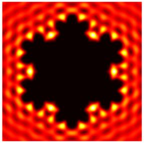
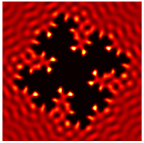
20 JUNE 2018, BOLOGNA

Scattering by fractal screens: functional analysis and computation



Andrea Moiola

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PAVIA



Joint work with

S.N. Chandler-Wilde (Reading), D.P. Hewett (UCL) and A. Caetano (Aveiro)

Acoustic wave scattering by a planar screen

Acoustic waves in free space governed by wave eq. $\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

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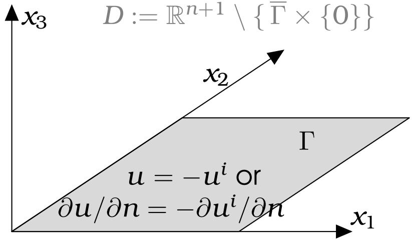
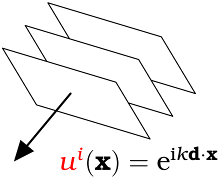
Γ bounded open subset of $\{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{R}^n$, $n = 1, 2$

$\Delta u + k^2 u = 0$

$D := \mathbb{R}^{n+1} \setminus \{\overline{\Gamma} \times \{0\}\}$

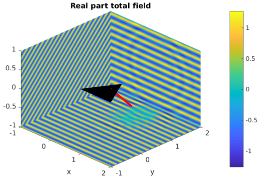


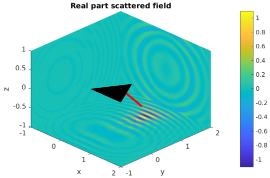


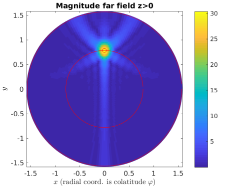
u satisfies Sommerfeld radiation condition (SRC) at infinity
(i.e. $\partial_r u - iku = o\left(r^{-(n-1)/2}\right)$ uniformly as $r = |\mathbf{x}| \to \infty$).

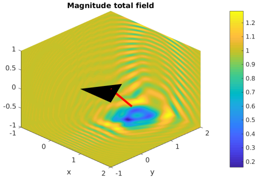
Scattering by Lipschitz and rough screens

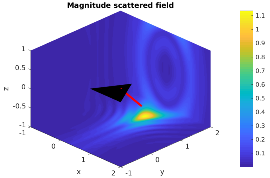
Incident field is plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}$, $|\mathbf{d}| = 1$. $\qquad u^{tot} = u + u^i$

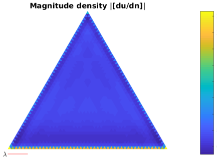












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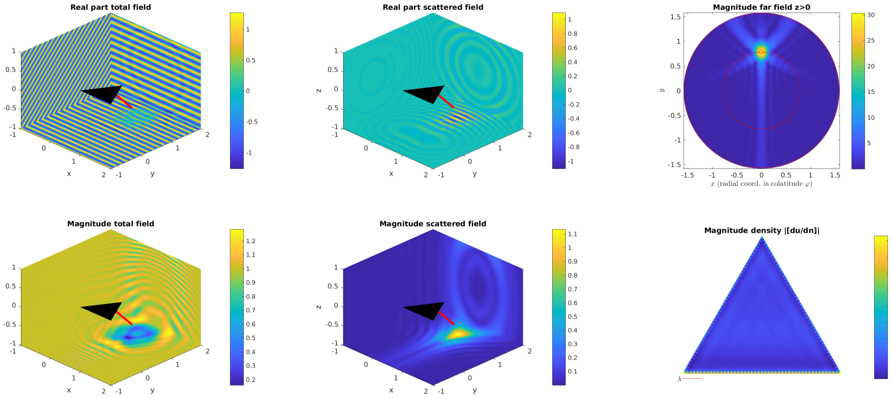

Classical problem when Γ is Lipschitz (Buffa, Christiansen, Costabel, Ha-Duong, Hiptmair, Holm, Jerez-Hanckes, Maischak, Stephan, Wendland, Urzúa-Torres, . . .)

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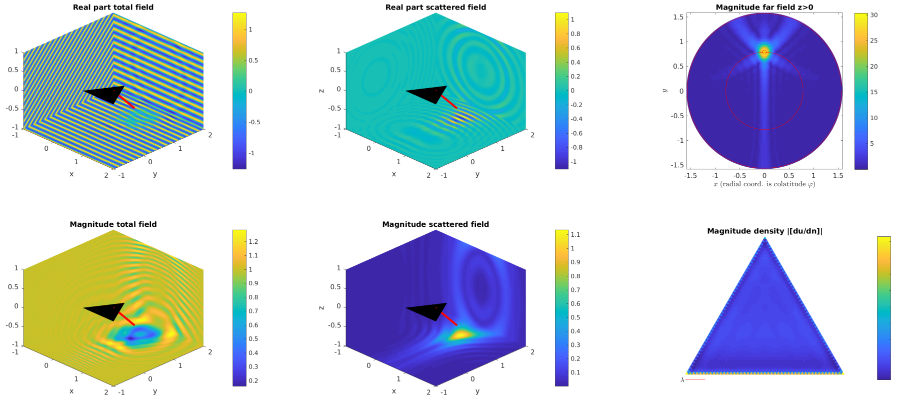

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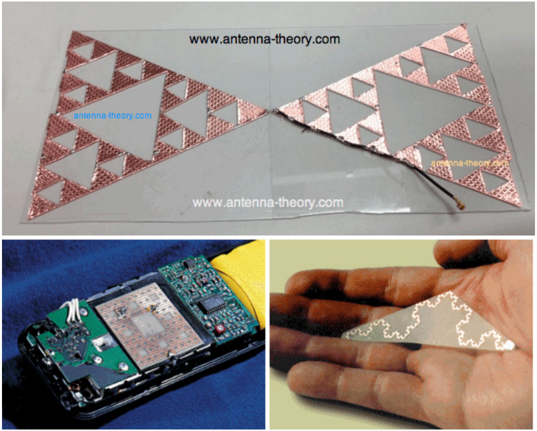
What happens for arbitrary (rougher than Lipschitz, e.g. fractal) Γ?

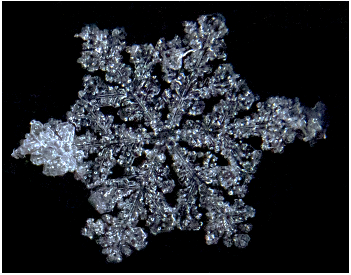
Fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

Fractal antennas are a popular topic in engineering:
Wideband/multiband, compact, cheap, metamaterials, cloaking. . .
Not yet analysed by mathematicians.

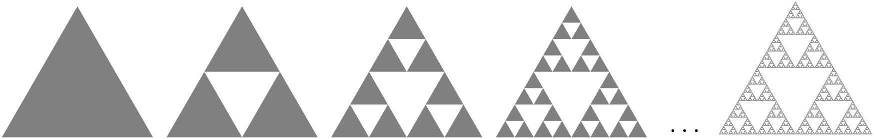
Other applications



Scattering by ice crystals
in atmospheric physics
e.g. C. Westbrook (Reading)

Fractal apertures in laser optics
e.g. J. Christian (Salford)

Scattering by fractal screens



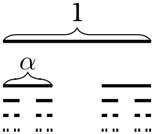
Lots of interesting mathematical questions:

- How to formulate well-posed BVPs?
  (What is the right function space setting? How to impose BCs?)
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?

- If the fractal has empty interior, does it scatter waves at all?
- How does the fractal (Hausdorff) dimension affect things?

Can you hear a Cantor dust?

For $0 < \alpha < 1/2$
let $C_\alpha \subset [0, 1]$ denote the
standard Cantor set:



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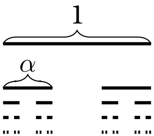
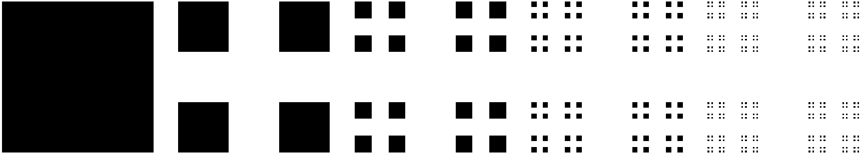


Let $C_\alpha^2 := C_\alpha \times C_\alpha \subset \mathbb{R}^2$ denote the associated "Cantor dust":



C_α^2 is uncountable, closed, with $\mathrm{int}(C_\alpha^2) = \emptyset$; in fact $m(C_\alpha^2) = 0$.

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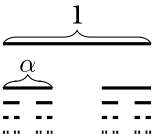

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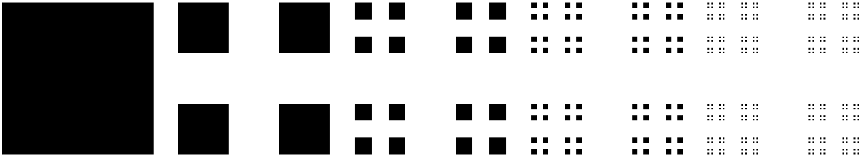

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Question: Is the scattered field zero or non-zero for the 3D Dirichlet scattering problem with $\Gamma = C_\alpha^2$?

Bibliography

I will discuss the answers we tried to give here:

(1) SNCW, DPH, *Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens,* IEOT, 2015.

(2) DPH, AM, *On the maximal Sobolev regularity of distributions supported by subsets of Euclidean space,* An. and Appl., 2017.

(3) SNCW, DPH, AM, *Sobolev spaces on non-Lipschitz subsets of* $\mathbb{R}^n$ *with application to BIEs on fractal screens,* IEOT, 2017.

(4) SNCW, DPH, *Well-posed PDE and integral equation formulations for scattering by fractal screens,* SIAM J. Math. Anal., 2018.

(5) SNCW, DPH, AM, *Scattering by fractal screens and apertures,* in preparation.

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▹ Scattering by open screens

(2) DPH, AM, *On the maximal Sobolev regularity of distributions supported by subsets of Euclidean space,* An. and Appl., 2017.

(3) SNCW, DPH, AM, *Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to BIEs on fractal screens,* IEOT, 2017.
▹ Sobolev spaces

(4) SNCW, DPH, *Well-posed PDE and integral equation formulations for scattering by fractal screens,* SIAM J. Math. Anal., 2018.
▹ Scattering by general screens

(5) SNCW, DPH, AM, *Scattering by fractal screens and apertures,* in preparation.
▹ BEM, convergence

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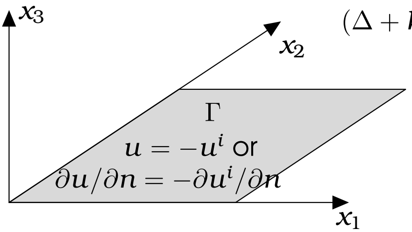
Part I

BVPs & BIEs

Boundary integral equations (BIEs)

BIEs provide a natural analytical and computational framework.



