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Space–time Trefftz discontinuous Galerkin methods for wave problems

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Minimal Trefftz example: Laplace equation

Imagine you want to approximate the solution u of Laplace eq.

$$\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \qquad (\text{+any BCs on } \partial\Omega),$$

using a standard discontinuous Galerkin (DG) method.
You seek the approximate solution in

$$\left\{ v_{hp} \in L^2(\Omega) : \; v_{hp|K} \in \mathbb{P}^p(K) \; \forall K \in \mathcal{T}_h \right\}$$

where $\mathbb{P}^p(K)$ is the space of polynomials of degree at most p on the element K of a mesh $\mathcal{T}_h$.

Why not use only (piecewise) harmonic polynomials

$$\left\{ v_{hp} \in L^2(\Omega) : \; v_{hp|K} \in \mathbb{P}^p(K), \; \Delta v_{hp|K} = 0 \; \forall K \in \mathcal{T}_h \right\} \quad ?$$

Comparable accuracy for $\mathcal{O}(p^{n-1} \cdot \#el)$ vs $\mathcal{O}(p^n \cdot \#el)$ DOFs.

(E.g., n=2, p=10: 21 vs 66 DOFs/el.; p=20: 41 vs 231 DOFs/el.)

Trefftz methods

Consider a linear PDE $\mathcal{L}u = 0$.

Trefftz methods are finite element schemes such that
test and trial functions are solutions of the PDE
in each element K of the mesh $\mathcal{T}_h$.

E.g.: piecewise harmonic polynomials if $\mathcal{L}u = \Delta u$.

Our main interest is in wave propagation, in:

- Frequency domain, Helmholtz eq. $-\Delta u - k^2 u = 0$
  lot of work done, $h/p/hp$-theory, Maxwell, elasticity...
  (recent survey: Hiptmair, AM, Perugia, arXiv:1506.04521)
- Time domain, wave equation $-\Delta U + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}U = 0$

  Trefftz methods are in space–time,
  as opposed to semi-discretisation + time-stepping.

Trefftz methods for wave equation

Why Trefftz methods? Comparing with standard DG,

- better accuracy per DOFs and higher convergence orders;
- PDE properties "known" by discrete space, e.g. dispersion;
- lower dimensional quadrature needed;
- simpler and more flexible, adapted bases and adaptivity...

No typical drawbacks of time-harmonic Trefftz (ill-cond., quad.).

Existing works on Trefftz for time-domain wave equation:

- MACIĄG, SOKALA, WAUER 2005–2011, LIU, KUO 2016, single element Trefftz;
- PETERSEN, FARHAT, TEZAUR, WANG 2009&2014, DG with Lagrange multipliers;
- EGGER, KRETZSCHMAR, SCHNEPP, TSUKERMAN, WEILAND 3×2014–2015, Maxwell equations;
  KRETZSCHMAR, MOIOLA, PERUGIA, SCHNEPP 2015, analysis;
  MOIOLA, PERUGIA, arXiv:1610.08002.
- BANJAI, GEORGOULIS, LIJOKA 2016, interior penalty-DG (see talk on Wednesday).

Simplest Trefftz space: Trefftz polynomials

Consider wave eq. $-\Delta U + c^{-2} U'' = 0$ in $K \subset \mathbb{R}^{n+1}$ (c const).

Choose Trefftz space of polynomials of deg. $\le p$ on element K:

$$\mathbb{U}^p(K) := \left\{ v \in \mathbb{P}^p(K),\ -\Delta v + c^{-2} v'' = 0 \right\}.$$

- Basis functions are easily constructed:

  $$b_{j,\ell}(\mathbf{x}, t) = (\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j$$

  for suitable propagation directions $\mathbf{d}_{j,\ell}$ ($|\mathbf{d}_{j,\ell}| = 1$).

- Orders of approximation in h are for free, because Taylor polynomial of (smooth) U belongs to $\mathbb{U}^p(K)$.

- $\dim\left(\mathbb{U}^p(K)\right) = \mathcal{O}_{p\to\infty}(p^n) \ll \dim\left(\mathbb{P}^p(K)\right) = \mathcal{O}_{p\to\infty}(p^{n+1})$.

Part II

Trefftz-DG for acoustic wave equations

Initial–boundary value problem

First order initial–boundary value problem (Dirichlet): find $(v, \boldsymbol{\sigma})$

$$\begin{cases} \nabla v + \dfrac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1},\ n \in \mathbb{N}, \\ \nabla \cdot \boldsymbol{\sigma} + \dfrac{1}{c^2}\dfrac{\partial v}{\partial t} = 0 & \text{in } Q, \\ v(\cdot,0) = v_0, \quad \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega, \\ v(\mathbf{x},\cdot) = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

From $-\Delta U + c^{-2}\frac{\partial^2}{\partial t^2}U = 0$, choose $v = \frac{\partial U}{\partial t}$ and $\boldsymbol{\sigma} = -\nabla U$.

Velocity c piecewise constant. $\Omega \subset \mathbb{R}^n$ Lipschitz bounded.

Extensions:

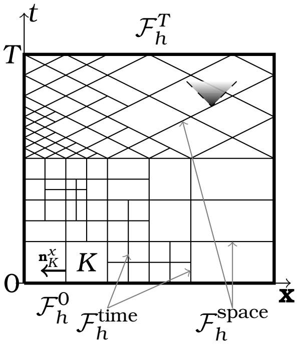
- Neumann $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ & Robin $\frac{\vartheta}{c} v - \boldsymbol{\sigma} \cdot \mathbf{n} = g$ BCs (✓),
- Maxwell equations (✓),
- elasticity,
- 1^{st} order hyperbolic systems ($\sim$),
- Maxwell equations in dispersive materials. . .

Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on Q.
Assume: $c = c(\mathbf{x})$ constant in elements.

Assume: each face $F = \partial K_1 \cap \partial K_2$ with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c|\mathbf{n}_F^x| < n_F^t$, denote $F \subset \mathcal{F}_h^{\mathrm{space}}$, or
- time-like: $n_F^t = 0$, denote $F \subset \mathcal{F}_h^{\mathrm{time}}$.



DG notation:

$$\{\!\{w\}\!\} := \frac{w_{|K_1} + w_{|K_2}}{2}, \qquad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}_{|K_1} + \boldsymbol{\tau}_{|K_2}}{2},$$

$$[\![w]\!]_{\mathbf{N}} := w_{|K_1}\mathbf{n}_{K_1}^x + w_{|K_2}\mathbf{n}_{K_2}^x,$$

$$[\![\boldsymbol{\tau}]\!]_{\mathbf{N}} := \boldsymbol{\tau}_{|K_1}\cdot\mathbf{n}_{K_1}^x + \boldsymbol{\tau}_{|K_2}\cdot\mathbf{n}_{K_2}^x,$$

$$[\![w]\!]_t := w_{|K_1} n_{K_1}^t + w_{|K_2} n_{K_2}^t = (w^- - w^+)n_F^t,$$

$$[\![\boldsymbol{\tau}]\!]_t := \boldsymbol{\tau}_{|K_1} n_{K_1}^t + \boldsymbol{\tau}_{|K_2} n_{K_2}^t = (\boldsymbol{\tau}^- - \boldsymbol{\tau}^+)n_F^t,$$

$$\mathcal{F}_h^0 := \Omega \times \{0\}, \qquad \mathcal{F}_h^T := \Omega \times \{T\},$$

$$\mathcal{F}_h^\partial := \partial\Omega \times [0, T].$$

DG elemental equation and numerical fluxes

Trefftz space:
$$\mathbf{T}(\mathcal{T}_h) := \Big\{(w,\boldsymbol{\tau}) \in L^2(Q), (w|_K, \boldsymbol{\tau}|_K) \in H^1(K)^{1+n},$$
$$\nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\tau} + c^{-2}\frac{\partial w}{\partial t} = 0 \ \forall K \in \mathcal{T}_h\Big\}.$$

Multiplying PDEs with test $(w, \boldsymbol{\tau})$, integrating by parts in K, using Trefftz property and summing over $K \in \mathcal{T}_h$: $\forall (w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h)$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \left((v\,\boldsymbol{\tau} + \boldsymbol{\sigma}\, w) \cdot \mathbf{n}_K^x + \left(\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \frac{1}{c^2}\, v\, w\right) n_K^t \right) \mathrm{d}S = 0.$$

We approximate skeleton traces of $(v, \boldsymbol{\sigma})$ with numerical fluxes $(\widehat{v}_{hp}, \widehat{\boldsymbol{\sigma}}_{hp})$, defined as $\alpha, \beta \in L^\infty(\mathcal{F}_h^{\mathrm{time}} \cup \mathcal{F}_h^{\partial})$

$$\widehat{v}_{hp} := \begin{cases} v_{hp}^- \\ v_{hp} \\ v_0 \\ \{\!\{v_{hp}\}\!\} + \beta [\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} \\ g \end{cases} \qquad \widehat{\boldsymbol{\sigma}}_{hp} := \begin{cases} \boldsymbol{\sigma}_{hp}^- & \text{on } \mathcal{F}_h^{\mathrm{space}}, \\ \boldsymbol{\sigma}_{hp} & \text{on } \mathcal{F}_h^{T}, \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^{0}, \\ \{\!\{\boldsymbol{\sigma}_{hp}\}\!\} + \alpha [\![v_{hp}]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\mathrm{time}}, \\ \boldsymbol{\sigma}_{hp} - \alpha(v-g)\mathbf{n}_\Omega^x & \text{on } \mathcal{F}_h^{\partial}. \end{cases}$$

$\alpha = \beta = 0 \to$ KRETZSCHMAR–S.–T.–W., $\quad \alpha\beta \geq \frac{1}{4} \to$ MONK–RICHTER.

Trefftz-DG formulation

Substituting the fluxes in the elemental equation and choosing any finite-dimensional $\mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$, write Trefftz-DG as:

Seek $(v_{hp}, \boldsymbol{\sigma}_{hp}) \in \mathbf{V}_p(\mathcal{T}_h)$ s.t., $\quad \forall (w, \boldsymbol{\tau}) \in \mathbf{V}_p(\mathcal{T}_h)$,

$\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) = \ell(w, \boldsymbol{\tau}) \qquad$ where

$$\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) := \int_{\mathcal{F}_h^{\text{space}}} \left(\frac{v_{hp}^{-} [\![w]\!]_t}{c^2} + \boldsymbol{\sigma}_{hp}^{-} \cdot [\![\boldsymbol{\tau}]\!]_t + v_{hp}^{-} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \boldsymbol{\sigma}_{hp}^{-} \cdot [\![w]\!]_{\mathbf{N}} \right) \mathrm{d}S$$

$$+ \int_{\mathcal{F}_h^{\text{time}}} \left(\{\!\{ v_{hp} \}\!\} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \{\!\{ \boldsymbol{\sigma}_{hp} \}\!\} \cdot [\![w]\!]_{\mathbf{N}} + \alpha [\![v_{hp}]\!]_{\mathbf{N}} \cdot [\![w]\!]_{\mathbf{N}} + \beta [\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \right) \mathrm{d}S$$

