INRIA/LJLL Paris — 2.11.2026

A discontinuous Galerkin method with fractal elements

Andrea Moiola

http://matematica.unipv.it/moiola/



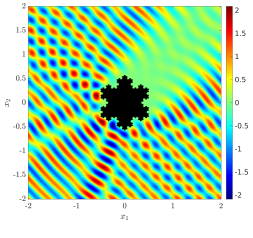
UNIVERSITÀ DI PAVIA
Department of Mathematics
"Felice Casorati"

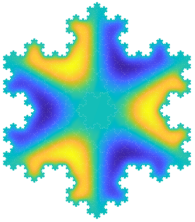
S. Gómez (Milano Bicocca) — D.P. Hewett (UCL)

arXiv:2604.11093 — :FractalDG

Aims and motivation

Aim: develop, implement and analyse numerical methods for the solution of PDEs on domains with fractal boundary.

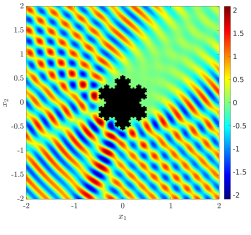


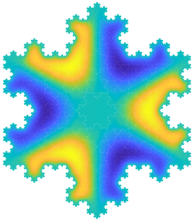


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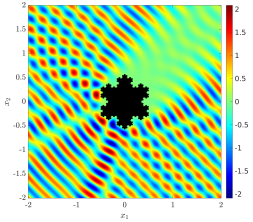
Why?

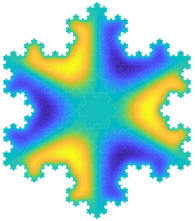




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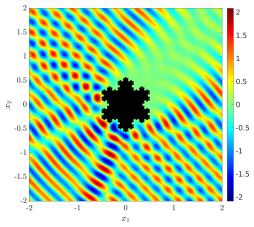
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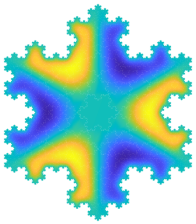
1. *Because it's there*: mathematical interest.



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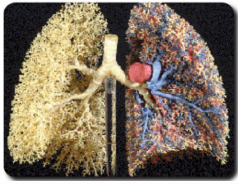


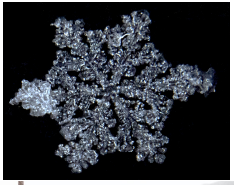
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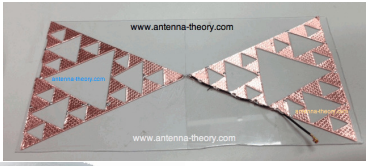
1. *Because it's there*: mathematical interest.
2. Fractals provide natural models for structures with "multiscale roughness".

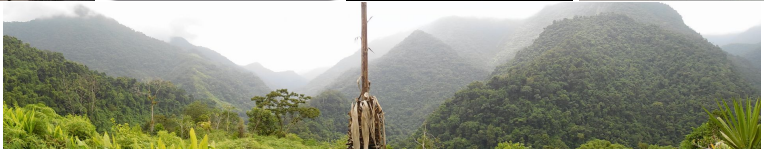












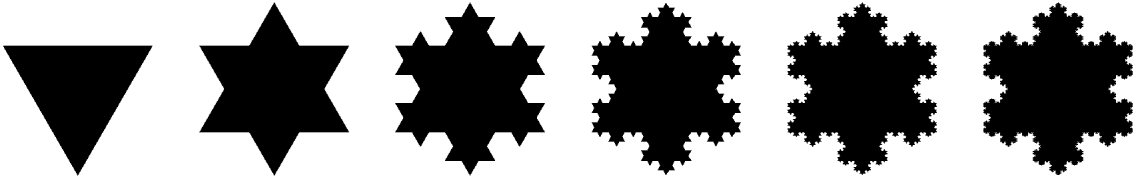
Fractals and prefractals

Fractals are often defined as the limit of a sequence of smoother "prefractals", generated by a recursive procedure.

Koch curve:



Koch snowflake:



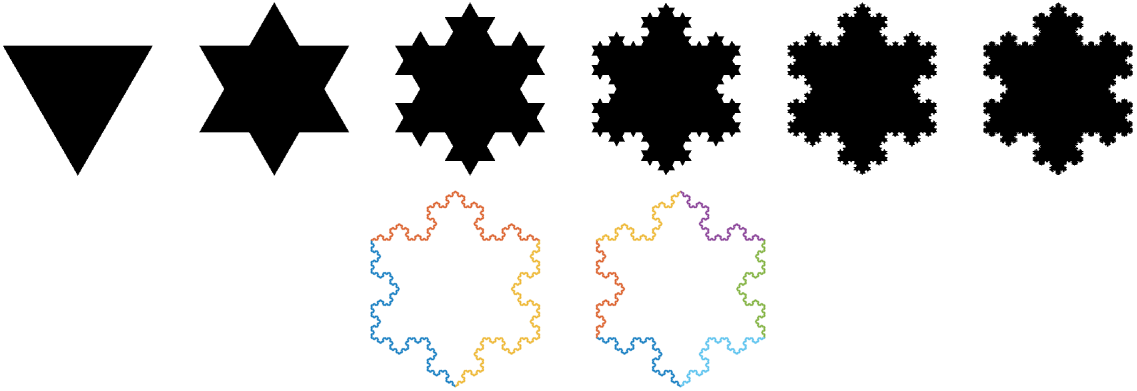
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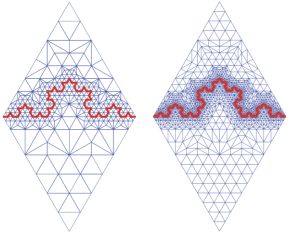


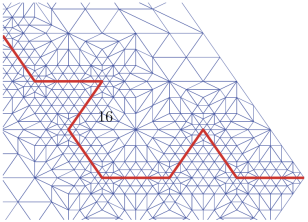
Numerics on prefractals

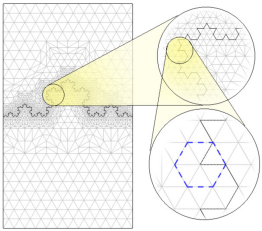
Idea: Approximate fractal by a polygonal prefractal and apply a classical method.

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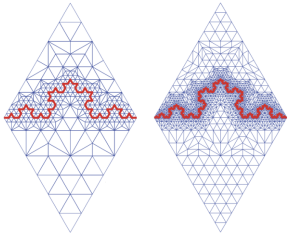


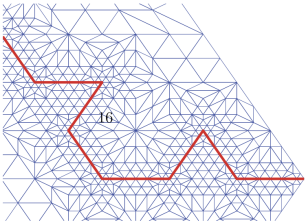
BAGNERINI, BUFFA, VACCA, *Mesh generation and numerical analysis of a Galerkin method for highly conductive prefractal layers*, Appl. Numer. Math., 2013.

