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A space–time continuous and coercive formulation for the wave equation

Paolo Bignardi — Andrea Moiola

https://matematica.unipv.it/moiola/

UNIVERSITÀ DI PAVIA
Department of Mathematics
"Felice Casorati"

Space–time methods

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$$\partial_{tt}^2 u - c^2 \Delta u = f$$

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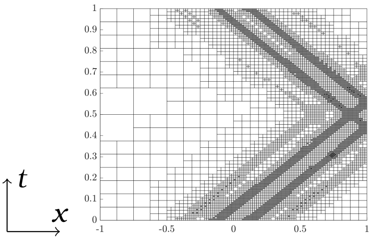
Why?

- local mesh refinement
- adaptivity
- parallelization
- moving boundaries and interfaces
- high-order solution at all $\mathbf{x}, t$
- use tools designed for elliptic PDEs
- ...



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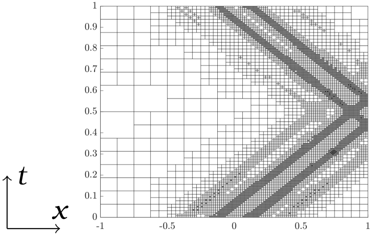

Existing conforming XT formulations for the wave equation

This slide mostly from Ferrari, Perugia, Zampa!

Existing conforming XT formulations for the wave equation

Second-order ($\partial_t^2 u - c^2\Delta u = f$) — Weak in space This slide mostly from Ferrari, Perugia, Zampa!

- (French, 1993), (Walkington, 2014), (Dong, Mascotto, Wang, 2024) CG(trial)-DG(test) in time
- (Ferrari, Perugia, 2025) splines in time, exp. weight

Second-order — Weak in space & time

- (Steinbach, Zank, 2019), (Zank, 2021) stab. with projection or penalty term
- (Ferrari, Fraschini, Loli, Moiola, Sangalli, 2024–2025x2) splines, stab. with penalty term
- (Löscher, Steinbach, Zank, 2023) modified Hilbert transform
- (Köthe, Löscher, Steinbach, 2023) least-squares

Second-order — Ultra-weak in space & time

- (Henning, Palitta, Simoncini, Urban, 2022) Petrov-Galerkin, L^2-H^2 regularity

First-order system

- (Bales, Lasiecka, 1994), (French, Peterson, 1996), (Gómez, 2025) CG-DG in time, $(u, \partial_t u)$
- (Ferrari, Fraschini, Loli, Perugia, 2025) CG-DG, splines, $(u, \partial_t u)$
- (Anselmann, Bause, Becher, Matthies, 2020x2) Petrov-Galerkin collocation, splines, $(u, \partial_t u)$
- (Berggren, Hägg, 2021) no numerical scheme, Friedrichs, $(\partial_t u, \nabla u)$
- (Führer, González, Karkulik, 2025) least squares (FOSLS), $(\partial_t u, \nabla u)$
- (Ferrari, Perugia, Zampa, 2025) exp. weight, $(u, \partial_t u)$

+ Plenty of XT-DG (Barucq, Diaz, Dörfler, Egger, Falk, Feistauer, Gopalakrishnan, Haber, Kretzschmar, Moiola, Monk, Perugia, Richter, Schnepp, Schöberl, van der Vegt, Wieners, Wintersteiger…)

Elliptic problems: variational form and Galerkin method

Dirichlet problem for Poisson eq.: $$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

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Abstract form: find $u \in V$ s.t. $b(u, v) = F(v) \quad \forall v \in V$

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Corresponds to square linear system: $\underline{\underline{\mathbf{B}}}\vec{\mathbf{U}} = \vec{\mathbf{F}}$, $\quad V_h = \mathrm{span}\{\varphi_1, \dots, \varphi_N\}$

$B_{i,j} := b(\varphi_j, \varphi_i), \quad F_i := F(\varphi_i), \quad u_h := \sum_{j=1}^N U_j \varphi_j$

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Galerkin well-posed and quasi-optimal for all subspaces $V_h \subset V = H^1_0(\Omega)$ by Lax–Milgram (e.g. piecewise polynomials of any degree on any mesh)

Requires continuity & coercivity: $\quad |b(u, v)| \le C_b \|u\|_V \|v\|_V, \quad b(v, v) \ge \underbrace{\gamma_b}_{>0} \|v\|^2_V \quad \forall u, v \in V$

Then $\underline{\underline{\mathbf{B}}}$ invertible and $\quad \|u - u_h\|_V \le \frac{C_b}{\gamma_b} \inf_{v_h \in V_h} \|u - v_h\|_V$

How to adapt techniques from elliptic PDEs to XT?