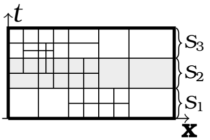
$$+ \int_{\mathcal{F}_h^{T}} (c^{-2} v_{hp} w + \boldsymbol{\sigma}_{hp} \cdot \boldsymbol{\tau})\, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} (\boldsymbol{\sigma}_{hp} \cdot \mathbf{n}_{\Omega} + \alpha v_{hp})\, w \, \mathrm{d}S,$$

$$\ell(w, \boldsymbol{\tau}) := \int_{\mathcal{F}_h^{0}} (c^{-2} v_0 w + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau})\, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} g(\alpha w - \boldsymbol{\tau} \cdot \mathbf{n}_{\Omega})\, \mathrm{d}S.$$

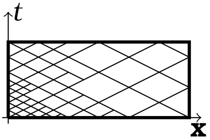
Global, implicit and explicit schemes

1 Trefftz-DG formulation is global in space–time domain Q:
large linear system! Might be good for adaptivity.

2 If mesh is partitioned in time-slabs $\Omega \times (t_{j-1}, t_j)$, matrix is block lower-triangular: for each time-slab a system can be solved sequentially: implicit method.



3 If mesh is suitably chosen, Trefftz-DG solution can be computed with a sequence of local systems: explicit method, allows parallelism!

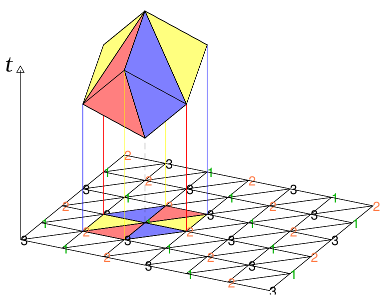


"Tent pitching algorithm" of Üngör–Sheffer, Monk–Richter, Gopalakrishnan–Monk–Sepúlveda, Gopalakrishnan–Schöberl–Wintersteiger. . . (See talk tomorrow.)

Versions 1–2–3 are algebraically equivalent (on the same mesh).

Tent-pitched elements

Tent-pitched elements/patches obtained from regular space meshes in 2+1D give parallelepipeds or octahedra+tetrahedra:



Trefftz requires quadrature on faces only:
only the shape of space elements matters.
Simplices around a tent pole can be merged in single element.

Part III

Trefftz-DG error analysis

Trefftz-DG norms

Assume $\alpha, \beta > 0$, $\gamma := c|\mathbf{n}_F^x|/n_F^t \in [0,1)$ on $\mathcal{F}_h^{\text{space}}$.

Define jump/averages seminorms on $H^1(\mathcal{T}_h)^{1+n}$:

$$
\begin{aligned}
|||(w,\boldsymbol{\tau})|||_{DG}^2 := \frac{1}{2}\Big(& \left\| c^{-1} w \right\|_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)}^2 + \left\| \boldsymbol{\tau} \right\|_{L^2(\mathcal{F}_h^0 \cup \mathcal{F}_h^T)^n}^2 \\
& + \left\| \left(\frac{1-\gamma}{n_F^t}\right)^{1/2} c^{-1} [\![w]\!]_t \right\|_{L^2(\mathcal{F}_h^{\text{space}})}^2 + \left\| \left(\frac{1-\gamma}{n_F^t}\right)^{1/2} [\![\boldsymbol{\tau}]\!]_t \right\|_{L^2(\mathcal{F}_h^{\text{space}})^n}^2 \Big) \\
& + \left\| \alpha^{1/2} [\![w]\!]_{\mathbf{N}} \right\|_{L^2(\mathcal{F}_h^{\text{time}})^n}^2 + \left\| \beta^{1/2} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \right\|_{L^2(\mathcal{F}_h^{\text{time}})}^2 + \left\| \alpha^{1/2} w \right\|_{L^2(\mathcal{F}_h^{\partial})}^2,
\end{aligned}
$$

$$
\begin{aligned}
|||(w,\boldsymbol{\tau})|||_{DG+}^2 := |||(w,\boldsymbol{\tau})|||_{DG}^2 & \\
+ 2 \left\| \left(\frac{n_F^t}{1-\gamma}\right)^{1/2} c^{-1} w^- \right\|_{L^2(\mathcal{F}_h^{\text{space}})}^2 & + 2 \left\| \left(\frac{n_F^t}{1-\gamma}\right)^{1/2} \boldsymbol{\tau}^- \right\|_{L^2(\mathcal{F}_h^{\text{space}})^n}^2 \\
+ \left\| \beta^{-1/2} \{\!\{ w \}\!\} \right\|_{L^2(\mathcal{F}_h^{\text{time}})}^2 & + \left\| \alpha^{-1/2} \{\!\{ \boldsymbol{\tau} \}\!\} \right\|_{L^2(\mathcal{F}_h^{\text{time}})^n}^2 + \left\| \alpha^{-1/2} \boldsymbol{\tau} \cdot \mathbf{n} \right\|_{L^2(\mathcal{F}_h^{\partial})}^2.
\end{aligned}
$$

They are norms on Trefftz space $\mathbf{T}(\mathcal{T}_h)$.