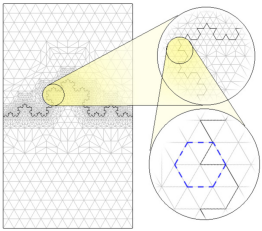
CEFALO, LANCIA, *An optimal mesh generation algorithm for domains with Koch type boundaries*, Math. Comput. Simul., 2014

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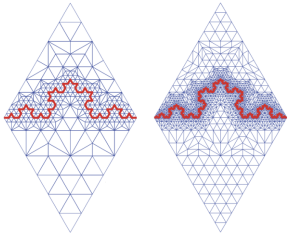
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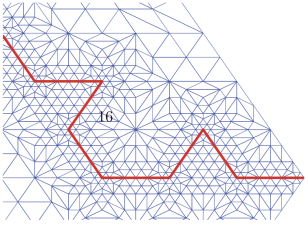
Drawbacks:

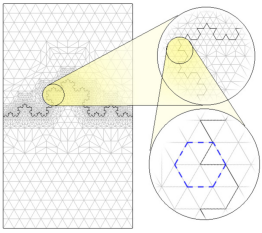
- may obtain extremely complicated meshes
- geometry approximation error worse than for smooth domains
- difficult to obtain error bounds quantifying the solution accuracy

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Question: Can we find an alternative approach, *avoiding* prefractal approximation?

Iterated function system (IFS) attractors

Our two examples are IFS attractors: (HUTCHINSON 1981)

Non-empty compact sets $K \subset \mathbb{R}^n$ such that $K = \bigcup_{m=1}^{M} s_m(K)$,

for a collection $\{s_1, \ldots, s_M\}$ of contracting similarities $s_m(\mathbf{x}) = \rho_m A_m \mathbf{x} + \delta_m$,

with $A_m \in \mathbb{R}^{n \times n}$ orthogonal, $\rho_m \in (0, 1)$ and $\delta_m \in \mathbb{R}^n$, $m = 1, \ldots, M$.

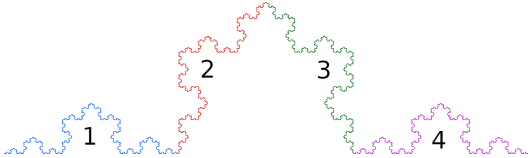
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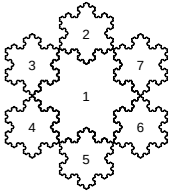
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Koch curve

$M = 4, \quad \rho_1 = \ldots = \rho_4 = 1/3$

$d = \log 4 / \log 3$



Koch snowflake

$M = 7, \quad \rho_1 = 1/\sqrt{3}, \rho_2 = \ldots = \rho_7 = 1/3$

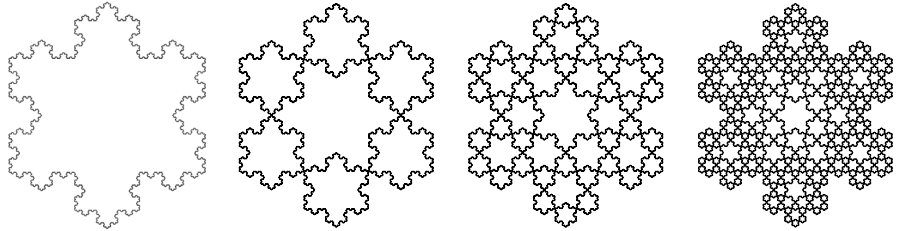
$d = 2$

In both cases, Hausdorff dimension $d = \dim_{\mathrm{H}}(K)$ is the unique solution of $\sum_{m=1}^{M} (\rho_m)^d = 1$.

Any compact $K \subset \mathbb{R}^n$ can be approximated by an IFS attractor. (FALCONER 2014)

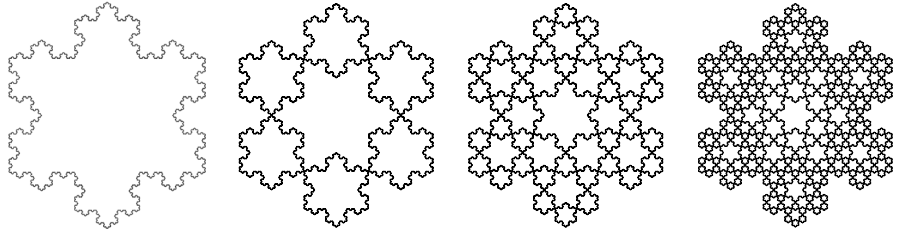
Numerics on fractals

Idea: Discretise snowflake by a geometry-conforming mesh with "fractal elements", obtained by repeated application of the IFS decomposition:

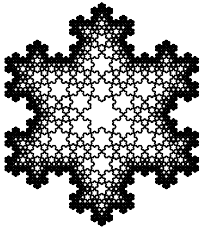


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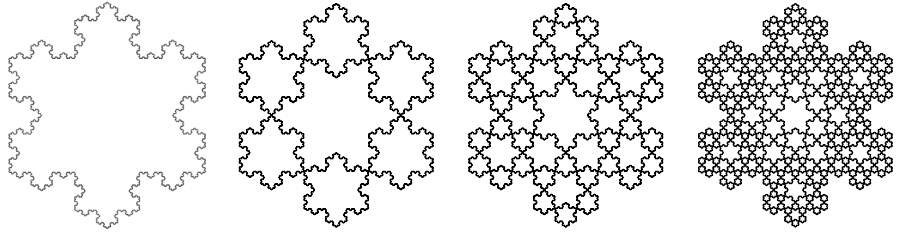


Local refinement possible:

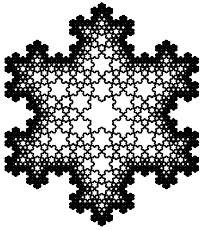


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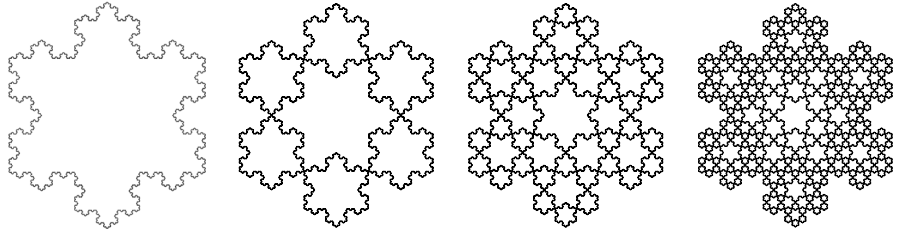
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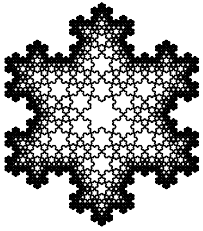
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- − Can't use continuous piecewise polynomial spaces.

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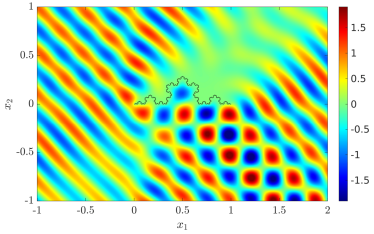
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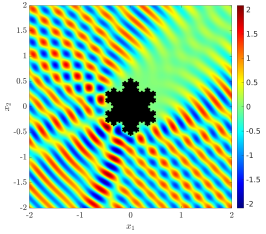
Issues to address:

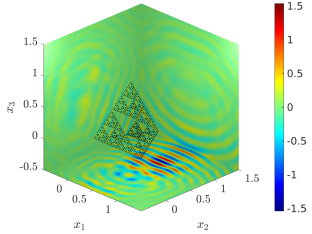
- Function spaces, regularity and approximation theory?
- Definition of "normal derivatives" on fractal interfaces?
- Quadrature?

Previous work

We have already applied geometry-conforming meshes in integral equation methods for acoustic wave scattering (Helmholtz eq. $\Delta u + k^2 u = 0$) by fractal scatterers:







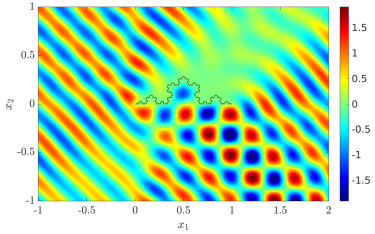
IE formulation & analysis, fully-discrete piecewise-constant Galerkin convergence rates.