The same recipe does not work for the wave equation
Integrating by parts $\partial_{tt}^2 u - c^2 \Delta u = f$ in $Q = \Omega \times (0, T)$ we don't have coercivity:

$$\int_Q (-\partial_t v \partial_t v + c^2 \nabla v \cdot \nabla v) + \text{boundary terms} \quad \not\geq \quad \gamma_b \|v\|^2$$

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No stable formulation for "general" XT discrete spaces is available
Most of those listed earlier require tensor-product spaces V_h

But see: FOSLS scheme in (Führer, González, Karkulik 2025)

Coercivity from Morawetz multipliers: Helmholtz equation

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$$\Delta u + k^2 u = f \qquad k \gg 1$$

which models time-harmonic waves $\qquad U(\mathbf{x}, t) = \Re\{e^{-i\omega t} u(\mathbf{x})\}, \; k = \frac{\omega}{c}$

Helmholtz is prototype of indefinite elliptic PDE:

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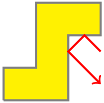
Trick: use Rellich–Morawetz multipliers and identities,
i.e. do integration by parts with a "special" test function (the multiplier)

Rellich–Morawetz multipliers are classical (1940, 1961) tools of scattering theory:
related to the "physics" of the problem, energy decay, ray (non)trapping,
require geometric conditions on the BVP



Abstract Morawetz argument

General linear BVP/IBVP: find $u \in V \subset L^2(D)$ s.t. $\begin{cases} \mathcal{L}u = f & \text{on } D \subset \mathbb{R}^d \\ \mathcal{B}u = g & \text{on } \partial D \end{cases}$

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Choose 1 multiplier $\mathcal{M} : V \to L^2(D)$ and 2 decomposition

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Choose 3 auxiliary Hermitian positive semi-definite $\ell : V \times V \to \mathbb{C}$, and $L_\ell \in V^*$
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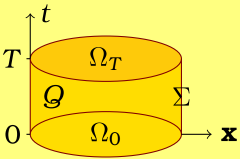
Impedance initial-boundary value problem

Interior impedance IBVP: $\Omega \subset \mathbb{R}^d$ Lipsch., bdd, constant $c, \theta > 0$

$$\begin{aligned} \partial_{tt}^2 u - c^2 \Delta u &= f && \text{on } Q := \Omega \times (0,T) \\ \partial_{\mathbf{n}} u + (\theta c)^{-1} \partial_t u &= g && \text{on } \Sigma := \partial\Omega \times (0,T) \\ u = u_0, \qquad \partial_t u &= u_1 && \text{on } \Omega_0 := \Omega \times \{0\} \end{aligned}$$



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Dirichlet = $\lim_{\theta \searrow 0}$ impedance Neumann = $\lim_{\theta \nearrow \infty}$ impedance

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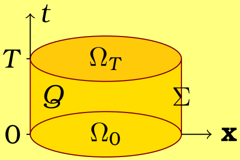

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Well-posedness and regularity: Faedo–Galerkin, adapting (LADYZHENSKAYA 1985)

$$f \in L^2(Q), g \in L^2(\Sigma), u_0 \in H^1(\Omega), u_1 \in L^2(\Omega) \quad \Rightarrow \quad \exists!\ u \in H^1(Q),\ \partial_t u \in L^2(\Sigma)$$

$$\partial_t f \in L^2(Q), \partial_t g \in L^2(\Sigma), u_0 \in H^2(\Omega), u_1 \in H^1(\Omega) \quad \Rightarrow \quad \partial_{tt}^2 u, \nabla \partial_t u \in L^2(Q), \partial_{tt}^2 u, \nabla u \in L^2(\Sigma)$$

$$\partial\Omega \text{ is } C^{1,1}, \quad \partial_t g \in L^2(0,T; H^{\frac{1}{2}}(\partial\Omega)) \quad \Rightarrow \quad u \in H^2(Q)$$