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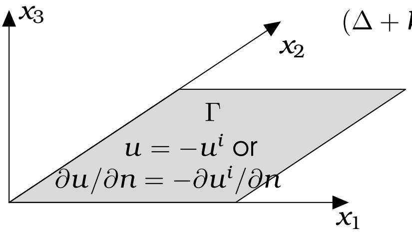

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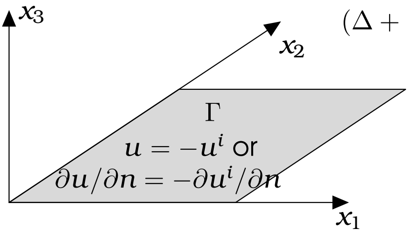

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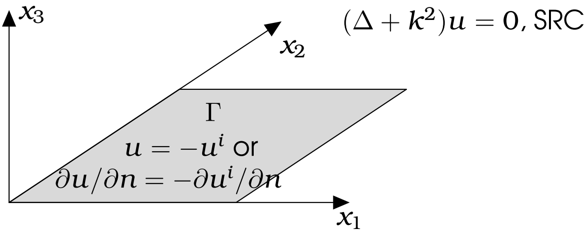

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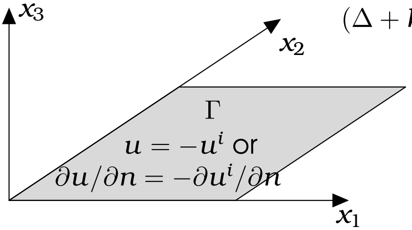

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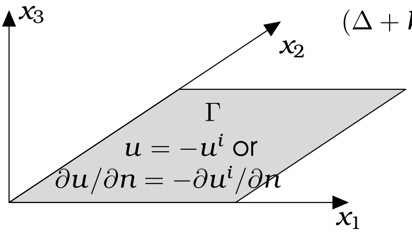

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- The jumps satisfy certain boundary integral equations
- The associated boundary integral operators are coercive, thus invertible, between appropriate spaces
  (Ha-Duong, Chandler-Wilde/Hewett)

Sobolev spaces on $\Gamma \subset \mathbb{R}^n$

BIEs require us to work in fractional (Bessel) Sobolev spaces on $\Gamma \subset \mathbb{R}^n$.
For $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^n) : \ \|u\|_{H^s(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)^s |\hat{u}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} < \infty \right\}.$$

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For $\Gamma \subset \mathbb{R}^n$ open and $F \subset \mathbb{R}^n$ closed define (McLean)

$$H^s(\Gamma) := \{ u|_\Gamma : u \in H^s(\mathbb{R}^n) \} \qquad \text{restriction}$$

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“Global” and “local” spaces:

$$\underbrace{\widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}}}_{\text{“0-trace”}} \subset H^s(\mathbb{R}^n) \subset \mathcal{D}^*(\mathbb{R}^n) \quad \xrightarrow[\text{restriction oper.}]{|_\Gamma} \quad H^s(\Gamma) \subset \mathcal{D}^*(\Gamma).$$

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- $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$ with equal norms
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For general non-Lipschitz Γ

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- ×
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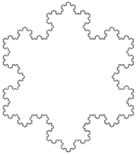

This has implications for the scattering problem!

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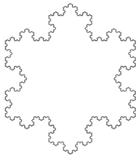


This has implications for the scattering problem!

There exist many works on Sobolev (Besov,. . .) spaces on rough sets; most use intrinsic definitions on (e.g.) d-sets.
Analogous to $W^s(\Gamma)$, based on $L^p(\Gamma, \mathcal{H}_d)$.
Related to spaces in $\mathbb{R}^n$ by traces. See: Jonsson–Wallin, Strichartz.

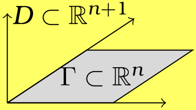
Our spaces are different, more suited for integral equations and BEM.

Dirichlet BVP (Lipschitz open $\Gamma \subset \mathbb{R}^n$)

Problem **D**

Given $g_D \in H^{1/2}(\Gamma)$ (e.g. $g_D = -u^i|_\Gamma$), find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$
\begin{aligned}
(\Delta + k^2)u &= 0 \qquad \text{in } D = \mathbb{R}^{n+1} \setminus \overline{\Gamma}, \\
u &= g_D \qquad \text{on } \Gamma,
\end{aligned}
$$



and u satisfies the Sommerfeld radiation condition.

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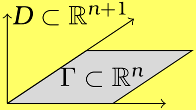

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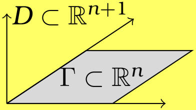

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Theorem (cf. Stephan and Wendland '84, Stephan '87)

If Γ is Lipschitz then **D** *has a unique solution for all $g_D \in H^{1/2}(\Gamma)$.*

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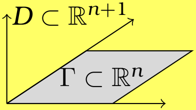

and u satisfies the Sommerfeld radiation condition.

Theorem (cf. Stephan and Wendland '84, Stephan '87)

If Γ is Lipschitz then **D** *has a unique solution for all $g_D \in H^{1/2}(\Gamma)$.*

BIE: $\mathrm{S}[\partial_n u] = -g_D$ representation: $u = -\mathcal{S}[\partial_n u]$ single-layer potential ($\mathcal{S}$) operator (S):

$\mathcal{S} : \widetilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^1_{loc}(D)$ $\qquad \mathcal{S}\phi(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D$

$\mathrm{S} : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ $\qquad \mathrm{S}\phi(\mathbf{x}) := \gamma^\pm \mathcal{S}\phi|_\Gamma(\mathbf{x}) \quad \mathbf{x} \in \Gamma$

S invertible, $\qquad \Phi(\mathbf{x},\mathbf{y}) := \mathrm{e}^{ik|\mathbf{x}-\mathbf{y}|}/4\pi|\mathbf{x}-\mathbf{y}|$ (in 3D)

Failure of BVP **D** for non-Lipschitz Γ

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  Then $\mathcal{S}\phi$ satisfies homogeneous problem.

We need to modify **D** to deal with this.

Dirichlet BVP (arbitrary open Γ)

Problem $\widetilde{\mathsf{D}}$

Given $g_\mathsf{D} \in H^{1/2}(\Gamma)$ (e.g. $g_\mathsf{D} = -u^i|_\Gamma$), find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\begin{aligned}
(\Delta + k^2)u &= 0 \qquad \text{in } D, \\
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[u] &= 0, \qquad\qquad (\mathsf{D}') \\
\left[\frac{\partial u}{\partial n}\right] &\in \widetilde{H}^{-1/2}(\Gamma), \qquad (\mathsf{D}'')
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and u satisfies the Sommerfeld radiation condition.

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Theorem (Chandler-Wilde & Hewett 2013)

For any bounded open Γ, $\widetilde{\mathbf{D}}$ has a unique solution for all $g_{\mathrm{D}} \in H^{1/2}(\Gamma)$.

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For any bounded open Γ, $\widetilde{\mathbf{D}}$ has a unique solution for all $g_D \in H^{1/2}(\Gamma)$.

If $H^{1/2}_{\partial\Gamma} = \{0\}$ then $\mathbf{D}'$ is superfluous.
If $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ then $\mathbf{D}''$ is superfluous. (E.g. if Γ is C^0.)

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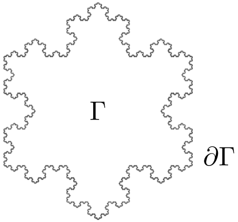
Two key questions: (i) when is $H^s_{\partial\Gamma} = \{0\}$? (ii) when is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?

Part II

Two Sobolev space questions

Key question #1: nullity

Given a compact set $K \subset \mathbb{R}^n$ with empty interior (e.g. $K = \partial\Gamma$), for which $s \in \mathbb{R}$ is $H^s_K \neq \{0\}$?



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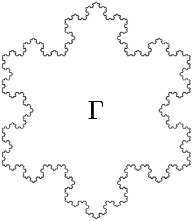


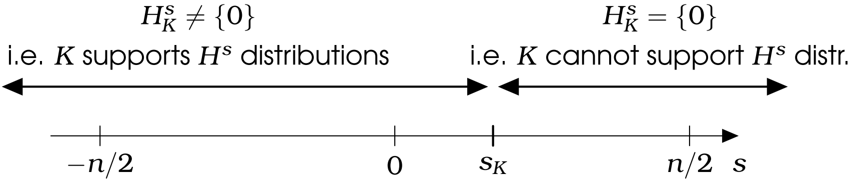
Terminology:

$H^s_K = \{0\} \iff \nexists$ non-zero elements of H^s supported inside K.
We call such a set K "s-null".

Other terminology exists: "$(-s)$-polar" (Maz'ya, Littman), "set of uniqueness for H^s" (Maz'ya, Adams/Hedberg).

Nullity threshold

For every compact $K \subset \mathbb{R}^n$ with $\mathrm{int}(K) = \emptyset$,
$\exists$ $s_K \in [-n/2, n/2]$, called the *nullity threshold* of K,
such that $H^s_K = \{0\}$ for $s > s_K$ and $H^s_K \neq \{0\}$ for $s < s_K$.



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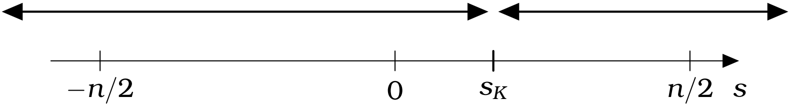
$H^s_K \neq \{0\}$ i.e. K supports H^s distributions

$H^s_K = \{0\}$ i.e. K cannot support H^s distr.



Theorem (H & M 2017)

If $m(K) = 0$ then

$$s_K = \frac{\dim_H(K) - n}{2} \leq 0$$

Theorem (Polking 1972)

$\exists$ compact K with $\mathrm{int}(K) = \emptyset$ and $m(K) > 0$ for which $H^{n/2}_K \neq \{0\}$, so that $s_K = n/2$.

Connection with $\dim_H$ comes from standard potential theory results (Maz'ya 2011, Adams & Hedberg 1996 etc.)

Nullity theory $\sim$complete for $m(K) = 0$, open problems for $m(K) > 0$.

Key question #2: identity of 0-trace spaces

Given an open set $\Gamma \subset \mathbb{R}^n$, when is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?

Equivalent to density of $C_0^\infty(\Gamma)$ in $\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Gamma}\}$.