Trefftz-DG a priori error analysis

From integration by parts and Cauchy–Schwarz:

$$\forall (v,\boldsymbol{\sigma}),(w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h): \qquad (\alpha,\beta>0)$$

$$\mathcal{A}(v,\boldsymbol{\sigma};v,\boldsymbol{\sigma}) \geq |||(v,\boldsymbol{\sigma})|||_{DG}^2 \qquad \text{coercivity,}$$

$$|\mathcal{A}(v,\boldsymbol{\sigma};w,\boldsymbol{\tau})| \leq 2\,|||(v,\boldsymbol{\sigma})|||_{DG^+}\ |||(w,\boldsymbol{\tau})|||_{DG} \qquad \text{continuity,}$$

$$\Downarrow$$

Existence & uniqueness of discrete solution (only for Trefftz!)
Unconditional stability and quasi-optimality:

$$|||(v-v_{hp},\boldsymbol{\sigma}-\boldsymbol{\sigma}_{hp})|||_{DG} \leq 3 \inf_{(w_{hp},\boldsymbol{\tau}_{hp})\in\mathbf{V}_p(\mathcal{T}_h)} |||(v-w_{hp},\boldsymbol{\sigma}-\boldsymbol{\tau}_{hp})|||_{DG^+}.$$

Can control L^2 norm of error on space-like faces, e.g. $L^2(\Omega\times\{t\})$.

Energy dissipation: (if $g=0$)

$$\frac{1}{2}\int_{\Omega\times\{T\}}(c^{-2}v_{hp}^2+|\boldsymbol{\sigma}_{hp}|^2)\,\mathrm{d}\mathbf{x} \leq \frac{1}{2}\int_{\Omega\times\{0\}}(c^{-2}v_0^2+|\boldsymbol{\sigma}_0|^2)\,\mathrm{d}\mathbf{x}.$$

Energy dissipation controlled by jumps and mismatch with BCs.

Stability and error bound in $L^2(Q)$ norm

$|||\cdot|||_{DG}$ controls jumps on mesh skeleton and traces on ∂Q.
Error bounded in mesh-independent $\mathbf{X}^*$ norm (e.g. $L^2(Q)^{1+n}$) if

$$\|(w,\boldsymbol{\tau})\|_{\mathbf{X}^*} \le C_{(\mathcal{T}_h,\alpha,\beta)} |||(w,\boldsymbol{\tau})|||_{DG} \qquad \forall (w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h). \qquad !$$

To prove this, consider auxiliary inhomogeneous IBVP

$$\begin{cases} \nabla z + \partial\boldsymbol{\zeta}/\partial t = \boldsymbol{\Phi} & \text{in } Q, \quad \boldsymbol{\Phi} \in L^2(Q)^n, \\ \nabla\cdot\boldsymbol{\zeta} + c^{-2}\,\partial z/\partial t = \psi & \text{in } Q, \quad \psi \in L^2(Q), \\ z(\cdot,0) = 0, \quad \boldsymbol{\zeta}(\cdot,0) = \mathbf{0} & \text{on } \Omega, \\ z(\mathbf{x},\cdot) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

! holds, if $\forall(\psi,\boldsymbol{\Phi}) \in \mathbf{X} \subset L^2(Q)^{1+n}$

$$2\left\|n_t^{\frac12}\frac{z}{c}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)} + 2\left\|n_t^{\frac12}\boldsymbol{\zeta}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)^n} + \left\|\frac{z}{\beta^{\frac12}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})} + \left\|\frac{\boldsymbol{\zeta}\cdot\mathbf{n}_K^x}{\alpha^{\frac12}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}}\cup\mathcal{F}_h^{\partial})}$$
$$\le C^2_{(\mathcal{T}_h,\alpha,\beta)}\,\|(\psi,\boldsymbol{\Phi})\|^2_{\mathbf{X}}$$

Here $\mathbf{X}^*$ is the norm dual to $\mathbf{X}$ $\quad (\mathbf{X} \subset L^2(Q)^{1+n} \subset \mathbf{X}^*)$.

Sketch of duality proof, à la Monk–Wang

$$\|(w,\boldsymbol{\tau})\|_{\mathbf{X}^*}=\sup_{\mathbf{0}\neq(\psi,\boldsymbol{\Phi})\in\mathbf{X}}\frac{\int_Q(w\psi+\boldsymbol{\tau}\cdot\boldsymbol{\Phi})\,\mathrm{d}x\,\mathrm{d}t}{\|(\psi,\boldsymbol{\Phi})\|_{\mathbf{X}}}.$$

$(w,\boldsymbol{\tau})$ Trefftz, TDG error,
$(\psi,\boldsymbol{\Phi})$ "dual" IBVP source,
$(z,\boldsymbol{\zeta})$ "dual" IBVP solution.