Morawetz multiplier and identities for the wave equation

Morawetz multiplier for the wave operator $\mathcal{W}v := \partial^2_{tt} v - c^2 \Delta v$:

$$\boxed{\mathcal{M}v := -\xi \mathbf{x} \cdot \nabla v + \beta(t - \nu T)\partial_t v} \qquad \xi, \beta > 0, \ \nu > 1.$$

Used in (MORAWETZ '61): energy decay of wave solutions outside star-shaped obstacles.

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Pointwise and integral identities:

$$\begin{aligned}\mathcal{M}u\,\mathcal{W}v+\mathcal{W}u\,\mathcal{M}v=\partial_t\Big[&\mathcal{M}u\,\partial_t v+\partial_t u\,\mathcal{M}v-\beta(t-\nu T)\left(\partial_t u\partial_t v-c^2\nabla u\cdot\nabla v\right)\Big]\\&-\nabla\cdot\Big[c^2\mathcal{M}u\,\nabla v+c^2\nabla u\,\mathcal{M}v-\xi\mathbf{x}\left(\partial_t u\partial_t v-c^2\nabla u\cdot\nabla v\right)\Big]\\&-(\beta+\xi d)\partial_t u\partial_t v-c^2\big(\beta+\xi(2-d)\big)\nabla u\cdot\nabla v\end{aligned}\qquad \boxed{u,v\in C^2}$$

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$$\begin{aligned}\int_Q(\mathcal{M}u\mathcal{W}v + \mathcal{W}u\mathcal{M}v) &= -\int_Q\left[(\beta+\xi d)\partial_t u\partial_t v + c^2(\beta+2\xi-d\xi)\nabla u\cdot\nabla v\right]\\ &\quad - \int_{\Omega_T}\left[\xi\mathbf{x}\cdot(\partial_t u\nabla v + \nabla u\partial_t v) - \beta(T-\nu T)(\partial_t u\partial_t v + c^2\nabla u\cdot\nabla v)\right]\\ \boxed{u,v\in H^2(Q)} &\quad + \int_{\Omega_0}\left[\xi\mathbf{x}\cdot(\partial_t u\nabla v + \nabla u\partial_t v) + \beta\nu T(\partial_t u\partial_t v + c^2\nabla u\cdot\nabla v)\right]\\ \text{"Morawetz–Green identity"} &\quad - \int_\Sigma\left[c^2\mathcal{M}u\partial_{\mathbf{n}}v + c^2\partial_{\mathbf{n}}u\mathcal{M}v + \xi\mathbf{x}\cdot\mathbf{n}(-\partial_t u\partial_t v + c^2\,\nabla u\cdot\nabla v)\right]\end{aligned}$$

XT variational formulation

Now follow the abstract framework: 1 $\mathcal{M}v = -\xi\mathbf{x}\cdot\nabla v + \beta(t-\nu T)\partial_t v$

2 decompose $\int_Q(\mathcal{W}u\mathcal{M}v + \mathcal{M}u\mathcal{W}v) = r(u,v) + s(u,v)$ with $s(u,v) = S_{u_0,u_1,g}(v)$,

3 introduce penalty $\ell(u,v) = A_Q T^2 \int_Q \mathcal{W}u\mathcal{W}v + A_{\Omega_0} T^{-1}\int_{\Omega_0} uv$ $A_Q, A_{\Omega_0} > 0$.

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- The solution u of the impedance IBVP satisfies $\boxed{b(u,v) = F(v)}$ $\forall v$ "smooth"

XT variational formulation

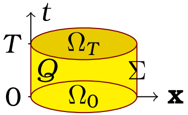
Now follow the abstract framework: 1 $\mathcal{M}v = -\xi\mathbf{x}\cdot\nabla v + \beta(t-\nu T)\partial_t v$

2 decompose $\int_Q(\mathcal{W}u\mathcal{M}v + \mathcal{M}u\mathcal{W}v) = r(u,v) + s(u,v)$ with $s(u,v) = S_{u_0,u_1,g}(v)$,

3 introduce penalty $\ell(u,v) = A_Q T^2 \int_Q \mathcal{W}u\mathcal{W}v + A_{\Omega_0}T^{-1}\int_{\Omega_0} uv$ $A_Q, A_{\Omega_0} > 0$.