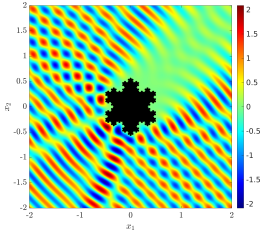
CAETANO, CHANDLER-WILDE, CLAEYS, GIBBS, HEWETT, MOIOLA, *Integral equation methods for acoustic scattering by fractals,* Proc. Roy. Soc. A, 2025

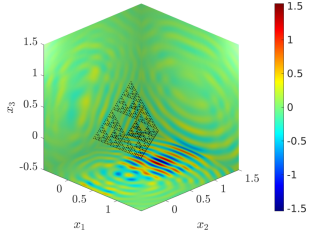
julia : IFSIntegrals

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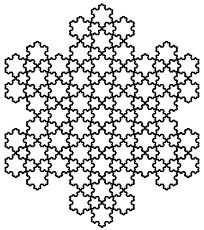
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Integral equation methods for acoustic scattering by fractals, julia : IFSIntegrals

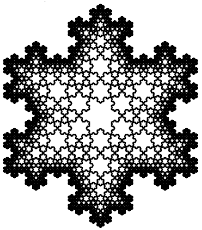
Fractional best-approximation estimates for discontinuous piecewise-polynomial spaces:

HEWETT, *Piecewise polynomial approximation on non-Lipschitz domains*, Math.Comp. 2026

Aim

Our goal is to use these meshes
in a new setting:
discontinuous Galerkin FEM

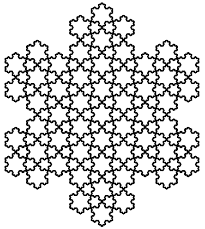


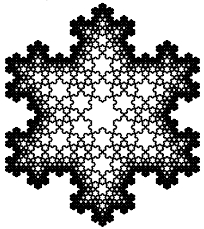


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discontinuous Galerkin FEM

- formulation
- well-posedness
- quadrature
- numerics
- error analysis





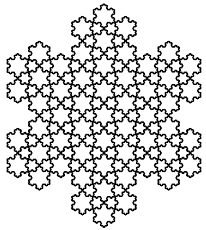
Aim

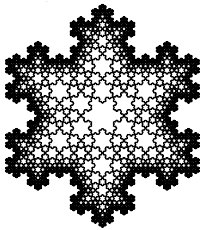
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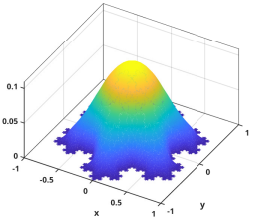
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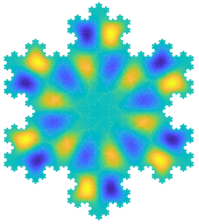
For simplicity we restrict attention to:

- the Poisson equation with Dirichlet BCs
  (and related eigenvalue problems)
- the Koch snowflake domain









Aim

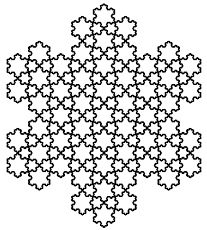
Our goal is to use these meshes
in a new setting:
discontinuous Galerkin FEM

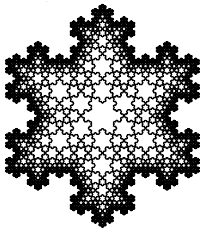
- formulation
- well-posedness
- quadrature
- numerics
- error analysis

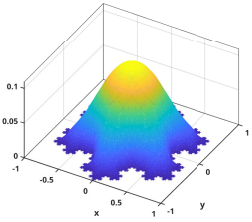
For simplicity we restrict attention to:

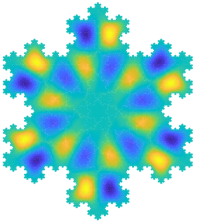
- the Poisson equation with Dirichlet BCs
  (and related eigenvalue problems)
- the Koch snowflake domain

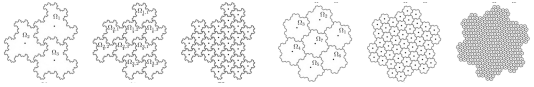
. . . but we expect the ideas generalise
to other PDEs, BCs and fractal domains











Part I

A DG-FEM with fractal elements

Poisson Dirichlet BVP and SIP-DG-FEM on a polygon

Given a bounded domain Ω and $f \in L_2(\Omega)$, find $u \in W_0^1(\Omega)$ such that

$$-\Delta u = f \quad \text{on } \Omega.$$

Poisson Dirichlet BVP and SIP-DG-FEM on a polygon

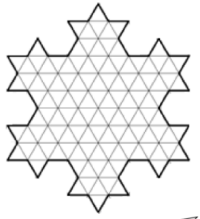
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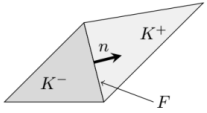
If Ω is a polygon, given a triangulation $\mathcal{T}$ of Ω and a polynomial degree $p \in \mathbb{N}_0$, the symmetric interior penalty (SIP) DG-FEM reads: $V_h = \mathbb{P}^p(\mathcal{T})$

$$\text{find} \quad u_h \in V_h \quad \text{such that} \quad a_{\text{SIP}}(u_h, v_h) = \int_\Omega f\, v_h \, \mathrm{d}\mathbf{x} \quad \forall v_h \in V_h, \quad \text{where}$$

$$a_{\text{SIP}}(u_h, v_h) := \sum_{K \in \mathcal{T}} \int_K \nabla u_h \cdot \nabla v_h \, \mathrm{d}\mathbf{x} + \sum_{F \in \mathcal{F}} \int_F \Big(\underbrace{\{\partial_{\mathbf{n}} u_h\}[v_h]}_{\text{consistency}} + \underbrace{\{\partial_{\mathbf{n}} v_h\}[u_h]}_{\text{symmetry}} + \underbrace{\alpha [u_h][v_h]}_{\text{penalty}} \Big) \, \mathrm{d}s,$$

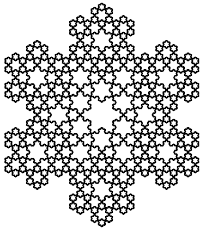


for some positive function α on the mesh skeleton.
If $F = \partial K_+ \cap \partial K_-$, then $\{v\} := \frac{1}{2}(v^+ + v^-)$ and $[v] := v^- - v^+$.
If F is a boundary face, $\{v\} = [v] = v$.



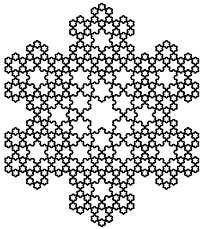
DG-FEM on a fractal mesh?

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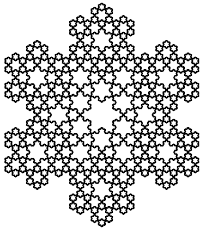
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Challenges:

DG-FEM on a fractal mesh?