$$\int_Q(w\psi+\boldsymbol{\tau}\cdot\boldsymbol{\Phi})\,\mathrm{d}V=\sum_{K\in\mathcal{T}_h}\int_K\left(w\nabla\cdot\boldsymbol{\zeta}+c^{-2}w\frac{\partial z}{\partial t}+\boldsymbol{\tau}\cdot\nabla z+\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\zeta}}{\partial t}\right)\mathrm{d}V$$

$$=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(w\boldsymbol{\zeta}\cdot\mathbf{n}_K^x+\boldsymbol{\tau}\cdot\mathbf{n}_K^x z+c^{-2}wzn_K^t+\boldsymbol{\tau}\cdot\boldsymbol{\zeta}n_K^t\right)\mathrm{d}S \qquad \text{(IBP \& Trefftz)}$$

$$=\int_{\mathcal{F}_h^{\text{space}}}\underbrace{[\![w\boldsymbol{\zeta}+\boldsymbol{\tau}z]\!]_{\mathbf{N}}+[\![c^{-2}wz+\boldsymbol{\tau}\cdot\boldsymbol{\zeta}]\!]_t}_{\le c^{-1}|[\![w]\!]_t|(\gamma|\boldsymbol{\zeta}|+c^{-1}|z|)+|[\![\boldsymbol{\tau}]\!]_t|(\gamma c^{-1}|z|+|\boldsymbol{\zeta}|)}\,\mathrm{d}S$$

$$+\int_{\mathcal{F}_h^T}\left(c^{-2}wz+\boldsymbol{\tau}\cdot\boldsymbol{\zeta}\right)\mathrm{d}S-\int_{\mathcal{F}_h^0}\left(c^{-2}w\underbrace{z}_{=0}+\boldsymbol{\tau}\cdot\underbrace{\boldsymbol{\zeta}}_{=\mathbf{0}}\right)\mathrm{d}S$$

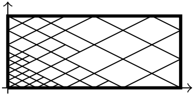
$$+\int_{\mathcal{F}_h^{\text{time}}}\underbrace{[\![w\boldsymbol{\zeta}+\boldsymbol{\tau}z]\!]_{\mathbf{N}}}_{=[\![w]\!]_{\mathbf{N}}\cdot\boldsymbol{\zeta}+[\![\boldsymbol{\tau}]\!]_{\mathbf{N}}z}\,\mathrm{d}S+\int_{\mathcal{F}_h^D}\left(w\boldsymbol{\zeta}\cdot\mathbf{n}_\Omega^x+\boldsymbol{\tau}\cdot\mathbf{n}_\Omega^x\underbrace{z}_{=0}\right)\mathrm{d}S$$

$$\le |||(w,\boldsymbol{\tau})|||_{\mathrm{DG}}\cdot\left(L^2\text{ norms of skeleton traces of } z,\boldsymbol{\zeta}\right)^{1/2}$$

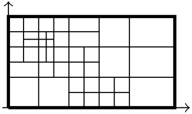
When does adjoint stability hold?

1. In 1D, using Gronwall + energy + integration by parts
$\Rightarrow$ explicit bound for $\mathbf{X} = \mathbf{X}^* = L^2(Q)^{1+n}$

2. nD, no time-like faces ($\mathcal{F}_h^{\text{time}} = \emptyset$), impedance BCs only,
$\Rightarrow$ explicit bound for $\mathbf{X} = \mathbf{X}^* = L^2(Q)^{1+n}$



3. 3D, Dirichlet BCs, "space $\times$ time" elements
$\Rightarrow$ $\mathbf{X}^* = H^{-1}(0,T;L^2(\Omega)) \times L^2(0,T;H^{-1}(\Omega)^3)$



Difficulty: bounding trace $\|\boldsymbol{\zeta}\cdot\mathbf{n}_x\|_{L^2(\mathcal{F}_h^{\text{time}})}$, $\boldsymbol{\zeta} \in L^2(0,T;H(\mathrm{div};\Omega))$.

In all cases, if $\alpha_{|K_1\cap K_2} \sim \beta_{|K_1\cap K_2} \sim \dfrac{\max_{K\in\mathcal{T}_h} h_K^x}{\min\{h_{K_1}^x, h_{K_2}^x\}}$, then
$C \sim (1/\max_{K\in\mathcal{T}_h}\{h_K^x\} + \#^{\text{space}}_{\text{interfaces}})^{1/2}$.

Part IV

Discrete Trefftz spaces

Polynomial Trefftz spaces

If $n \geq 2$, not all solutions $(v, \boldsymbol{\sigma})$ of $\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}$, $\nabla \cdot \boldsymbol{\sigma} + \frac{1}{c^2}\frac{\partial v}{\partial t} = 0$ satisfy $(v, \boldsymbol{\sigma}) = (\frac{\partial}{\partial t}U, -\nabla U)$ for U solution of $\Delta U - c^{-2}\frac{\partial^2 U}{\partial t^2} = 0$
(e.g. if $\operatorname{curl} \boldsymbol{\sigma}_0 \neq \mathbf{0}$).

Define two local polynomial Trefftz spaces on mesh element K

$$\begin{aligned}
\mathbb{T}^p(K) &:= \mathbf{T}(K) \cap \mathbb{P}^p(\mathbb{R}^{n+1})^{1+n} \\
&= \left\{(w, \boldsymbol{\tau}) \in \mathbb{P}^p(\mathbb{R}^{n+1})^{1+n} : \nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} = \mathbf{0}, \nabla \cdot \boldsymbol{\tau} + c^{-2}\frac{\partial w}{\partial t} = 0\right\} \\
\mathbb{W}^p(K) &:= \left\{\left(\frac{\partial U}{\partial t}, -\nabla U\right) :\ U \in \mathbb{P}^{p+1}(\mathbb{R}^{n+1}), \Delta U + c^{-2}\frac{\partial^2 U}{\partial t^2} = 0\right\}.
\end{aligned}$$

$$\mathbb{W}^p(K) \subset \mathbb{T}^p(K), \qquad \text{equality} \iff n = 1.$$

We can use $\mathbb{W}^p$ if IBVP at hand comes from a 2nd-order IBVP.