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Let $\Gamma \subset \mathbb{R}^n$ be C^0. Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

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Let $\Gamma \subset \mathbb{R}^n$ be C^0. Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

1st class of sets: "regular except at a few points", e.g. prefractal 

Theorem (C-W, H & M 2017)

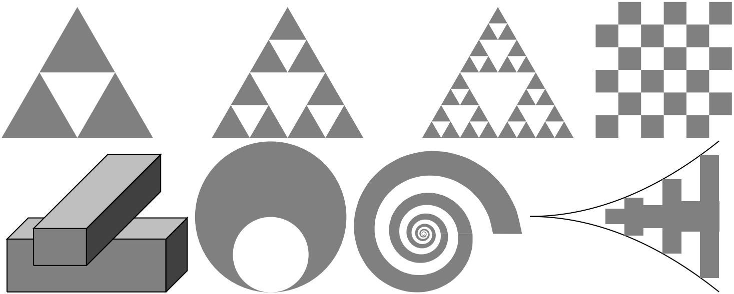
Let $n \geq 2$, $\Gamma \subset \mathbb{R}^n$ open and C^0 except at finite $P \subset \partial\Gamma$.
Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ for $|s| \leq 1$.

- For $n = 1$ the same holds for $|s| \leq 1/2$.
- Can take countable $P \subset \partial\Gamma$ with finitely many limit points in every bounded subset of $\partial\Gamma$.

Proof uses sequence of special cutoffs for $s = 1$, duality, interpolation.

Examples of non-C^0 sets with $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \le 1$

E.g. union of disjoint C^0 open sets, whose closures intersect only in P.

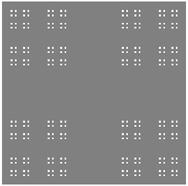


Sierpinski triangle prefractals, (unbounded) checkerboard, double brick, inner and outer (double) curved cusps, spiral, Fraenkel's "rooms and passages".

Constructing counterexamples

Consider another class of sets:
"nice domain minus small holes".

E.g. when $\text{int}(\overline{\Gamma})$ is smooth.



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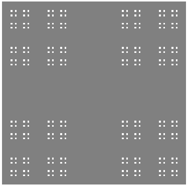

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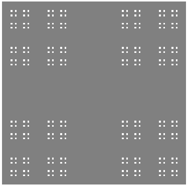

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Corollary

For every $n \in \mathbb{N}$*, there exists a bounded open set* $\Gamma \subset \mathbb{R}^n$ *such that,*

$$\widetilde{H}^s(\Gamma) \subsetneqq H^s_{\overline{\Gamma}}, \qquad \forall s \geq -n/2$$

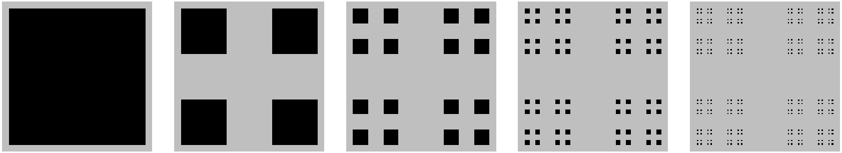
Proof: take a ball and remove a Polking set (not s-null for any $s \leq n/2$)

(Can also have $\widetilde{H}^s(\Gamma) \subsetneqq \{u \in H^s : u = 0 \text{ a.e. in } \Gamma^c\} \subsetneqq H^s_{\overline{\Gamma}} \quad \forall s > 0$.)

Part III

Formulations on general screens

Prefractal convergence



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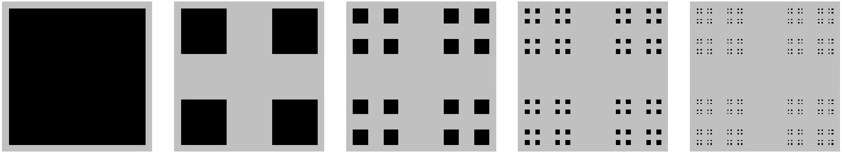

Theorem (C-W, H & M 2017)

Consider a bounded sequence of nested open screens $\Gamma_1 \subset \Gamma_2 \subset \cdots$
For each j *let* u_j *denote the solution of problem* $\widetilde{\mathbf{D}}$ *for* Γ_j.
Let $\Gamma := \bigcup_{j\in\mathbb{N}} \Gamma_j$ *and let* u *denote the solution of problem* $\widetilde{\mathbf{D}}$ *for* Γ.
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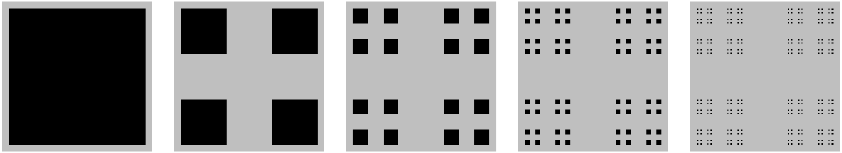

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Proof:

$$\widetilde{H}^s(\Gamma_1) \subset \widetilde{H}^s(\Gamma_2) \subset \cdots \qquad \text{and} \qquad \widetilde{H}^s\Big(\bigcup_{j\in\mathbb{N}} \Gamma_j\Big) = \overline{\bigcup_{j\in\mathbb{N}} \widetilde{H}^s(\Gamma_j)}.$$

Then write BIEs in variational form and apply Céa's Lemma.

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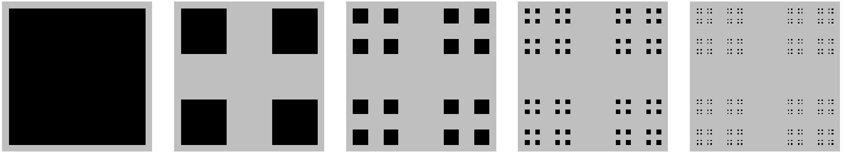

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Then write BIEs in variational form and apply Céa's Lemma.

What if we want to use $\Gamma_1 \supset \Gamma_2 \supset \cdots \to \Gamma$? e.g. Cantor dust
Need framework for closed screens.

What about general screens?

For an open screen Γ, we imposed the BC by restriction to Γ:

$$(\gamma^{\pm} u)|_{\Gamma} = g_{\mathsf{D}}$$

and viewed S as an operator $\mathsf{S} : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \cong (\widetilde{H}^{-1/2}(\Gamma))^{*}$.

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But since $H^{1/2}(\mathbb{R}^n) \supset (H^{1/2}_{\Gamma^c})^{\perp} \xrightarrow[\text{isomorphism}]{|_{\Gamma}} H^{1/2}(\Gamma)$

we could equivalently impose the BC by orthogonal projection:

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This viewpoint suggests a way of writing down BVP formulations for general screens (even with $\mathrm{int}(\Gamma) = \emptyset$):

- replace $\widetilde{H}^{-1/2}(\Gamma)$ by some $V^- \subset H^{-1/2}(\mathbb{R}^n)$
- characterise $(V^-)^*$ as a subspace $V^+_* \subset H^{1/2}(\mathbb{R}^n)$
- impose BC by orthogonal projection onto V^+_*
- view S as an operator $\mathsf{S} : V^- \to V^+_*$

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Here we are using the following fact:

Let $H, \mathcal{H}$ be Hilbert spaces with $H^* \cong \mathcal{H}$ (unit. isom.).
(E.g. $H = H^{-1/2}(\mathbb{R}^n)$, $\mathcal{H} = H^{1/2}(\mathbb{R}^n)$.)
If $V \subset H$ is a closed subspace, $V^* \cong (V^{a,\mathcal{H}})^{\perp,\mathcal{H}}$ (with inherited duality pairing)

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Problem $\mathbf{D}(V^-)$

Given $g_\mathrm{D} \in V_*^+$ (e.g. $g_\mathrm{D} = -P_{V_*^+} u^i$),
find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$(\Delta + k^2)u = 0 \quad \text{in } D,$$
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$$[u] = 0,$$
$$[\partial_n u] \in V^-,$$

SRC at infinity.

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and define $V_*^+ \cong (V^-)^*$ by $V_*^+ := ((V^-)^a)^\perp \subset H^{1/2}(\mathbb{R}^n)$.

Problem **D**(V^-)

Given $g_D \in V_*^+$ (e.g. $g_D = -P_{V_*^+} u^i$), find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$(\Delta + k^2)u = 0 \quad \text{in } D,$$
$$P_{V_*^+}\gamma^\pm u = g_D,$$
$$[u] = 0,$$
$$[\partial_n u] \in V^-,$$

SRC at infinity.

Theorem (C-W & H 2016)

Problem **D**(V^-) *is well-posed for any choice of* V^-.

Operator $\mathsf{S} : V^- \to V_*^+$ inherits coercivity!

Which formulation to use?

For any bounded Γ, each choice $\quad \widetilde{H}^{-1/2}(\mathrm{int}(\Gamma)) \subset V^- \subset H^{-1/2}_{\overline{\Gamma}}$
gives its own well-posed formulation **D**(V^-).

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If $\widetilde{H}^{-1/2}(\mathrm{int}(\Gamma)) = H_{\overline{\Gamma}}^{-1/2}$ *there is only one such formulation.*
If $\widetilde{H}^{-1/2}(\mathrm{int}(\Gamma)) \neq H_{\overline{\Gamma}}^{-1/2}$ $\exists$ *infinitely many formulations with* $\neq$ *solutions!*

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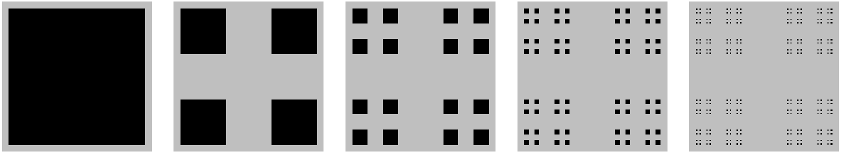

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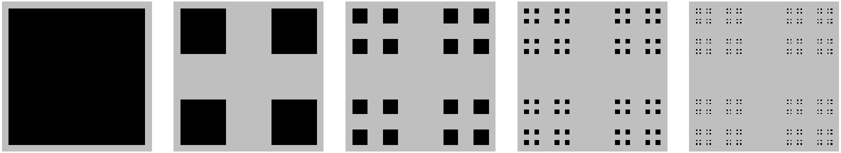

- $\Gamma_1 \subset \Gamma_2 \subset \cdots$ open and "nice" (e.g. Lipschitz)
- $\Gamma := \bigcup_j \Gamma_j$ open (gray part),

$\to$ natural choice is

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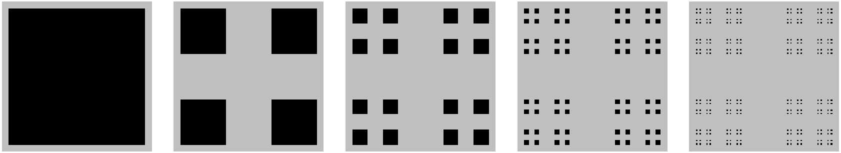

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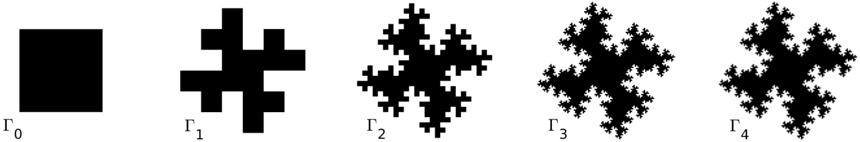
What if prefractals are not nested?