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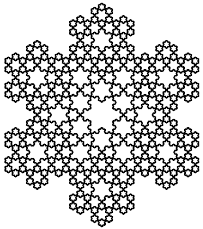


Challenges:

- **Face integrals:** The faces $F \in \mathcal{F}$ are fractal curves.
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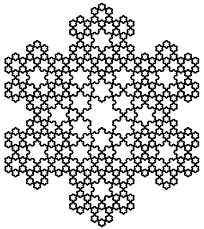


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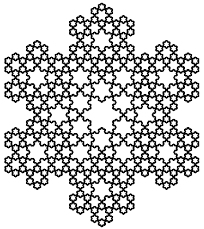


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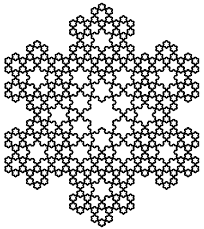


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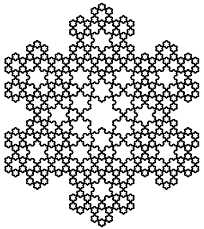


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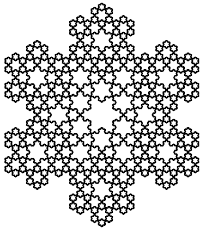


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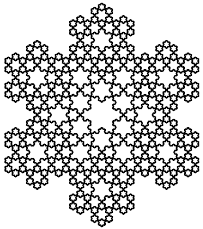


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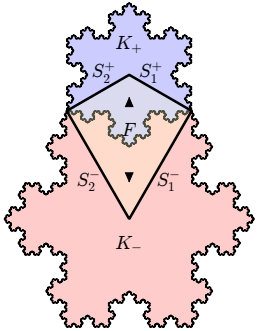
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- **Penalty term:** How should we choose α to ensure well-posedness?
  Take $\alpha = \dfrac{\eta}{h_F^d}$ for sufficiently large $\eta > 0$, $h_F := \mathrm{diam}(F)$

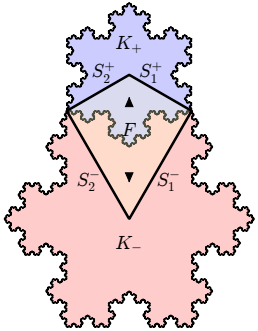
Normal derivative terms



$\vee := \boldsymbol{S}_2^- \cup \boldsymbol{S}_1^-$,
$\wedge := \boldsymbol{S}_2^+ \cup \boldsymbol{S}_1^+$

Assume mesh is "locally quasi-uniform" (LQU): $(h_K := \mathrm{diam} K)$
each internal face $F = \partial K_+ \cap \partial K_-$ has $h_{K_-}/h_{K_+} = \sqrt{3}$.

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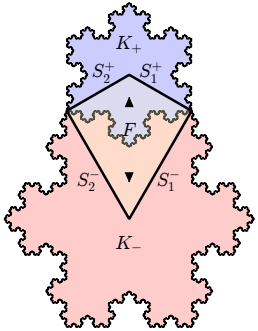
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Need to define, on each such F, proxies for

$$\int_F \partial_{\mathbf{n}_-} w\, v \, \mathrm{d}s \qquad \text{and} \qquad \int_F \partial_{\mathbf{n}_+} w\, v \, \mathrm{d}s,$$

with $\mathbf{n}_-$ and $\mathbf{n}_+$ pointing outward from K_- and K_+.

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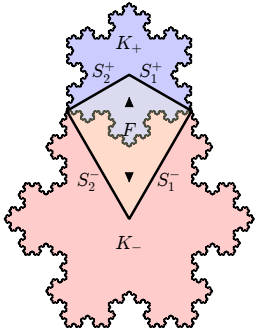
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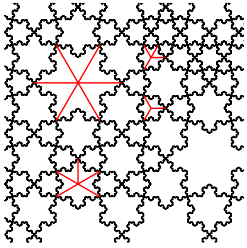
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For each F, introduce "wedges" $\blacktriangledown \subset K_-$ and $\blacktriangle \subset K_+$,
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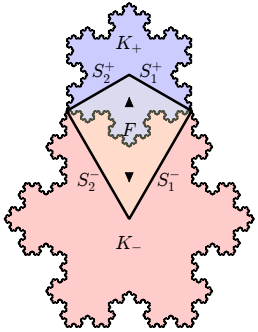
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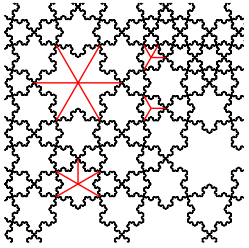
For each F, introduce "wedges" $\blacktriangledown \subset K_-$ and $\blacktriangle \subset K_+$,
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Each element splits into 3, 4, 5, or 6 wedges.

Normal derivative terms



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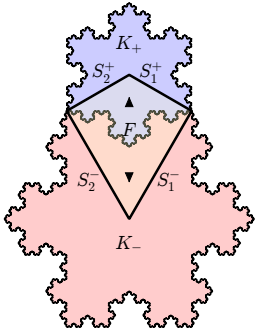
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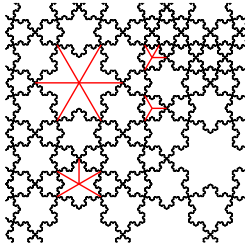
If F were a Lipschitz curve, then

$$\begin{aligned}
\int_F \partial_{\mathbf{n}_-} w\, v \mathrm{d}s &= \int_{\blacktriangledown} \mathrm{div}[\nabla w\, v] \mathrm{d}\mathbf{x} - \int_{\vee} \partial_{\mathbf{n}} w\, v \mathrm{d}s, \\
\int_F \partial_{\mathbf{n}_+} w\, v \mathrm{d}s &= \int_{\blacktriangle} \mathrm{div}[\nabla w\, v] \mathrm{d}\mathbf{x} - \int_{\wedge} \partial_{\mathbf{n}} w\, v \mathrm{d}s.
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$\wedge := S_2^+ \cup S_1^+$



Assume mesh is "locally quasi-uniform" (LQU): $(h_K := \mathrm{diam} K)$
each internal face $F = \partial K_+ \cap \partial K_-$ has $h_{K_-}/h_{K_+} = \sqrt{3}$.

Need to define, on each such F, proxies for

$$\int_F \partial_{\mathbf{n}_-} w\, v \, \mathrm{d}s \qquad \text{and} \qquad \int_F \partial_{\mathbf{n}_+} w\, v \, \mathrm{d}s,$$

with $\mathbf{n}_-$ and $\mathbf{n}_+$ pointing outward from K_- and K_+.

For each F, introduce "wedges" $\blacktriangledown \subset K_-$ and $\blacktriangle \subset K_+$,
with boundaries $F \cup \vee$ and $F \cup \wedge$.
Each element splits into 3, 4, 5, or 6 wedges.

If F were a Lipschitz curve, then

$$\int_F \partial_{\mathbf{n}_-} w\, v \mathrm{d}s := \int_{\blacktriangledown} \mathrm{div}[\nabla w\, v] \mathrm{d}\mathbf{x} - \int_{\vee} \partial_{\mathbf{n}} w\, v \mathrm{d}s,$$
$$\int_F \partial_{\mathbf{n}_+} w\, v \mathrm{d}s := \int_{\blacktriangle} \mathrm{div}[\nabla w\, v] \mathrm{d}\mathbf{x} - \int_{\wedge} \partial_{\mathbf{n}} w\, v \mathrm{d}s.$$

In the fractal case, we take the above formulas as the definitions of $\int_F \partial_{\mathbf{n}_-} w\, v \mathrm{d}s$ and $\int_F \partial_{\mathbf{n}_+} w\, v \mathrm{d}s$!