$$\begin{aligned}
&\dim\left(\mathbb{W}^p(K)\right) = \tfrac{2p+n+2}{p+1}\tbinom{p+n}{n} - 1 = \mathcal{O}_{p\to\infty}(p^n) \\
\leq &\dim\left(\mathbb{T}^p(K)\right) = (n+1)\tbinom{p+n}{n} = \mathcal{O}_{p\to\infty}(p^n) \\
\ll &\dim\left(\mathbb{P}^p(K)^{1+n}\right) = (p+n+1)\tbinom{p+n}{n} = \mathcal{O}_{p\to\infty}(p^{n+1})
\end{aligned}$$

Bases for $\mathbb{T}^p(K)$ and $\mathbb{W}^p(K)$

We generate basis of $\mathbb{T}^p$ by "evolving" polynomial initial conditions. Elements are in the form

$$(v, \boldsymbol{\sigma}) = \sum_{k\in\mathbb{N}_0, \boldsymbol{\alpha}\in\mathbb{N}_0^n,\ k+|\boldsymbol{\alpha}|\le p} \Big(a_{v,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k,\ a_{\sigma_1,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k,\ \dots,\ a_{\sigma_n,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k\Big),$$

for $a_{v,k,\boldsymbol{\alpha}}, a_{\sigma_1,k,\boldsymbol{\alpha}}, \dots, a_{\sigma_n,k,\boldsymbol{\alpha}} \in \mathbb{R}$ satisfying recurrence relations

$$\begin{aligned} a_{v,k,\boldsymbol{\alpha}} &= -\frac{c^2}{k}\sum_{m=1}^{n} (\alpha_m+1) a_{\sigma_m,k-1,\boldsymbol{\alpha}+\mathbf{e}_m}, & k &= 1,\dots,p, \\ a_{\sigma_m,k,\boldsymbol{\alpha}} &= -\frac{1}{k}(\alpha_m+1) a_{v,k-1,\boldsymbol{\alpha}+\mathbf{e}_m}, & |\boldsymbol{\alpha}| &\le p-k-1. \end{aligned}$$

Basis of $\mathbb{W}^p$: $\quad \mathbb{W}^p(K) := \operatorname{span}\left\{\left(\frac{\partial U_{j,\ell}}{\partial t}, -\nabla U_{j,\ell}\right),\ {\scriptstyle 1\le j\le p+1, \atop 1\le \ell \le \frac{2j+n-1}{j+n-1}\binom{j+n-1}{j}}\right\}$

$$U_{j,\ell}(\mathbf{x},t) := (\mathbf{d}_{j,\ell}\cdot\mathbf{x} - ct)^j.$$

Choice of directions $\mathbf{d}_{j,\ell}$: (corresponding to homog. polyn. deg. j)

- $n=1$, left/right directions $\mathbf{d}_{j,1}=1, \mathbf{d}_{j,2}=-1$, $\mathbb{T}^p(K)=\operatorname{span}\{(x\pm ct)^j\}$;
- $n=2$, any distinct $\{\mathbf{d}_{j,\ell}\}_{\ell=1,\dots,2j+1}$ give a basis;
- $n\ge 3$, $(\mathbf{d}_{j,\ell}\cdot\mathbf{x} - ct)^j$ linearly indep. $\iff$ $[Y_N^m(\mathbf{d}_{j,\ell})]_{N\le j, m;\ell}$ full rank.

h-approximation and convergence

Explicit h-approximation bounds follow from Bramble–Hilbert, since Taylor polynomials are in $\mathbb{T}^p/\mathbb{W}^p$.

On K star-shaped wrt $\{(\mathbf{x},t), |\mathbf{x}|^2+c^2|t|^2<\rho_K^2 h_K^2\}$, $\quad h_K=\operatorname{diam} K$

$(\upsilon,\boldsymbol{\sigma})\in\mathbf{T}(K)\cap H^{s+1}(K)^{1+n} \Rightarrow \exists(w_{hp},\boldsymbol{\tau}_{hp})\in\mathbb{T}^p(K)$ s.t. $\forall j\le m:=\min\{p,s\}$

$$|(\upsilon-w_{hp},\boldsymbol{\sigma}-\boldsymbol{\tau}_{hp})|_{H_c^j(K)}\le 2\binom{n+j}{n}\frac{(n+1)^{m+1-j}}{(m-j)!}\frac{h_K^{m+1-j}}{\rho^{(n+1)/2}}|(\upsilon,\boldsymbol{\sigma})|_{H_c^{m+1}(K)}.$$

If $(\upsilon,\boldsymbol{\sigma})=(\frac{\partial U}{\partial t},-\nabla U)$, same bound holds with $(w_{hp},\boldsymbol{\tau}_{hp})\in\mathbb{W}^p(K)$ (up to factor $\frac{(n+j+1)\min\{n+1,j\}}{j+1}$).

Combined with quasi-optimality $\to$ convergence bounds, e.g.