- The solution u of the impedance IBVP satisfies $b(u,v) = F(v)$ $\forall v$ "smooth", with

$$\begin{aligned}
b(u,v) := &\int_Q \left[\mathcal{M}u\mathcal{W}v + (\beta+\xi d)\partial_t u\partial_t v + c^2(\beta+2\xi-d\xi)\nabla u\cdot\nabla v\right]\\
&+\int_{\Omega_T}\left[\xi\mathbf{x}\cdot(\partial_t u\,\nabla v + \nabla u\,\partial_t v) - \beta(T-\nu T)(\partial_t u\partial_t v + c^2\nabla u\cdot\nabla v)\right]\\
&+\int_\Sigma\left[c^2\mathcal{M}u\,\partial_{\mathbf{n}}v - c\theta^{-1}\partial_t u\mathcal{M}v + \xi\mathbf{x}\cdot\mathbf{n}(-\partial_t u\partial_t v + c^2\nabla u\cdot\nabla v)\right]\\
&+A_Q T^2\int_Q \mathcal{W}u\mathcal{W}v \quad + A_{\Omega_0}T^{-1}\int_{\Omega_0} uv
\end{aligned}$$



$$\begin{aligned}
F(v) := &\int_Q -f\mathcal{M}v - \int_\Sigma c^2 g\mathcal{M}v + A_Q T^2\int_Q f\mathcal{W}v + A_{\Omega_0}T^{-1}\int_{\Omega_0} u_0 v\\
&+\int_{\Omega_0}\left[\xi\mathbf{x}\cdot(u_1\,\nabla v + \nabla u_0\,\partial_t v) + \beta\nu T\left(u_1\partial_t v + c^2\nabla u_0\cdot\nabla v\right)\right]
\end{aligned}$$

Norm, continuity and coercivity

Simplest norm to ensure (explicit) continuity of $b(\cdot,\cdot), F(\cdot)$: $(L := \operatorname{diam}\Omega)$

$$
\begin{aligned}
\|v\|_V^2 := {}& \|\partial_t v\|_Q^2 + \|c\nabla v\|_Q^2 + T^2\|\mathcal{W}v\|_Q^2 \\
&+ T\|\partial_t v\|_{\Omega_0\cup\Omega_T}^2 + T\|c\nabla v\|_{\Omega_0\cup\Omega_T}^2 + T^{-1}\|v\|_{\Omega_0}^2 \\
&+ L\|\partial_t v\|_\Sigma^2 + L\|c\nabla v\|_\Sigma^2
\end{aligned}
$$

$$\|v\|_{H^1(Q)} \lesssim \|v\|_V \lesssim \|v\|_{H^2(Q)}$$

$$|b(u,v)| \le C_b\,\|u\|_V\,\|v\|_V$$

$$|F(v)| \le C_F\,\|v\|_V$$

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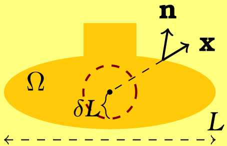
Coercivity

If Ω is star-shaped wrt $B_{\delta L}$ (i.e. $\mathbf{n}\cdot\mathbf{x} \ge \delta L$ a.e. on $\partial\Omega$),

$$\xi, A_Q, A_{\Omega_0} > 0, \quad \nu > 1, \quad \beta \ge \max\begin{cases}\xi(d-1)\\ \xi\frac{1}{\nu-1}\left(\frac{L}{cT}+1\right)\\ \xi\frac{1}{\nu-1}\frac{L}{cT}\left(\theta+\frac{1}{\delta\theta}\right)\end{cases}$$



Then, $\forall u$ with $\|u\|_V < \infty$, $\qquad \boxed{b(u,u) \ge \min\left\{\frac{\xi\delta}{4},\ A_Q,\ A_{\Omega_0}\right\}\|u\|_V^2}$

The proof is elementary: only uses Cauchy–Schwarz, Young, $\mathbf{n}\cdot\mathbf{x}\ge\delta L \ldots$

Star-shapedness assumption is related to non-trapping of rays.