SIP-DG-FEM on the Koch snowflake

Find $u_h \in V_h = \mathbb{P}^p(\mathcal{T})$ such that $a_{\mathrm{SIP}}(u_h, v_h) = \int_\Omega f\, v_h \mathrm{d}\mathbf{x}$ $\forall v_h \in V_h$, where

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$$a_{\mathrm{SIP}}(w_h, v_h) := \sum_{K\in\mathcal{T}} \int_K \nabla w_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} \qquad w_h^\pm := w_h|_{K_\pm}$$



$$+ \sum_{F\in\mathcal{F}_I} \Big[-\frac{1}{2}\Big(\mathcal{I}_{\blacktriangledown}(w_h^-, v_h^- - v_h^+) - \mathcal{I}_{\blacktriangle}(w_h^+, v_h^- - v_h^+) \Big\} \text{consistency} \sim \int_F \{\partial_{\mathbf{n}} w_h\}[v_h]$$

$$+ \mathcal{I}_{\blacktriangledown}(v_h^-, w_h^- - w_h^+) - \mathcal{I}_{\blacktriangle}(v_h^+, w_h^- - w_h^+)\Big) \Big\} \text{symmetry} \sim \int_F \{\partial_{\mathbf{n}} v_h\}[w_h]$$

$$+ \frac{\eta}{h_F^d} \int_F (w_h^- - w_h^+)(v_h^- - v_h^+) \mathrm{d}\mathcal{H}^d \Big] \Big\} \text{penalty} \sim \int_F \frac{\eta}{h_F^d}[w_h][v_h]$$

$$+ \sum_{F\in\mathcal{F}_B} \Big[-\mathcal{I}_{\blacktriangledown}(w_h, v_h) - \mathcal{I}_{\blacktriangledown}(v_h, w_h) + \frac{\eta}{h_F^d} \int_F w_h\, v_h \,\mathrm{d}\mathcal{H}^d \Big] \Big\} \text{boundary} \sim \ldots$$

$\int_F \partial_{\mathbf{n}_\pm} w\, v \,\mathrm{d}s$ proxies

$$\mathcal{I}_{\blacktriangledown}(w, v) := \int_{\blacktriangledown} \mathrm{div}[\nabla w\, v]\,\mathrm{d}\mathbf{x} - \int_{\vee} \partial_{\mathbf{n}} w\, v \,\mathrm{d}s, \qquad \mathcal{I}_{\blacktriangle}(w, v) := \int_{\blacktriangle} \mathrm{div}[\nabla w\, v]\,\mathrm{d}\mathbf{x} - \int_{\wedge} \partial_{\mathbf{n}} w\, v \,\mathrm{d}s.$$

Coercivity and well-posedness

$$\|v\|_{\mathrm{DG}}^2 := \sum_{K\in\mathcal{T}} \|\nabla v\|_{L_2(K)}^2 + \sum_{F\in\mathcal{F}_I} \frac{1}{h_F^d}\int_F (v^- - v^+)^2 \mathrm{d}\mathcal{H}^d + \sum_{F\in\mathcal{F}_B} \frac{1}{h_F^d}\int_F v^2 \mathrm{d}\mathcal{H}^d \qquad \forall v \in V_h.$$

Let $\mathcal{T}$ be an LQU mesh and let $p \in \mathbb{N}$.
If $\eta > 0$ is sufficiently large then $a_{\mathrm{SIP}}(\cdot,\cdot)$ is coercive:

$$a_{\mathrm{SIP}}(w_h, w_h) \geq \frac{1}{2}\|w_h\|_{\mathrm{DG}}^2 \qquad \forall w_h \in V_h.$$

For any $f \in L_2(\Omega)$, the SIP-DG-FEM admits a unique solution $u_h \in V_h$.

Part II

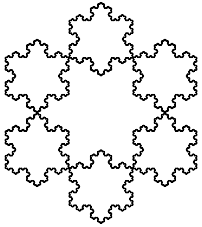
Quadrature by self-similarity

Quadrature by self-similarity

Implementation of the DG-FEM requires evaluation

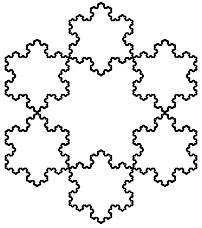
of integrals $\int_K F \, \mathrm{d}\mu$ of polynomials F

over an IFS attractor K (or a subset of it, e.g. ▼ or ▲)
with respect to a certain measure μ (either $|\cdot|$ or $\mathcal{H}^d$).



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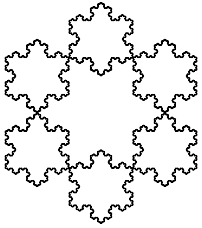


Integrals of polynomials on any IFS attractor K
can be evaluated exactly using self-similarity of K, as solutions of a small linear system.

(GIBBS, HEWETT, MOIOLA, Num. Alg. 2023),
(BARNSLEY, DEMKO 1985), (VRSCAY, ROEHRIG 1989), (STRICHARTZ 2000)...

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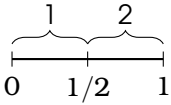
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For
$K = \bigcup_{m=1}^{M} s_m(K)$,
$s_m(\mathbf{x}) = \rho_m A_m \mathbf{x} + \delta_m$,
$d = \dim_{\mathrm{H}}(K)$,
$\mu = \mathcal{H}^d$,

$$\int_K F \, \mathrm{d}\mu = \sum_{m=1}^{M} \int_{s_m(K)} F \, \mathrm{d}\mu = \sum_{m=1}^{M} \rho_m^d \int_K F \circ s_m \, \mathrm{d}\mu$$

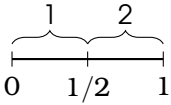
Illustrative example on $[0, 1]$

$$I[x^n] := \int_0^1 x^n \, \mathrm{d}x = \underbrace{\int_0^{1/2} x^n \, \mathrm{d}x}_{=:I_1[x^n]} + \underbrace{\int_{1/2}^1 x^n \, \mathrm{d}x}_{=:I_2[x^n]}.$$



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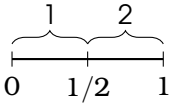


Similarities

$$s_1 : [0, 1] \to [0, 1/2], \qquad s_1(x) = x/2,$$
$$s_2 : [0, 1] \to [1/2, 1], \qquad s_2(x) = (1 + x)/2.$$

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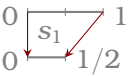


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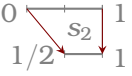
$\qquad\qquad\quad s_2 : [0,1] \to [1/2,1], \qquad s_2(x) = (1+x)/2.$

Using binomial theorem and integrand homogeneity $(F(cx) = c^n F(x))$:

$$I_1[x^n] = \int_0^{1/2} x^n \,\mathrm{d}x \overset{x=s_1(\tilde{x})}{=} \int_0^1 \left(\frac{\tilde{x}}{2}\right)^n \frac{\mathrm{d}\tilde{x}}{2} = \frac{1}{2^{n+1}} I[x^n],$$

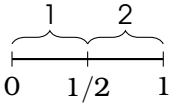


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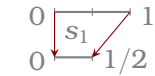


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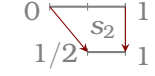
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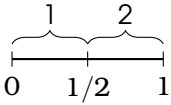
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$I[x^n] = I_1[x^n] + I_2[x^n]$ gives $\qquad \left(1 - \frac{1}{2^n}\right) I[x^n] = \frac{1}{2^{n+1}} \sum_{j=0}^{n-1} \binom{n}{j} I[x^j].$

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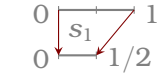


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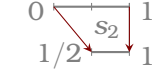
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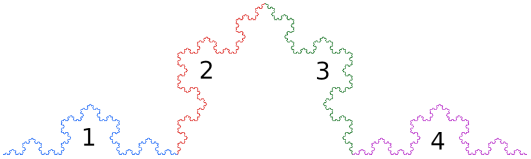


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Assuming knowledge of $I[1] = 1$ we can determine $I[x], I[x^2], \ldots, I[x^n]$ sequentially (equivalently, as the solution of a triangular linear system).