What if prefractals Γ_j are neither increasing nor decreasing? $\Gamma_j \overset{\not\subset}{\not\supset} \Gamma_{j+1}$



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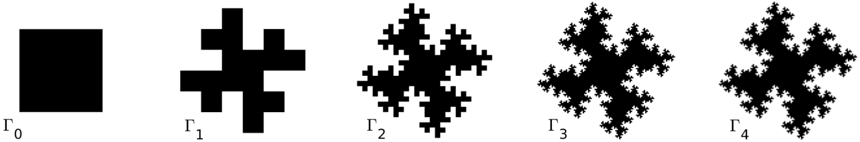

Key tool is Mosco convergence (Mosco 1969):

V_j, V closed subspaces of Hilbert space H, $j \in \mathbb{N}$, then $V_j \xrightarrow{\mathcal{M}} V$ if:

- $\forall v \in V, j \in \mathbb{N}, \exists v_j \in V_j$ s.t. $v_j \to v$ (strong approximability)
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Think: $H = H^{-1/2}(\mathbb{R}^n)$, $V_j = \widetilde{H}^{-1/2}(\Gamma_j)$, $\widetilde{H}^{-1/2}(\mathrm{int}(\Gamma)) \subset V \subset H^{-1/2}_{\overline{\Gamma}}$

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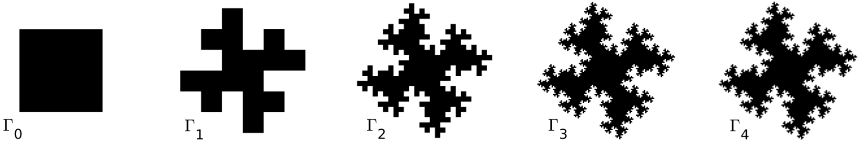

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Theorem (C-W, H & M, 2018)

If $V_j \xrightarrow{\mathcal{M}} V \subset H^{-1/2}(\mathbb{R}^n)$ then solution of **D**(V_j) *converges to sol.n of* **D**(V)

Holds for square snowflake above with $V = \widetilde{H}^{-1/2}(\mathrm{int}(\Gamma)) = H^{-1/2}_{\overline{\Gamma}}$

When is $u = 0$?

Theorem (C-W & H 2018)

Let Γ be closed with empty interior and let $V^- = H_\Gamma^{-1/2}$.

- *If $\dim_{\mathrm{H}}\Gamma < n - 1$ then $u = 0$ for every incident direction $\mathbf{d}$.*
- *If $\dim_{\mathrm{H}}\Gamma > n - 1$ then $u \neq 0$ for a.e. incident direction $\mathbf{d}$.*

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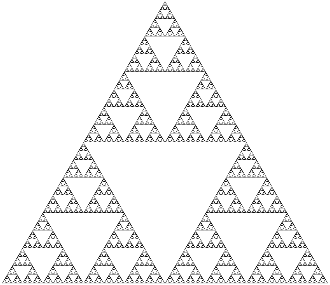
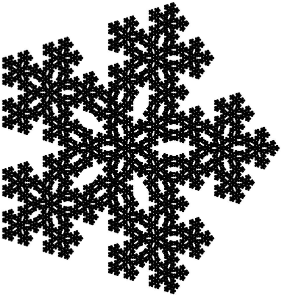
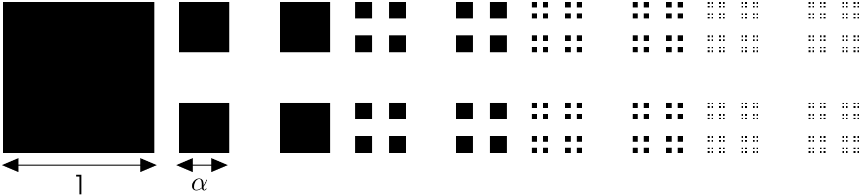
So both the Sierpinski triangle ($\dim_{\mathrm{H}} = \log 3 / \log 2$) and pentaflake ($\dim_{\mathrm{H}} = \log 6 / \log((3 + \sqrt{5})/2)$) generate a non-zero scattered field:





Back to the Cantor dust

Let $C_\alpha^2 := C_\alpha \times C_\alpha \subset \mathbb{R}^2$ denote the "Cantor dust" $(0 < \alpha < 1/2)$:



Question: Is the scattered field u zero or non-zero for the 3D Dirichlet scattering problem with $\Gamma = C_\alpha^2$?

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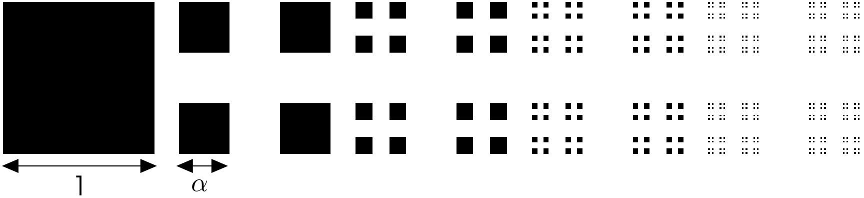


Question: Is the scattered field u zero or non-zero for the 3D Dirichlet scattering problem with $\Gamma = C_\alpha^2$?

$$\dim_{\mathrm{H}}(C_\alpha^2) = \frac{\log(4)}{\log(1/\alpha)}$$

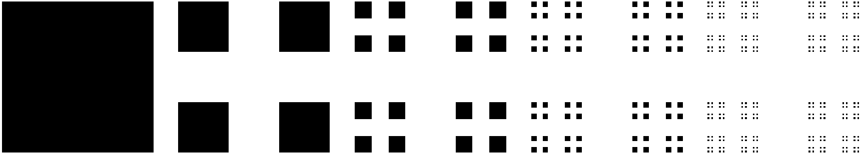
Answer:
$u = 0$, if $0 < \alpha \leq 1/4$;
$u \neq 0$, in general, if $1/4 < \alpha < 1/2$.

($u = 0$ for all α for Neumann BCs)

Part IV

Numerical approximation

Boundary element method (BEM)



For each prefractal Γ_j, the BIE $\mathrm{S}[\partial u/\partial n] = -g_\mathrm{D}$ can be solved using a standard BEM space, e.g. piecewise constants on a mesh of width h_j.

Let w_j denote the Galerkin BEM solution on Γ_j.

Let $l_j = \alpha^j$ be the width of each component of Γ_j (4^j of them).

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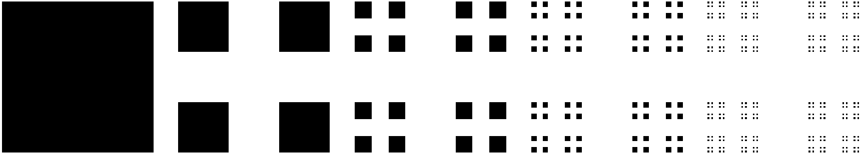

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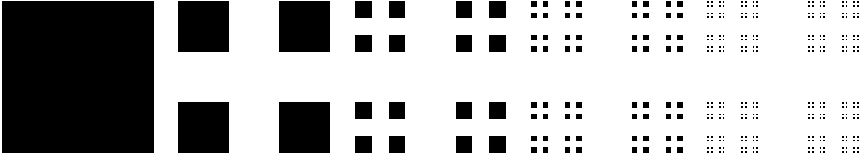

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Follows from Mosco convergence of BEM spaces.
This requires approximability ($\forall v \in H_\Gamma^{-1/2} \exists v_j \in \widetilde{H}^{-1/2}(\Gamma_j), v_j \to v$):
proved with mollification, L^2 projection, partition of unity, . . .

Convergence results for the Cantor dust

Theorem (C-W, H & M 2018)

Suppose $\exists\, -1/2 < t < 0$ *such that* H^t_Γ *is dense in* $H^{-1/2}_\Gamma$.
Then $\exists\, \mu = \mu(t) > 0$ *such that if* $h_j/l_j = O(\mathrm{e}^{-\mu j})$ *then* $w_j \to u$ *as* $j \to \infty$.

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If $h_j = o(l_j)$ *then* $\tilde{w}_j \to u$ *as* $j \to \infty$.

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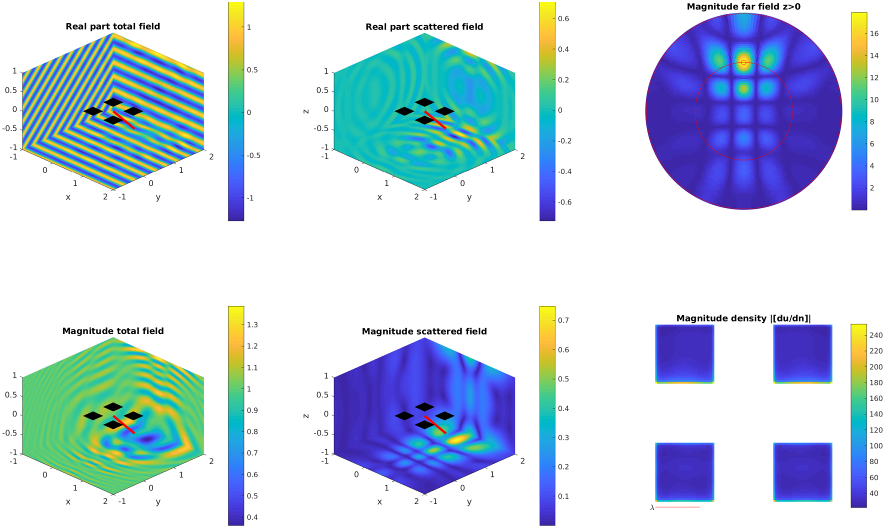
For simplicity, I'll show results on prefractals for #DOF fixed but large.