With $\xi = A_Q = A_{\Omega_0} = 1$, $\nu = 2$: coercivity constant = $\delta/4$.

Function spaces

We have a C&C formulation in norm $\|\cdot\|_V$, but... what is the right function space?

$$V := \overline{C^\infty(\overline{Q})}^{\|\cdot\|_V}, \qquad W := \{v \in L^2(Q),\ \|v\|_V < \infty\}$$

$$H^2(Q) \overset{\text{dense}}{\subsetneq} V \overset{\text{closed}}{\subset} W \overset{\text{dense}}{\subsetneq} H^1(Q)$$

Is $V = W$?

Equivalently: is $C^\infty(\overline{Q})$ dense in W?

Related question for 1st-order systems in (Berggren, Hägg, JDE 2021).

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Let u be the impedance IBVP solution.

- If $u \in V \quad \Rightarrow \quad u$ solves $b(u, v) = F(v)\ \forall v \in V$.
- If $u \notin W \quad \Rightarrow \quad$ formulation is not applicable.
- What if $u \in W \setminus V$? Is it possible at all?

Galerkin discretisation

The formulation and any conforming Galerkin scheme are well-posed: $\forall V_N \subset V$

$$\exists!\ u_N \in V_N \qquad \text{such that} \qquad b(u_N, v_N) = F(v_N) \qquad \forall v_N \in V_N.$$

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Explicit quasi-optimality:
with coefficients $A_Q = A_{\Omega_0} = \xi = 1, \quad \nu = 2, \quad \beta \geq \beta^\star := d + \left(\theta + \frac{1}{\theta\delta}\right)\frac{L}{cT}$

$$\|u - u_N\|_V \leq C_{qo} \inf_{v_N \in V_N} \|u - v_N\|_V$$

$$C_{qo} = \frac{4\sqrt{3}}{\delta} \max\left\{3\beta + d,\ \left(1 + \frac{1}{\theta}\right)\left(1 + 2\beta\frac{cT}{L}\right)\right\} \qquad \text{(linear in } T\text{)}$$

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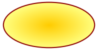
Norm $\|v\|_V$ contains residual term $\|\mathcal{W}v\|_Q$:
Piecewise-smooth space V_N is V-conforming $\iff$ $V_N \subset C^1(\overline{Q})$.
E.g.: IGA spaces made of splines.

What's left

- Extensions
- Numerics: smooth and singular solutions
- A posteriori estimator and adaptivity
- C^0-interior penalty formulation

Extensions: variable wavespeed and scattering problems

1 Smooth $c \in C^1(\Omega)$: add to bilinear form $b(u, v)$ the term $-\int_Q 2\xi(c^{-3}\nabla c \cdot \mathbf{x})\partial_t u \partial_t v$



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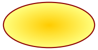

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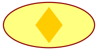

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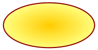

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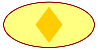

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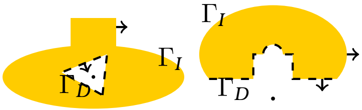

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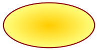

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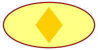

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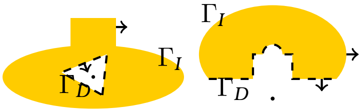

Radially decreasing wavespeed **1–2** (increasing refraction index)
and star-shaped obstacle **3** are non-trapping assumptions!

Linear system size

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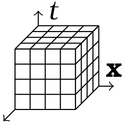
Take maximal-regularity IGA space $\mathbb{Q}^p(\mathcal{T}_h) \cap C^{p-1}(Q)$ on uniform Cartesian mesh:



$$Q = (0,L)^d \times (0,T), \qquad L \sim T \sim c \sim 1, \qquad N_x = L/h_x, \qquad N_t = T/h_t,$$

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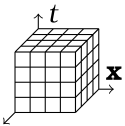

Compare with a single time-step of an implicit method, using $\mathbb{Q}^{p_{\mathrm{TS}}} \cap C^{p_{\mathrm{TS}}-1}(\Omega)$ in $\mathbf{x}$ only.
Assume time-step does not introduce any error other than approximation in space.
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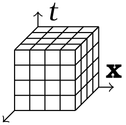