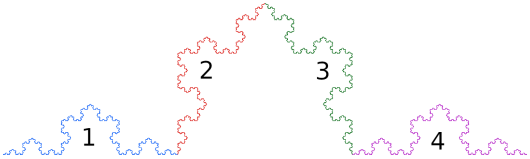
Quadrature by self-similarity: integrals over faces

We need integrals of polynomials over faces, w.r.t. $\mathcal{H}^d$, $d = \log 4 / \log 3$.



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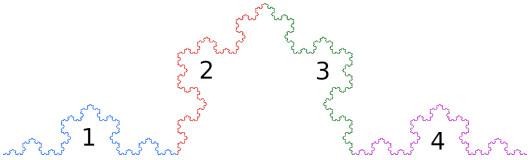


Let Γ be the Koch curve,
with $\mathcal{H}^d$ normalised so $J[1] = 1$.

$$J[F] := \int_\Gamma F \, \mathrm{d}\mathcal{H}^d = \sum_{m=1}^{4} \frac{1}{4} \int_\Gamma F \circ \boldsymbol{s}_m \, \mathrm{d}\mathcal{H}^d$$

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$$J[x] = \frac{1}{2}, \quad J[y] = \frac{1}{6\sqrt{3}},$$

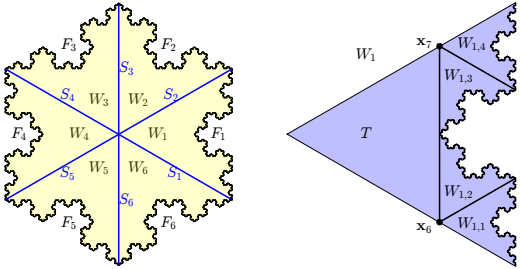
$$J[x^2] = \frac{19}{60}, \quad J[xy] = \frac{1}{12\sqrt{3}}, \quad J[y^2] = \frac{1}{60},$$

$$J[x^3] = \frac{9}{40}, \quad J[x^2y] = \frac{13}{280\sqrt{3}}, \quad J[xy^2] = \frac{1}{120}, \quad J[y^3] = \frac{1}{168\sqrt{3}},$$

$$J[x^4] = \frac{92983}{542640}, \; J[x^3y] = \frac{47}{1680\sqrt{3}}, \; J[x^2y^2] = \frac{47}{10640}, \; J[xy^3] = \frac{1}{336\sqrt{3}}, \; J[y^4] = \frac{83}{108528}.$$

Quadrature by self-similarity: integrals over wedges

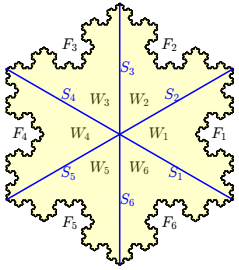
We also need the evaluation of integrals of polynomials over wedges:

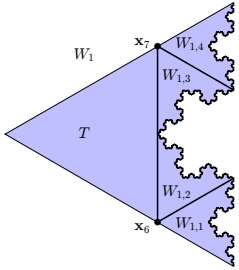


Let $I_i[F] := \int_{W_i} F \, \mathrm{d}\mathbf{x}$ for $i = 1, \ldots, 6$.

Quadrature by self-similarity: integrals over wedges

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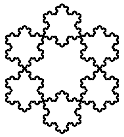
Let $I_i[F] := \int_{W_i} F \, \mathrm{d}\mathbf{x}$ for $i = 1, \ldots, 6$. Then,

$$I_1[1] = \frac{\sqrt{3}}{5}, \quad I_1[x] = \frac{11}{60}, \quad I_1[y] = 0,$$
$$I_1[x^2] = \frac{281\sqrt{3}}{4400}, \quad I_1[xy] = 0, \quad I_1[y^2] = \frac{39\sqrt{3}}{4400},$$

and $I_i[\cdot]$, $i = 2, \ldots, 6$, are a linear combination of these.

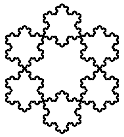
Integrals over elements

$\int_K \nabla w_h \cdot \nabla v_h \, \mathrm{d}\mathbf{x}$ term in DG-FEM formulation computed using 7 contractions:



$$\int_\Omega 1 \, \mathrm{d}\mathbf{x} = \frac{6\sqrt{3}}{5}, \qquad \int_\Omega x \, \mathrm{d}\mathbf{x} = \int_\Omega y \, \mathrm{d}\mathbf{x} = \int_\Omega xy \, \mathrm{d}\mathbf{x} = 0, \qquad \int_\Omega x^2 \, \mathrm{d}\mathbf{x} = \int_\Omega y^2 \, \mathrm{d}\mathbf{x} = \frac{12\sqrt{3}}{55}.$$

Integrals over elements



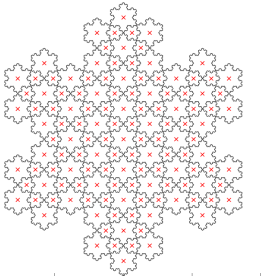
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$$\int_\Omega 1 \, \mathrm{d}\mathbf{x} = \frac{6\sqrt{3}}{5}, \qquad \int_\Omega x \, \mathrm{d}\mathbf{x} = \int_\Omega y \, \mathrm{d}\mathbf{x} = \int_\Omega xy \, \mathrm{d}\mathbf{x} = 0, \qquad \int_\Omega x^2 \, \mathrm{d}\mathbf{x} = \int_\Omega y^2 \, \mathrm{d}\mathbf{x} = \frac{12\sqrt{3}}{55}.$$

The only integral involving non-polynomial integrands is source term $\int_\Omega f \, v_h \, \mathrm{d}\mathbf{x}$.

We use composite barycentre rule:
evaluate integrand in barycentres of IFS uniform refinement

Applicable to all IFS using only contraction parameters,
error estimates in:
(GIBBS, HEWETT, MOIOLA, Num. Alg. 2023)

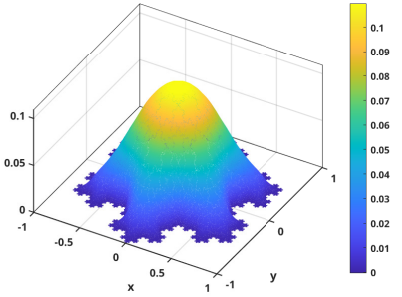


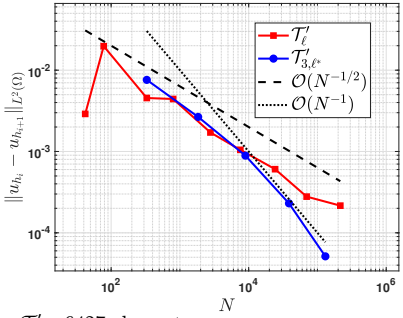
Part III

Numerical results

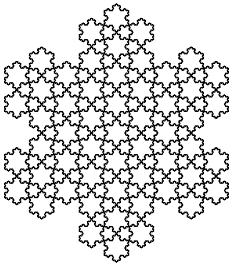
Poisson problem

Numerical solution of $-\Delta u = 1$ on Ω with $u = 0$ on $\partial\Omega$, for $p = 2$, $\eta = 10$:

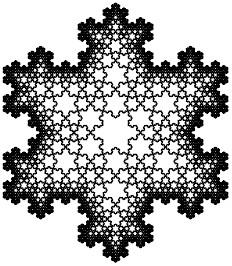




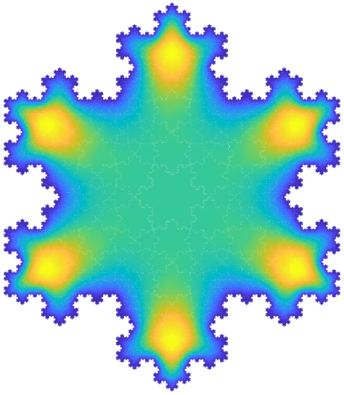
$\mathcal{T}'_4$, 133 elements



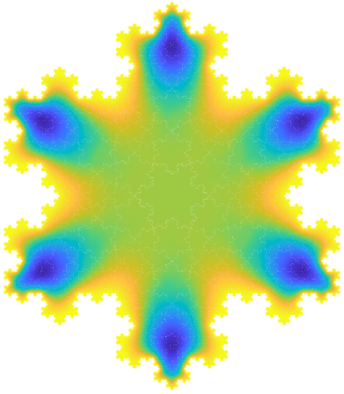
$\mathcal{T}'_{3,3}$, 6427 elements



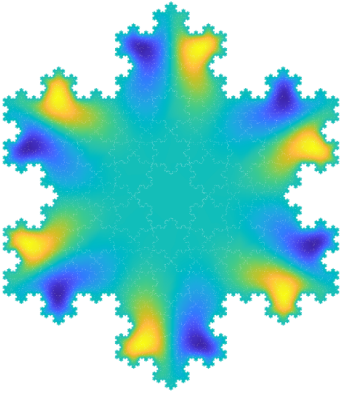
Other solutions



$-\Delta u = |\mathbf{x}|^6$



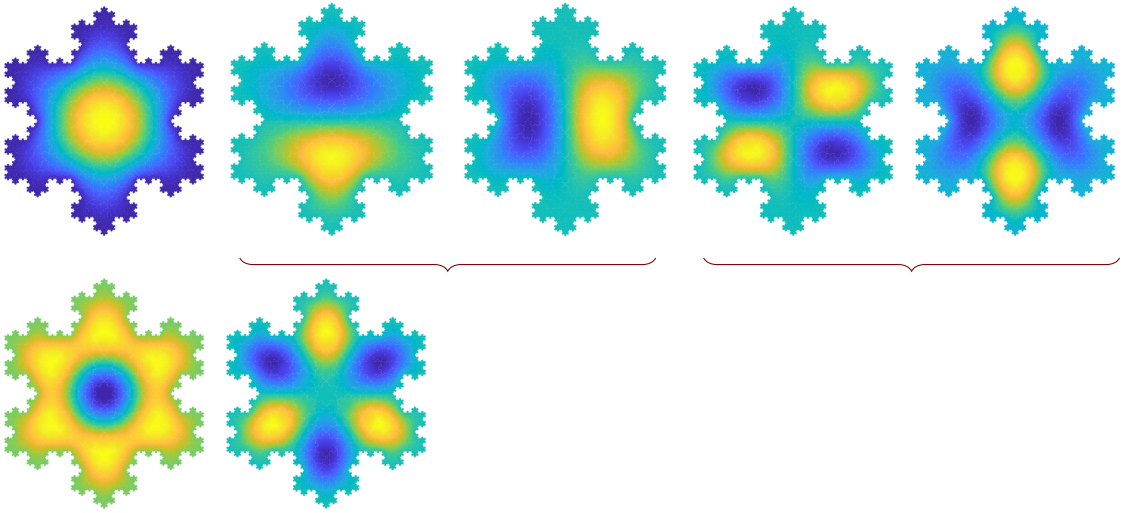
$-\Delta u = \Re\{(x+\mathrm{i}y)^6\}$



$-\Delta u = \Im\{(x+\mathrm{i}y)^6\}$

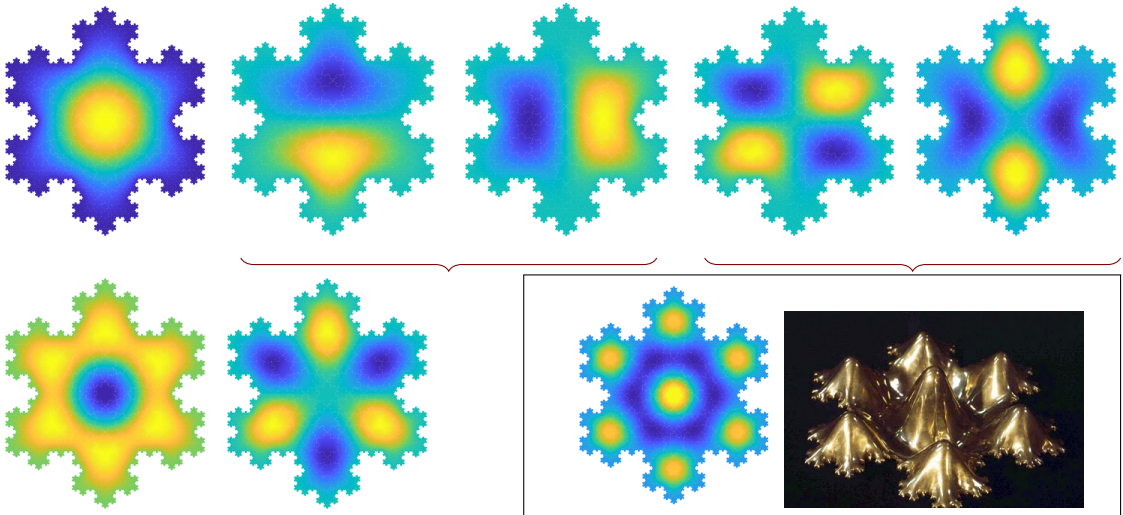
Eigenfunctions

Approximate solutions of $-\Delta u = \lambda u$ on Ω, with $u = 0$ on $\partial\Omega$



Eigenfunctions

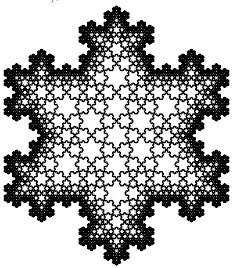
Approximate solutions of $-\Delta u = \lambda u$ on Ω, with $u = 0$ on $\partial\Omega$



Bronze sculpture of 13th eigenfunction
by Helaman Ferguson

Eigenvalues

$\mathcal{T}'_{4,3}$, 6793 elements



		$\mathrm{card}(\mathcal{T}'_{4,3}) = 6793$		$\mathrm{card}(\mathcal{T}'_{9}) = 105469$	
k	$\lambda_{\mathrm{ref}}^{(k)}$ [8]	$\widetilde{\lambda}_h^{(k)}$	Relative error	$\widetilde{\lambda}_h^{(k)}$	Relative error
1	39.348	39.118	5.85e-03	39.126	5.64e-03
2	97.436	96.892	5.58e-03	96.860	5.91e-03
3	97.436	96.921	5.29e-03	96.889	5.61e-03
4	165.406	164.322	6.55e-03	164.241	7.04e-03
5	165.406	164.661	4.50e-03	164.560	5.11e-03
6	190.370	189.672	3.67e-03	189.212	6.08e-03
7	208.608	207.385	5.86e-03	207.243	6.54e-03
8	272.406	271.258	4.21e-03	270.543	6.84e-03
9	272.406	271.491	3.36e-03	270.763	6.03e-03
10	312.353	312.148	6.56e-04	310.650	5.45e-03

$p = 2$, $\min h_K = \frac{1}{3^5}\mathbf{diam}(\Omega)$, ↑ Locally-refined mesh ↑ Quasi-uniform mesh

Reference results (by polygonal prefractal approximation and conformal mapping via Schwarz–Christoffel transformations) from:

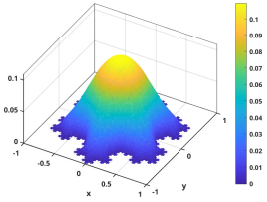
BANJAI, *Eigenfrequencies of fractal drums*, JCAM, 2007

Part IV

Error analysis

Solution regularity

For general $f \in L_2(\Omega)$, the solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ is not in $H^2(\Omega)$.