Numerical results: Cantor dust $\alpha = 1/3$ $(u \neq 0)$

$k = 25$, 4096 DOFs, prefractal level 1



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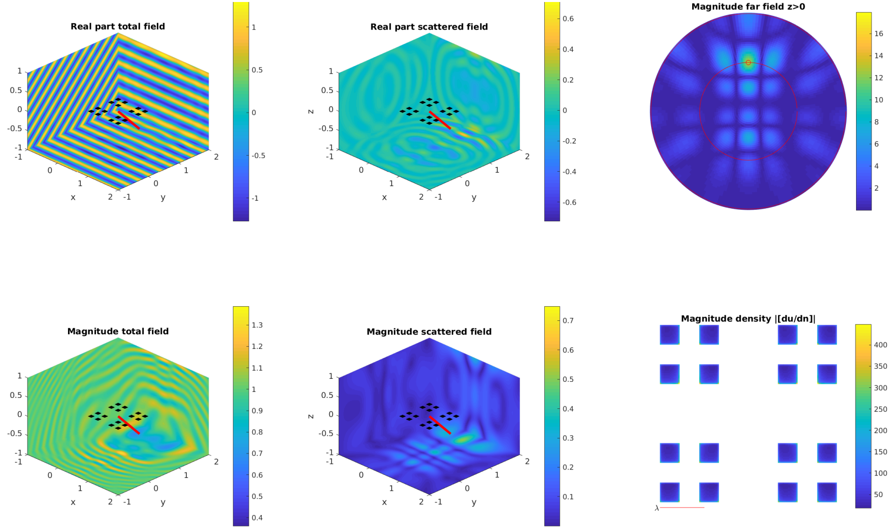
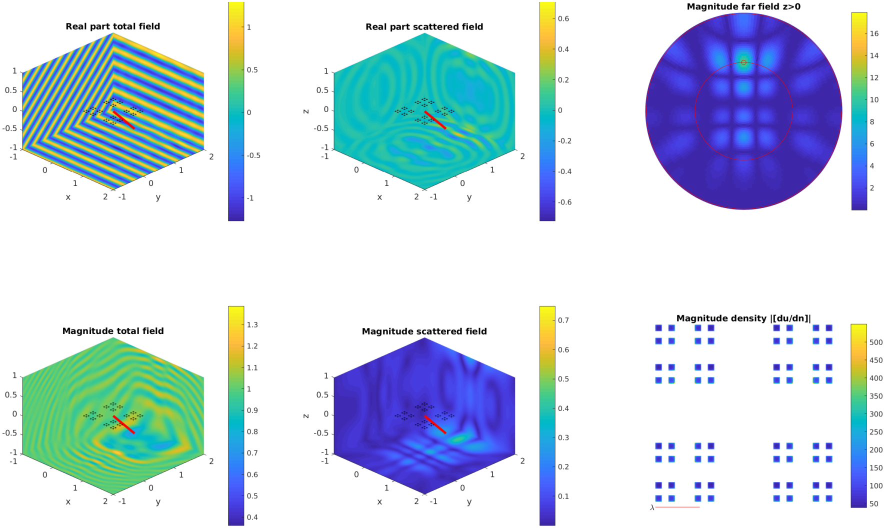
$k = 25$, 4096 DOFs, prefractal level 2



Numerical results: Cantor dust $\alpha = 1/3$ $(u \neq 0)$

$k = 25$, 4096 DOFs, prefractal level 3



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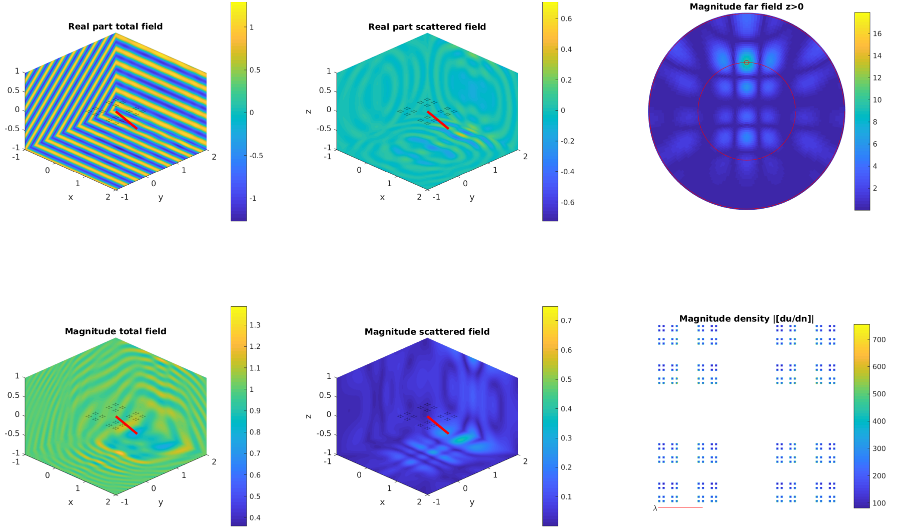
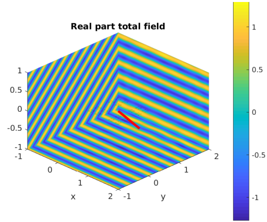
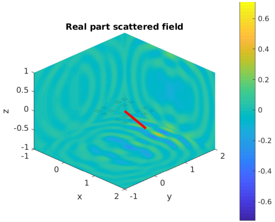
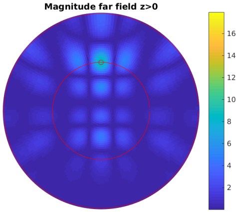
$k = 25$, 4096 DOFs, prefractal level 4

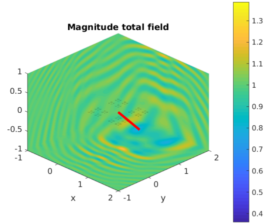


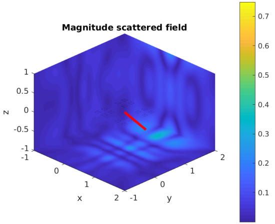
$k = 25$, 4096 DOFs, prefractal level 5

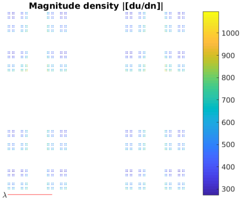






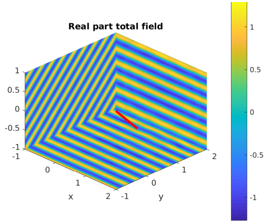


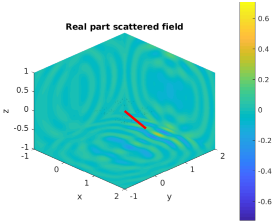


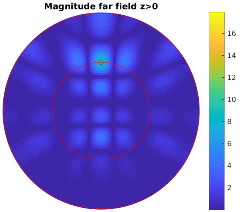


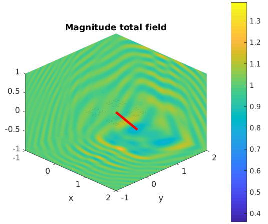
Numerical results: Cantor dust $\alpha = 1/3$ $(u \neq 0)$

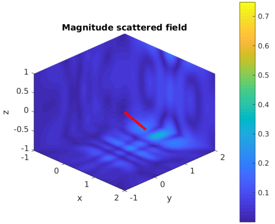
$k = 25$, 4096 DOFs, prefractal level 6

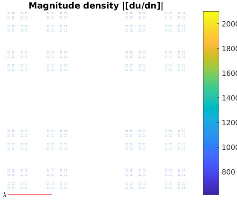






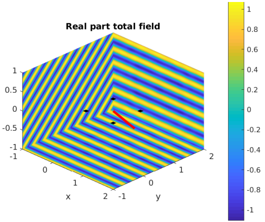


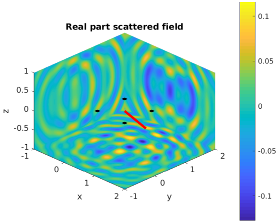


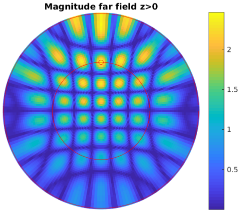


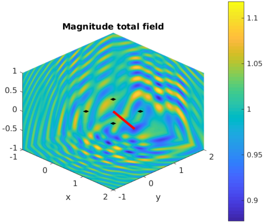
Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

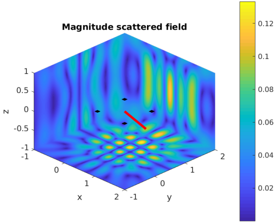
$k = 25$, 4096 DOFs, prefractal level 1

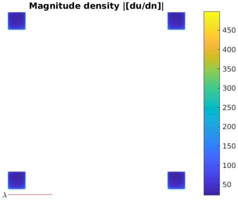






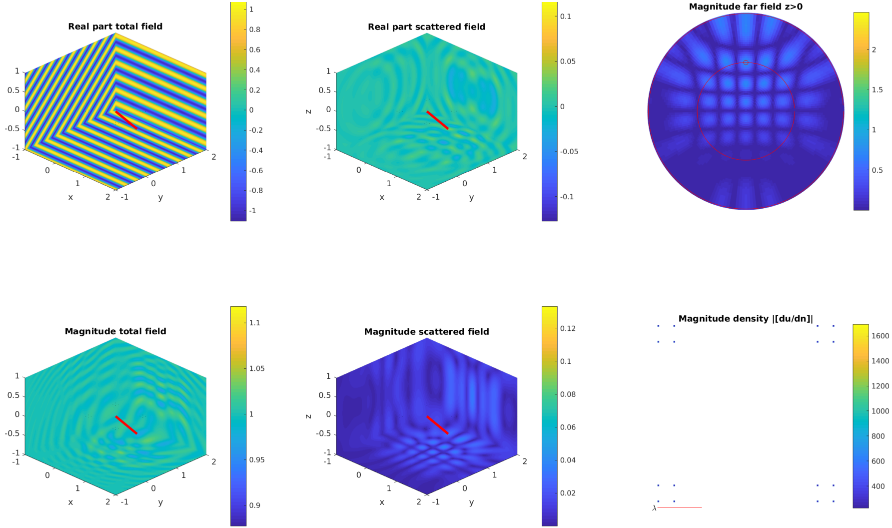






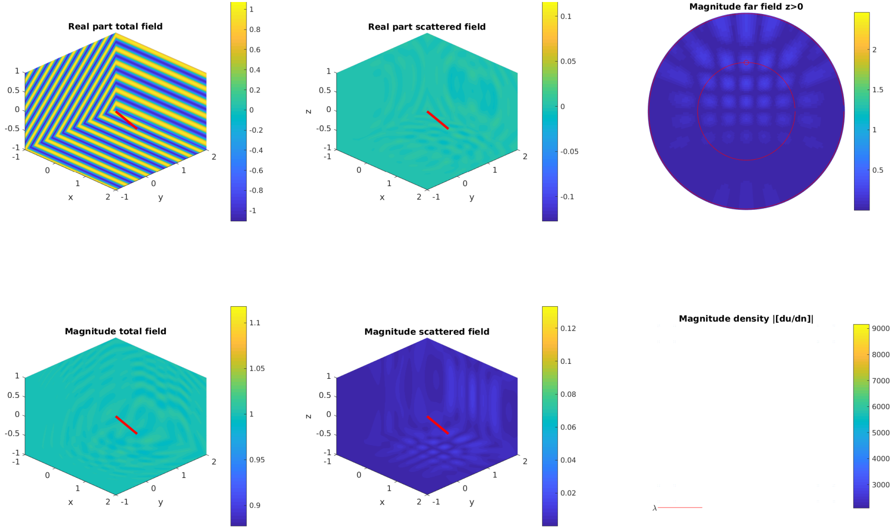
Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

