPs in star-shaped domains
- extends Helmholtz results from (GRAHAM, PEMBERY, SPENCE 2019)

Some open questions:

- resonance-free strip in complex k plane?
- presence of quasi-resonances blow up for "wrong" coefficients?
- rougher ($W^{1,\infty}(\Omega_i; SPD), L^\infty$) coefficients?
- relation with shape-differentiation and UQ?

Preprint coming soon. . .

Thank you!