Solution regularity

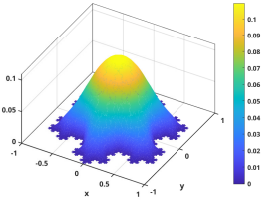
For general $f \in L_2(\Omega)$, the solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ is not in $H^2(\Omega)$.

It is known, however, that $u \in V^\mu$ for all $\mu > \mu_* \approx 0.828$, where

$$V^\mu := \{ v \in H^1(\Omega) : \quad \Delta w \in L_2(\Omega), \quad \delta^\mu D^2 v \in L_2(\Omega) \},$$

$$D^2 v := \left(|\partial_{x_1}^2 v|^2 + |\partial_{x_1} \partial_{x_2} v|^2 + |\partial_{x_2}^2 v|^2 \right)^{\frac{1}{2}},$$

$$\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial\Omega) \in C^{0,1}(\overline{\Omega}).$$



Is μ_* sharp?
V^μ analogous to Babuška–Guo $H_\mu^{2,2}(\Omega)$ space for non-convex polygons.

NYSTRÖM, *Smoothness properties of solutions to Dirichlet problems in domains with a fractal boundary*, PhD thesis, University of Umeå, 1994

CAPITANELLI, VIVALDI, *Weighted estimates on fractal domains*, Mathematika, 2015

Nyström also proves $u \in C^{0,1/3}(\overline{\Omega})$ and $u \in W_3^1(\Omega) := \{ u \in L_3(\Omega) : \nabla u \in L_3(\Omega)^2 \}$.

Quasi-optimality

Follow SIP-DG framework in (DIPIETRO, ERN 2011).

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Quasi-optimality

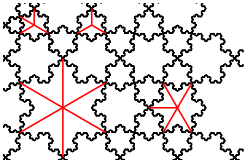
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$$\|w\|_{\mathrm{DG}^+}^2 := \|w\|_{\mathrm{DG}}^2 + \sum_K \left(h_K^2 \|\Delta w\|_{L_2(K)}^2 + h_K^{2-2/q} \|\partial_{\mathbf{n}} w\|_{L_q(*_K)}^2 \right),$$

$1 < q < \min\{2, 1/\mu\}$ ↑ proxy of $h\|\partial_{\mathbf{n}} w\|_{L_2(F)}^2$



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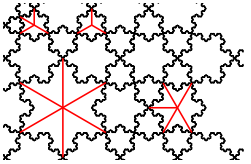
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$w \in V^\mu \implies \partial_{\mathbf{n}} w \in L_q(*_K)$ for every K, using a Hardy inequality



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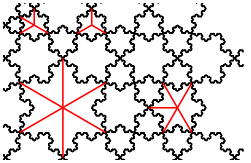
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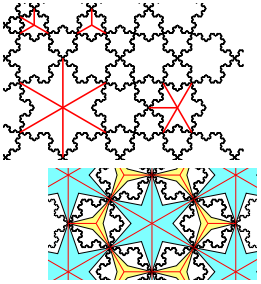
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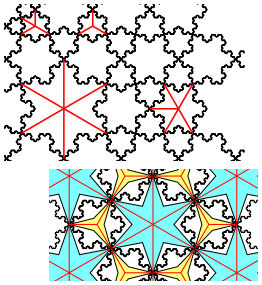
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- Quasi-optimality: $\mu \in [0, 1)$, $u \in V^\mu \implies$ $\|u - u_h\|_{\mathrm{DG}} \le C_{p,q} \inf_{v_h \in V_h} \|u - v_h\|_{\mathrm{DG}^+}$

$\mathbb{P}^1$ approximation and SIP-DG convergence rates

Quasi-optimality: $\|u - u_h\|_{\mathrm{DG}} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{\mathrm{DG}^+}$.

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Key: compactness of $V^\mu \subset H^1(\Omega)$ (Chua, Rodney, Wheeden, Pacific J. Math. 2013)

$\Rightarrow$ Poincaré estimate $\inf_{P \in \mathbb{P}^1(\Omega)} \big(\|v - P\|_{L_2(\Omega)} + \|\nabla(v - P)\|_{L_2(\Omega)} \big) \lesssim \|\delta^\mu D^2 v\|_{L_2(\Omega)} \quad \forall v \in V^\mu.$

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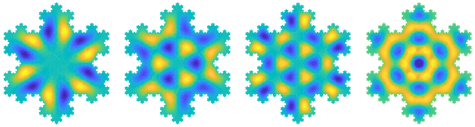
SIP-DG-FEM error estimate: if $u \in V^\mu(\Omega), \quad \mu \in [0, 1)$

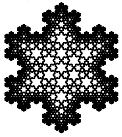
$$\|u - u_h\|_{\mathrm{DG}}^2 \le C_p \Big(\sum_{K \in \mathcal{T}} h_K^2 \|\Delta u\|_{L_2(K)}^2 + \sum_{\substack{K \in \mathcal{T}, \\ \text{on bdry}}} h_K^{2-2\mu} \|\delta^\mu D^2 u\|_{L_2(K)}^2 + \sum_{\substack{K \in \mathcal{T}, \\ \text{interior}}} h_K^2 \|D^2 u\|_{L_2(K)}^2 \Big).$$

Recall: $f \in L_2(\Omega) \implies u \in V^\mu$ with $\mu \gtrapprox 0.828$ (Nyström 1994, Capitanelli, Vivaldi 2015)

Outlook

- DG-FEM using fractal elements
- Quadrature by self-similarity
- Well-posedness and error estimate



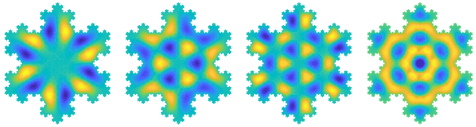


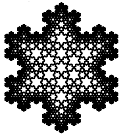
Much more to do:

- more general IFS attractors
- sharp regularity exponent μ
- analysis for boundary refinement
- other PDEs, BCs, interfaces, cracks, coefficients
- applications to spectral analysis
- mesh couplings
- fast solvers
- ...

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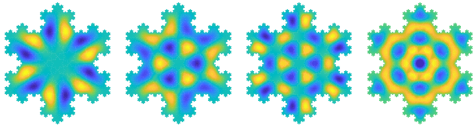
S. Gómez, D.P. Hewett, A. Moiola, *A DG method with fractal elements*, arXiv:2604.11093

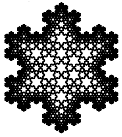
:FractalDG

Slides available on `https://euler.unipv.it/moiola/#slides`

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Thank you!

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