$k = 25$, 4096 DOFs, prefractal level 2



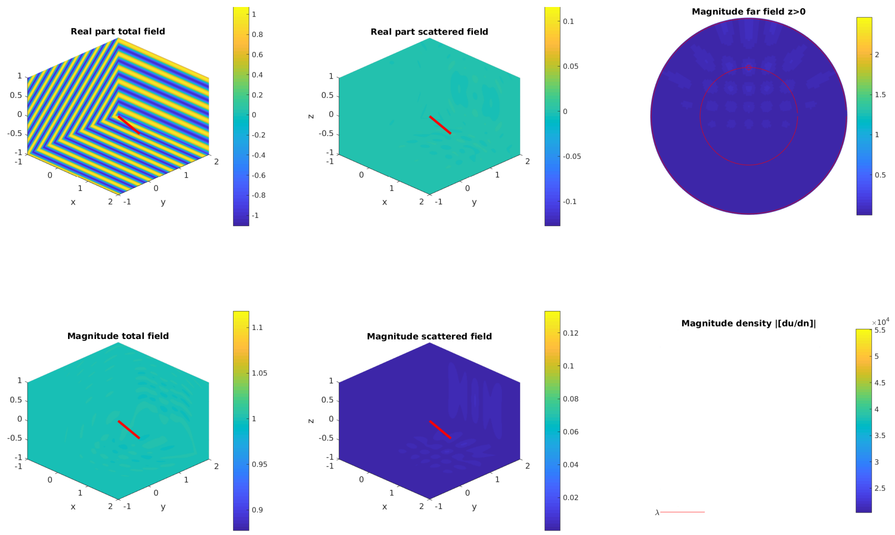
Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

$k = 25$, 4096 DOFs, prefractal level 3



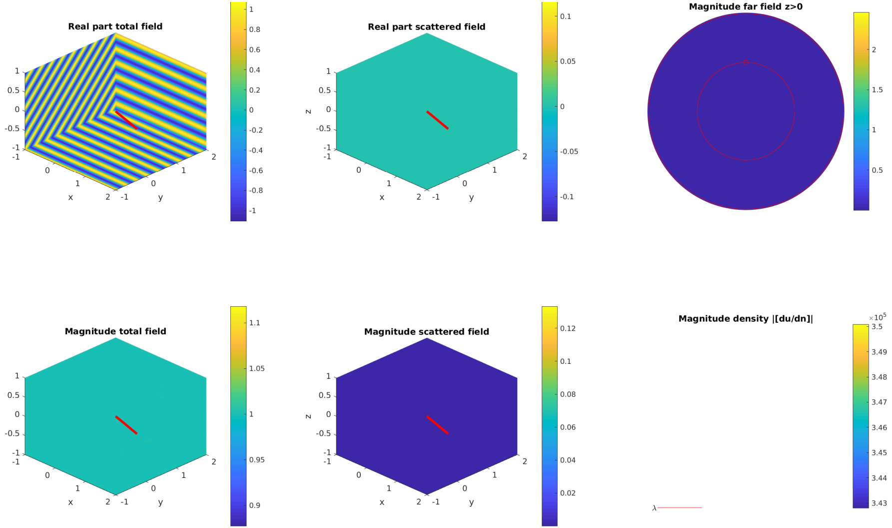
Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

$k = 25$,   4096 DOFs,   prefractal level 4



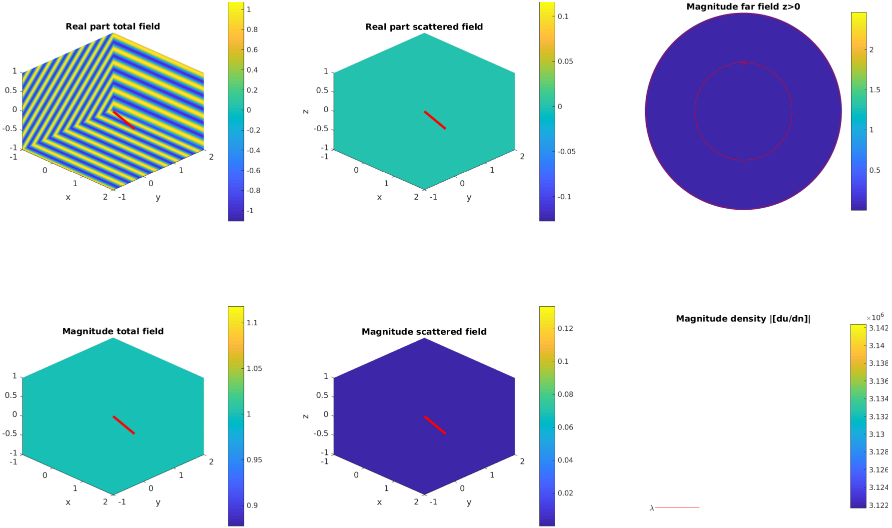
Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

$k = 25$, 4096 DOFs, prefractal level 5

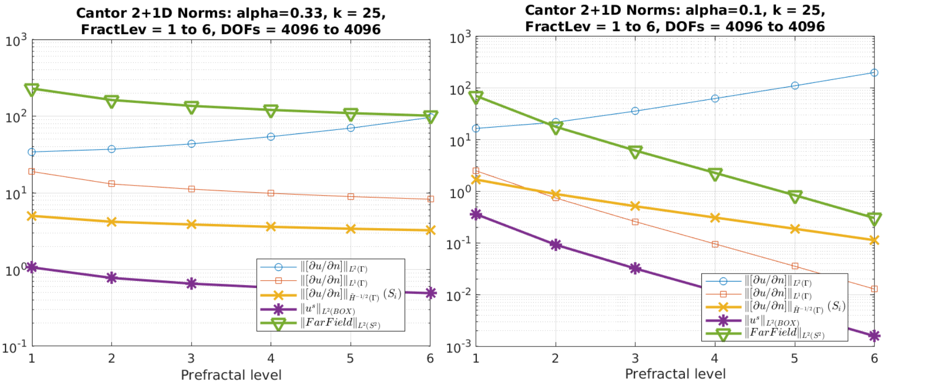


Numerical results: Cantor dust $\alpha = 0.1$ $(u = 0)$

$k = 25$, 4096 DOFs, prefractal level 6



Convergence of BEM solution norms: Cantor dust



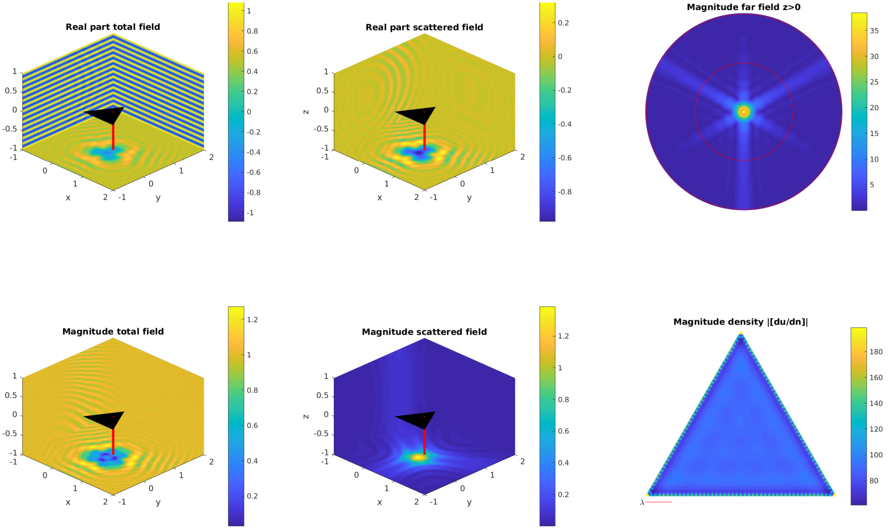
Norms of the solution on the prefractals converge:

- to a positive constant values for $\alpha = 1/3$ (left),
- to 0 for $\alpha = 1/10$ (right).

Numerical results: Sierpinski triangle

$k = 45,$ prefractal level 0, 2209 DOFs

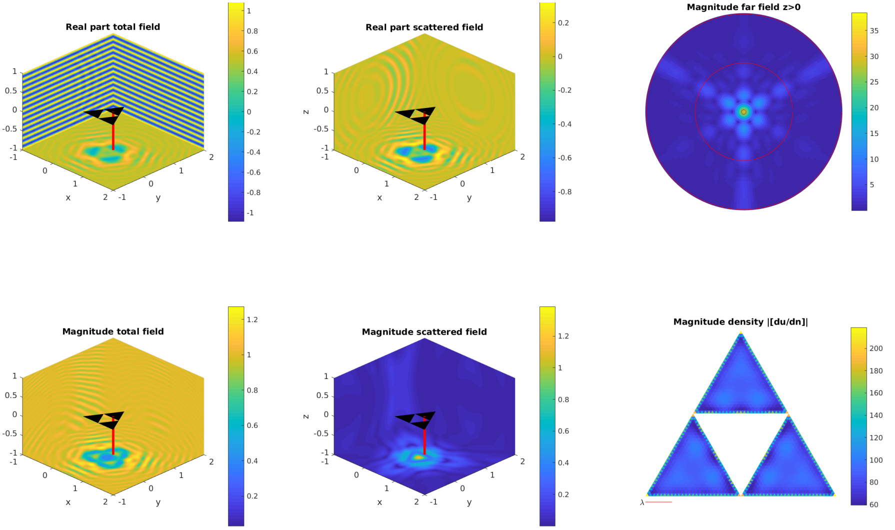
(Pr. levels 0 and 1 are not colour-scaled)