$|||(\upsilon-\upsilon_{hp},\boldsymbol{\sigma}-\boldsymbol{\sigma}_{hp})|||_{DG}$ (using $\mathbf{V}_p(\mathcal{T}_h)=\mathbb{T}^p(\mathcal{T}_h)$)

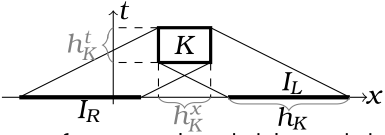
$$\le\sum_{K\in\mathcal{T}_h}\frac{36\sqrt{2}}{\rho_K^{1+n/2}}\frac{\big((n+1)h_K\big)^{s_K+\frac12}}{(s_K-1)!}\Big|(c^{-1/2}\upsilon,c^{1/2}\boldsymbol{\sigma})\Big|_{H_c^{s_K+1}(K)}$$

$\rho_K=$ “chunkiness”, $\alpha^{-1}=\beta=c$, $1\le s_K\le p_K$, (Cartesian mesh).

hp-approximation and convergence in 1+1D

In 1D, solution expands in left- and right-propagating waves:

$$\begin{cases} v(x,t) = \frac{c}{2}\big(u_R(x-ct) + u_L(x+ct)\big), \\ \sigma(x,t) = \frac{1}{2}\big(u_R(x-ct) - u_L(x+ct)\big), \end{cases} \qquad \begin{cases} u_R = \frac{v}{c} + \sigma, \\ u_L = \frac{v}{c} - \sigma. \end{cases}$$



u_R, u_L are functions of one real variable on intervals I_R, I_L:
1D polynomial hp-approx. bounds "transported" to $\mathbb{T}^p(K)$.

1D hp-convergence estimate

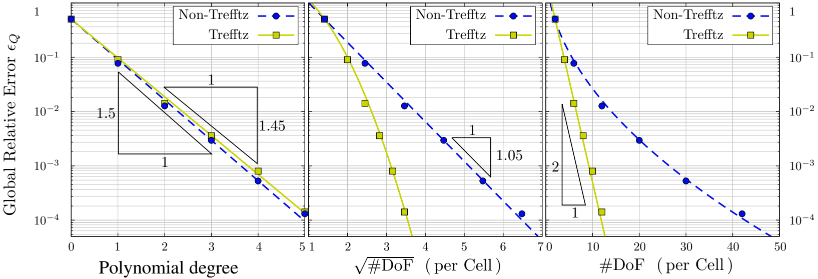
$$|||(v - v_{hp}, \sigma - \sigma_{hp})|||_{DG} \le 87 \sum_{K \in \mathcal{T}_h} \frac{(2h_K)^{s_K + \frac{3}{2}}}{p_K^{s_K}} \left|(c^{-1}v, \sigma)\right|_{W_c^{s_K+1,\infty}(K)}$$

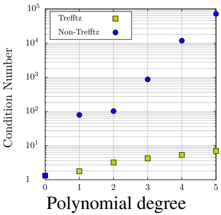
with $K = (x_K, x_K + h_K) \times (t_K, t_K + h_K/c)$, $\alpha^{-1} = \beta = c$, $1 \le s_K \le p_K$

- Exponential convergence for analytic solutions:
  $\sim \exp(-b\#\mathrm{DOFs})$ instead of $\exp(-b\sqrt{\#\mathrm{DOFs}})$.

Numerical example

Gaussian wave, uniform mesh of squares, p-convergence:





Very weak dependence on flux parameters, even for $\alpha, \beta = 0$.

Part V

Extensions

Maxwell's equations

$$\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} = \mathbf{0}, \qquad \nabla \times \mathbf{H} - \frac{\partial(\epsilon \mathbf{E})}{\partial t} = \mathbf{0} \quad \text{in } Q \subset \mathbb{R}^{3+1},$$

$$\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E} = \mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}(\mathbf{x}, t) \qquad \text{Dirichlet/PEC BCs,}$$

$$\begin{cases} [\![\mathbf{v}]\!]_t := (\mathbf{v}^- - \mathbf{v}^+) \\ [\![\mathbf{v}]\!]_{\mathbf{T}} := \mathbf{n}_{K_1}^{\mathbf{x}} \times \mathbf{v}_{|_{K_1}} + \mathbf{n}_{K_2}^{\mathbf{x}} \times \mathbf{v}_{|_{K_2}} \end{cases} \qquad \text{(tangential) jumps.}$$

Trefftz-DG formulation:

$$\begin{aligned}
\mathcal{A}_\mathcal{M}(\mathbf{E}_{hp}, \mathbf{H}_{hp}; \mathbf{v}, \mathbf{w}) = & \int_{\mathcal{F}_h^{\text{space}}} \Big(\epsilon \mathbf{E}_{hp}^- \cdot [\![\mathbf{v}]\!]_t + \mu \mathbf{H}_{hp}^- \cdot [\![\mathbf{w}]\!]_t - \mathbf{E}_{hp}^- \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \mathbf{H}_{hp}^- \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} \Big) \, \mathrm{d}S \\
& + \int_{\mathcal{F}_h^T} (\epsilon \mathbf{E}_{hp} \cdot \mathbf{v} + \mu \mathbf{H}_{hp} \cdot \mathbf{w}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} \big(\mathbf{H}_{hp} + \alpha(\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E}_{hp})\big) \cdot (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v}) \, \mathrm{d}S \\
& + \int_{\mathcal{F}_h^{\text{time}}} \Big(- \{\!\{\mathbf{E}_{hp}\}\!\} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \{\!\{\mathbf{H}_{hp}\}\!\} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \alpha [\![\mathbf{E}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \beta [\![\mathbf{H}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} \Big) \, \mathrm{d}S, \\
\ell_\mathcal{M}(\mathbf{v}, \mathbf{w}) = & \int_{\mathcal{F}_h^0} (\epsilon \mathbf{E}_0 \cdot \mathbf{v} + \mu \mathbf{H}_0 \cdot \mathbf{w}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}) \cdot \big(- \mathbf{w} + \alpha(\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v}) \big) \, \mathrm{d}S.
\end{aligned}$$

Well-posedness and stability identical to wave equation.
Explicit approximation bounds in h. Impedance BCs also fine.
Error bounds in $L^2(Q)^6$ for tent-pitched meshes and impedance.