Numerical results: Sierpinski triangle

$k = 45$, prefractal level 1, 2187 DOFs

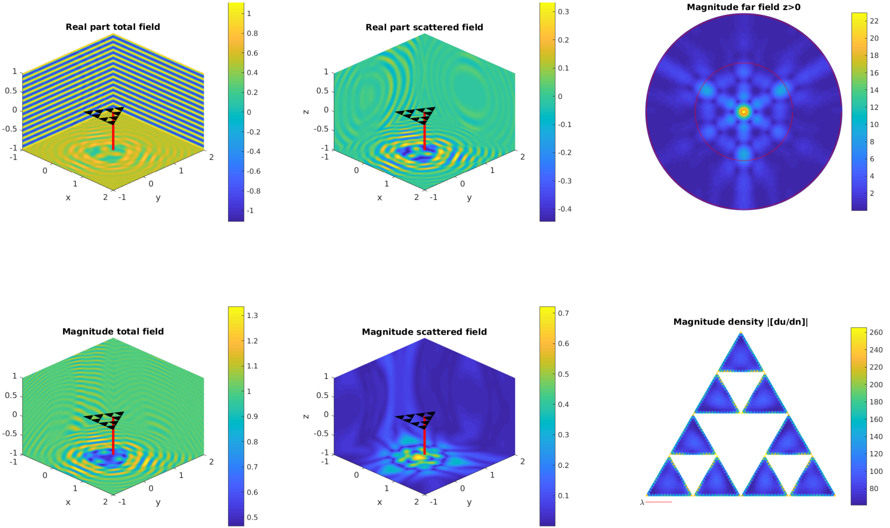
(Pr. levels 0 and 1 are not colour-scaled)



Numerical results: Sierpinski triangle

$k = 45$, prefractal level 2, 2304 DOFs

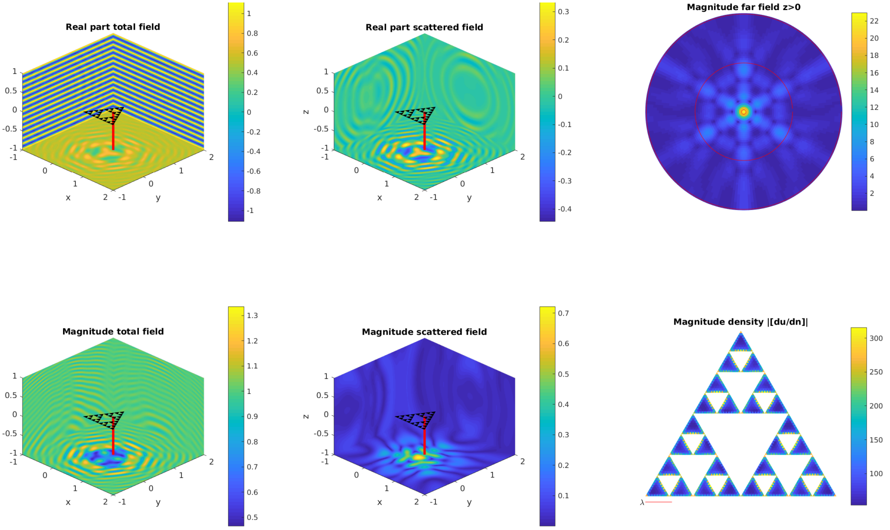
(Pr. levels 0 and 1 are not colour-scaled)



Numerical results: Sierpinski triangle

$k = 45$, prefractal level 3, 2187 DOFs

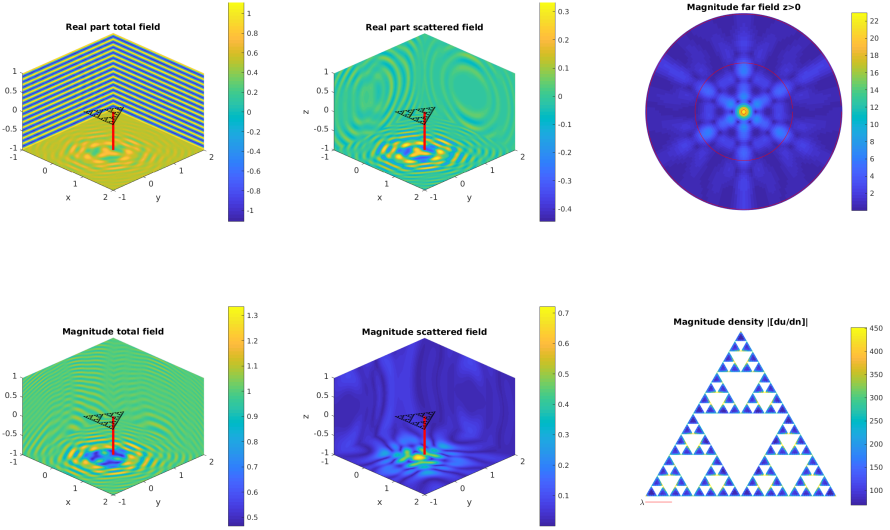
(Pr. levels 0 and 1 are not colour-scaled)



Numerical results: Sierpinski triangle

$k = 45$, prefractal level 4, 2916 DOFs

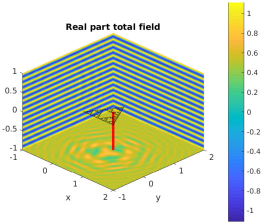
(Pr. levels 0 and 1 are not colour-scaled)

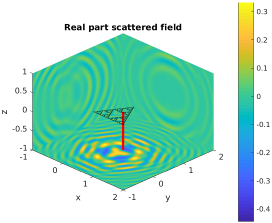


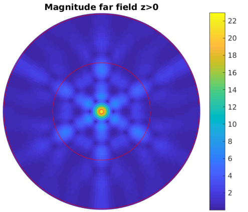
Numerical results: Sierpinski triangle

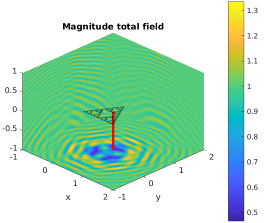
$k = 45$, prefractal level 5, 2187 DOFs

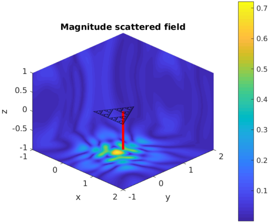
(Pr. levels 0 and 1 are not colour-scaled)

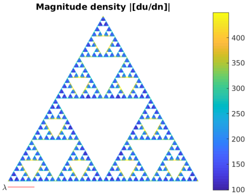








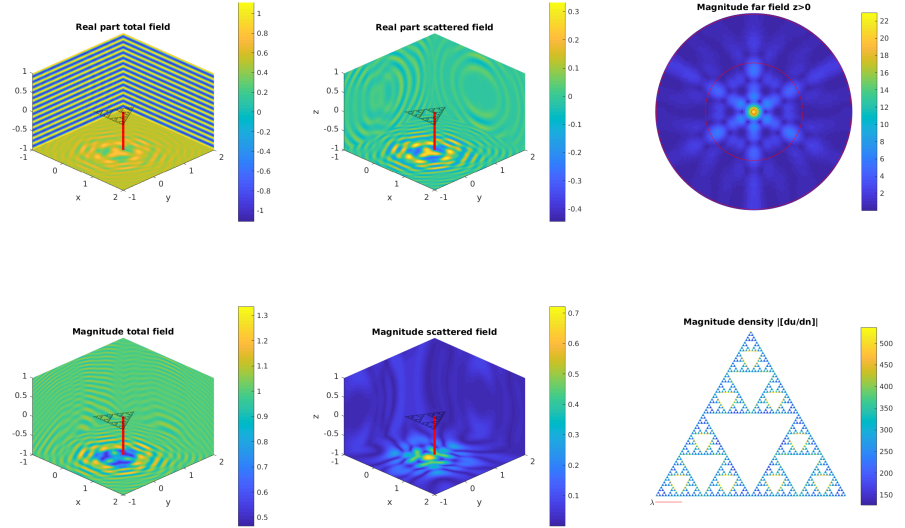




Numerical results: Sierpinski triangle

$k = 45$, prefractal level 6, 2916 DOFs

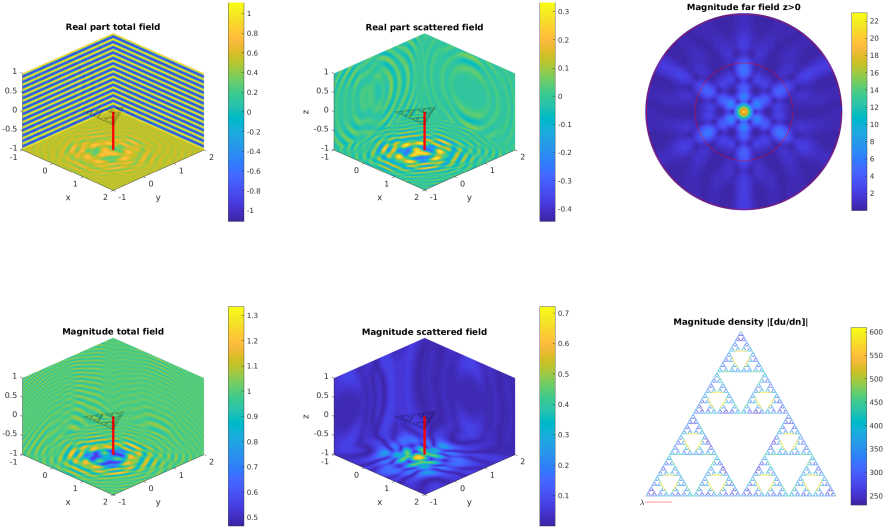
(Pr. levels 0 and 1 are not colour-scaled)



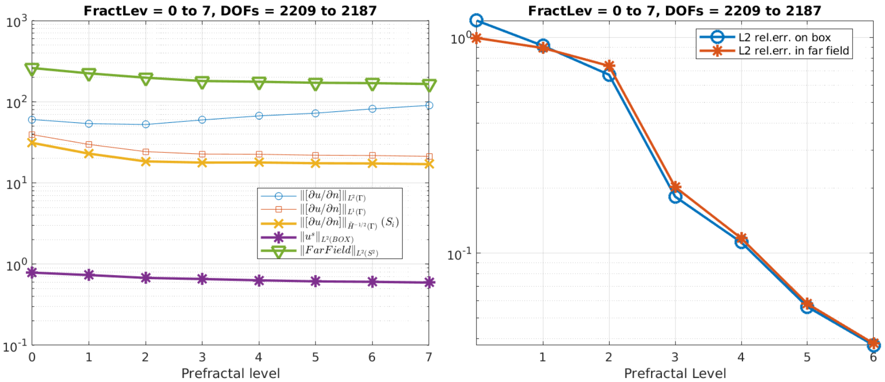
Numerical results: Sierpinski triangle

$k = 45$, prefractal level 7, 2187 DOFs

(Pr. levels 0 and 1 are not colour-scaled)



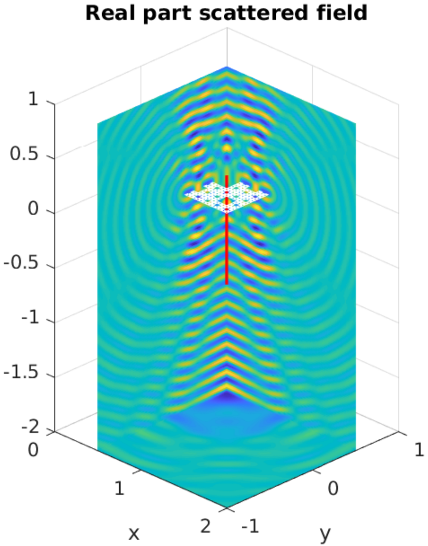
Convergence of BEM solutions: Sierpinski triangle



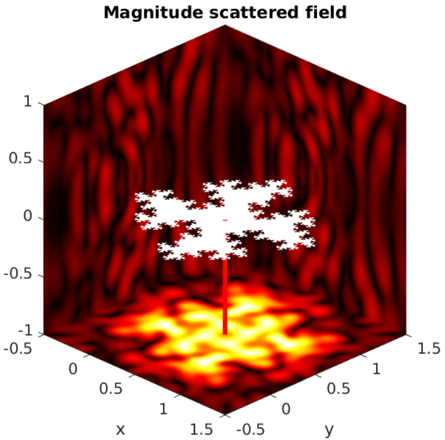
Right: $$\frac{\|w_j - w_7\|_{L^2(BOX)}}{\|w_7\|_{L^2(BOX)}}, \quad \frac{\|w_j - w_7\|_{L^2(FarField)}}{\|w_7\|_{L^2(FarField)}}.$$

(Prefractal level 3 is when density maxima are located and all wavelength-size prefractal features are resolved: big error reduction!)

Other shapes



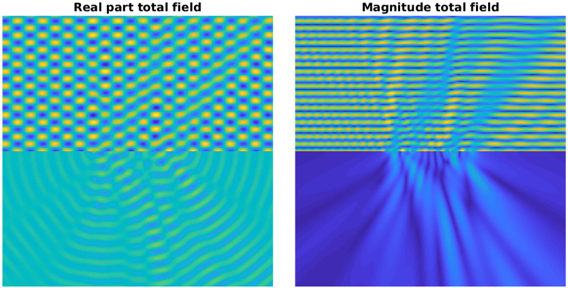
◁ Sierpinski carpet.



△ "Square snowflake",
limit of non-monotonic prefractals.

Apertures

Field through bounded apertures in unbounded Neumann screens computed via Babinet's principle.



$n = 1$, Cantor set $\alpha = 1/3$, prefractal level 12:
field through 0-measure holes!

Apertures

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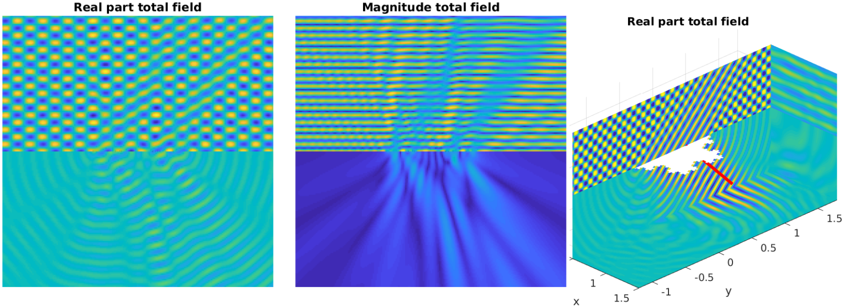

$n = 1$, Cantor set $\alpha = 1/3$, prefractal level 12:
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Koch snowflake-shaped aperture.

Experimental functional analysis!

Question: for Γ the open Koch snowflake, is $\widetilde{H}^{\pm 1/2}(\Gamma) = H_{\overline{\Gamma}}^{\pm 1/2}$?

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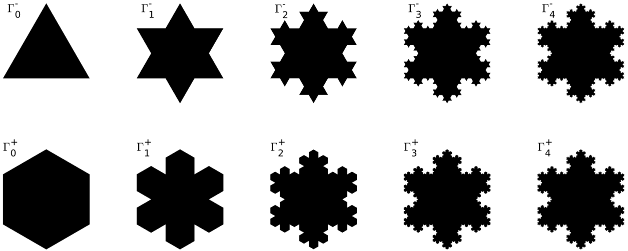
We can approximate Γ from inside and outside with polygons $\Gamma_j^{\pm}$:

$$\underbrace{\Gamma_1^- \subset \Gamma_2^-}_{\text{open}} \subset \Gamma_3^- \subset \cdots \subset \bigcup_{j\in\mathbb{N}} \Gamma_j^- = \Gamma \subset \overline{\Gamma} = \bigcap_{j\in\mathbb{N}} \Gamma_j^+ \subset \cdots \subset \Gamma_3^+ \subset \underbrace{\Gamma_2^+}_{\text{closed}} \subset \Gamma_1^+.$$



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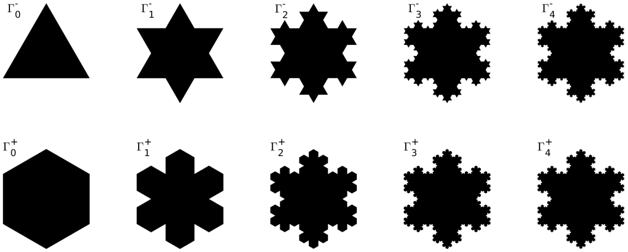

For a scattering BVP, $u_j^- \to u^- \in \widetilde{H}^{-1/2}(\Gamma)$, $u_j^+ \to u^+ \in H^{-1/2}_{\overline{\Gamma}}$,
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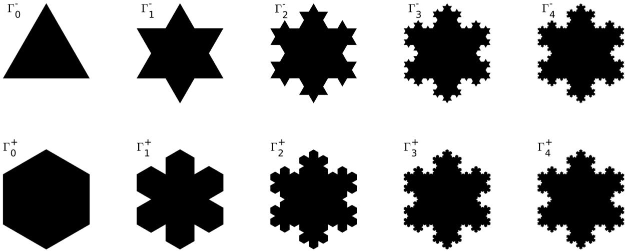

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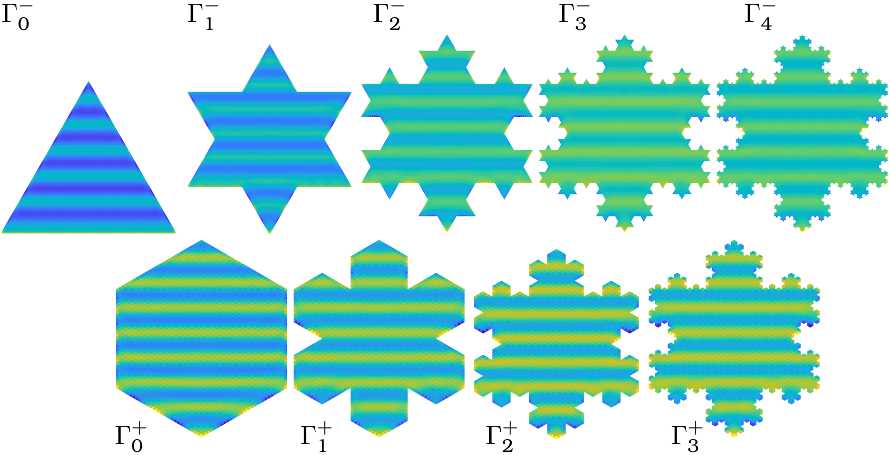
We study numerically if $u^- \stackrel{?}{=} u^+$, i.e. if inner and outer limits coincide.

Real part of fields on inner and outer prefractals



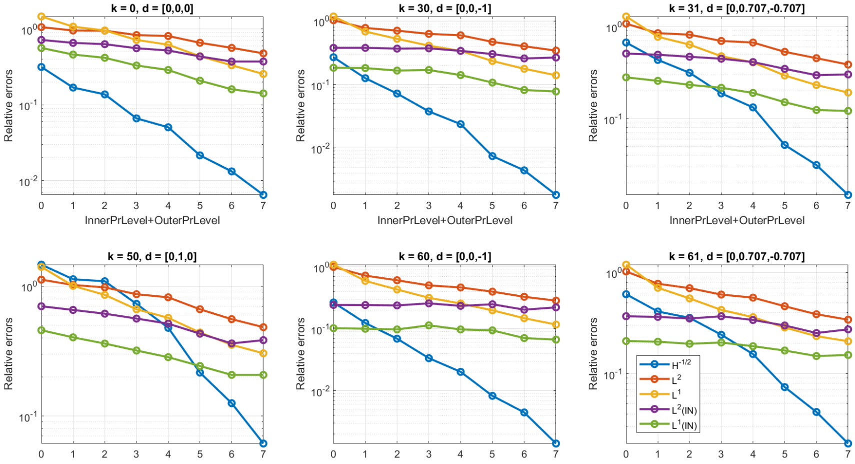
$k = 61$, $\mathbf{d} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^\top$, 3576 to 10344 DOFs, different colour scales.

Now I compare w_j^- against w_{j-1}^+ and w_j^+.

Inner and outer snowflake approximations

Blue lines are $\|w_j^- - w_l^+\|_{H^{-1/2}(\mathbb{R}^2)}$, converging fast to 0!

Evidence for $\widetilde{H}^{\pm 1/2}(\Gamma) = H^{\pm 1/2}_{\overline{\Gamma}}$?



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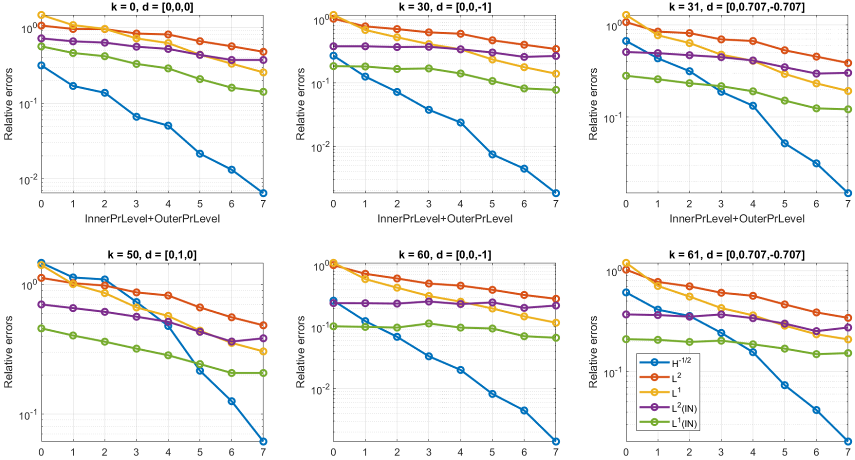

We can now prove $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ $\forall s \in \mathbb{R}$ for a class of snowflakes!

(Caetano + H + M, 2018)

Open questions

- How best to do numerical analysis in the joint limit of prefractal level and mesh refinement?
- Rates of convergence?
- Regularity theory for the fractal solution?
- Relation with “intrinsic” spaces?
- Approximation on fractals!
- What about curved screens?
- What about the Maxwell case?
  Other PDEs? (Laplace, reaction–diffusion already covered.)
- ...

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Thank you!