Symmetric hyperbolic systems

As in MONK–RICHTER: piecewise-constant $\mathsf{A} > 0$, constant A_j

$$\mathsf{A}\mathbf{u}_t + \sum_j \mathsf{A}_j \mathbf{u}_{x_j} = \mathbf{0} \qquad \text{in } \Omega \times (0,T),$$
$$(\mathsf{D}-\mathsf{N})\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega \times (0,T),$$
$$\mathbf{u} = \mathbf{u}_0 \qquad \text{on } \Omega \times \{0\},$$

$$\mathsf{D}|_{\partial K} := \sum_j n_K^j \mathsf{A}_j,$$
+conditions on N.

Decomposition $\quad \mathsf{M}|_{\partial K} := n_K^t \mathsf{A} + \sum_j n_K^j \mathsf{A}_j = \mathsf{M}_K^+ + \mathsf{M}_K^- \quad$ such that
$M^+ \geq 0$, $\; M^- \leq 0$, $\; \mathsf{M}_{K_1}^+ + \mathsf{M}_{K_2}^- = 0$ on $\partial K_1 \cap \partial K_2$, leads to

$$\mathcal{A}(\mathbf{u},\mathbf{w}) = \sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} \mathbf{u}_1 \cdot \mathsf{M}_{K_1}^+ (\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}S + \int_{\mathcal{F}_h^T} \mathbf{u} \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S$$
$$+ \frac{1}{2} \int_{\partial\Omega \times (0,T)} (\mathsf{D}+\mathsf{N})\mathbf{u} \cdot \mathbf{w}\,\mathrm{d}S,$$

$$\ell(\mathbf{w}) = -\int_{\mathcal{F}_h^0} \mathbf{u}_0 \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S - \frac{1}{2} \int_{\partial\Omega \times (0,T)} \mathbf{g} \cdot \mathbf{w}\,\mathrm{d}S.$$

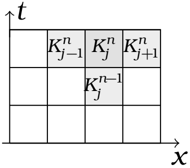
$$|||\mathbf{u}|||_{DG}^2 := \mathcal{A}(\mathbf{u},\mathbf{u}) = \sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} (\mathbf{u}_1 - \mathbf{u}_2) \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}S$$
$$+ \int_{\mathcal{F}_h^T \cup \mathcal{F}_h^0} \mathbf{u} \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} \mathbf{u}\,\mathrm{d}S + \frac{1}{2} \int_{\partial\Omega \times (0,T)} \mathbf{u} \cdot \mathsf{N}\mathbf{u}\,\mathrm{d}S.$$

Relation with UWVF and finite differences

With $\alpha c = \beta / c = \delta = 1/2$, TDG operator reads $(\mathrm{Id} - F^* \Pi)$, F isometry, Π "trace-flipping", as in Cessenat–Despres' UWVF. True in 1+1D; only formally because of a trace issue in n+1D...

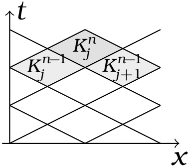
In 1+1D, without BCs, with piecewise constant basis, on Cartesian-product mesh, (implicit) TDG reads:

$$\frac{1}{c^2}\,\frac{v_j^n - v_j^{n-1}}{h_t} + \frac{\sigma_{j+1}^n - \sigma_{j-1}^n}{2h_x} = \alpha h_x\,\frac{v_{j-1}^n + v_{j+1}^n - 2v_j^n}{h_x^2},$$
$$\frac{\sigma_j^n - \sigma_j^{n-1}}{h_t} + \frac{v_{j+1}^n - v_{j-1}^n}{2h_x} = \beta h_x\,\frac{\sigma_{j-1}^n + \sigma_{j+1}^n - 2\sigma_j^n}{h_x^2},$$



On a uniform rhombic mesh, with piecewise constant basis, (explicit) TDG is Lax–Friedrichs:

$$v_j^n = \frac{v_j^{n-1} + v_{j+1}^{n-1}}{2} - c^2 h_t \frac{\sigma_{j+1}^{n-1} - \sigma_j^{n-1}}{2h_x},$$
$$\sigma_j^n = \frac{\sigma_j^{n-1} + \sigma_{j+1}^{n-1}}{2} - h_t \frac{v_{j+1}^{n-1} - v_j^{n-1}}{2h_x},$$



Extensions and open problems

We have described and (a priori) analysed a Trefftz scheme for the wave equation.
Basis functions are piecewise-solution polynomials.

See arXiv:1610.08002.

- More general space–time meshes (not aligned to t);
- non/less dissipative methods (is our dissipation too much?);
- analysis of non-penalised methods ($\alpha = \beta = 0$);
- mesh-independent stability in more general cases;
- Maxwell, elasticity, first-order hyperbolic systems, dispersive/Drude-type models for plasmas, . . . ;
- Trefftz hp-approximation theory in dimensions > 1;
- other bases: non-polynomial, trigonometric, directional. . . ;
- (directional) adaptivity;
- . . .

Thank you!