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We get comparable accuracy and #DOFs for full XT method on Q or 1 implicit time-step if we use slightly higher polynomial degree and coarser elements in the XT method:

$$p \sim \frac{d+1}{d} p_{TS}, \qquad N_x \sim N_t \sim N_{TS}^{\frac{p_{TS}}{p}} \sim N_{TS}^{\frac{d}{d+1}} \qquad (d \in \{1,2,3\})$$

Numerical test: smooth solution, parameter sensitivity

Matlab, $d = 1$, uniform tensor-product meshes and splines, $\mathbb{Q}^3(\mathcal{T}_h) \cap C^1(Q)$

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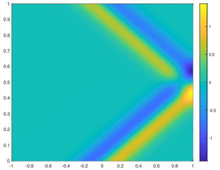
Difference of 2 Gaussians bouncing against impedance wall,
$f = g = 0$, $c = 2$, $\theta = 10$, $Q = (-1, 1) \times (0, 1)$



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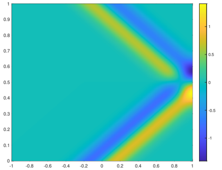
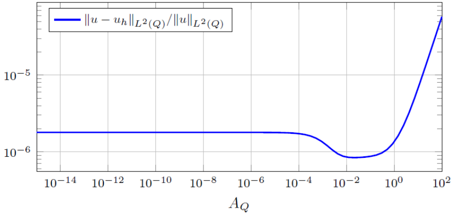
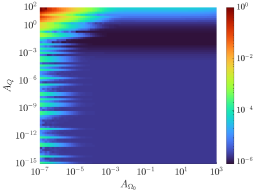


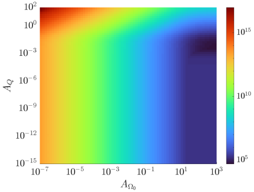
Parameter study: $\xi = 1, \nu = 2, \beta = \beta^\star$, $N_x = N_t = 32$
vary penalty terms A_Q, A_{Ω_0}:



◀ Can take $A_Q = 0$, drop $\int_Q \mathcal{W}u\mathcal{W}v$!

▼ Relative $L^2(Q)$ error and condition #





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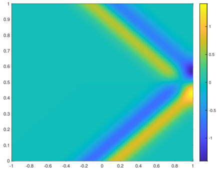

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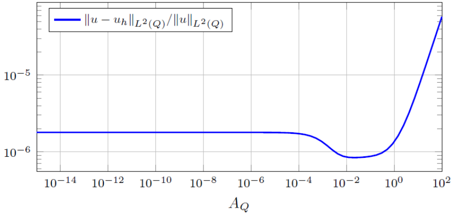

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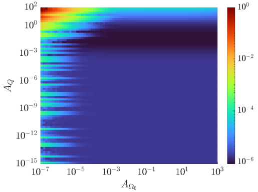
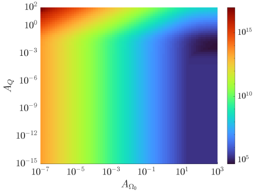
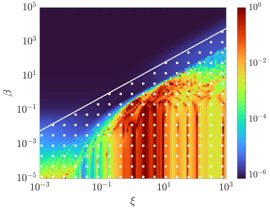
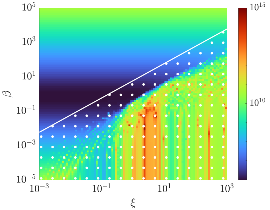




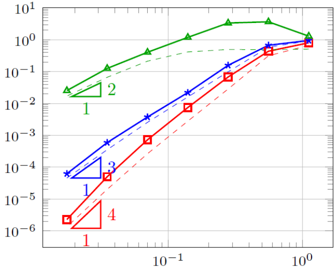
Vary ξ, β:

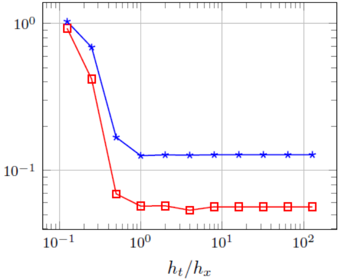




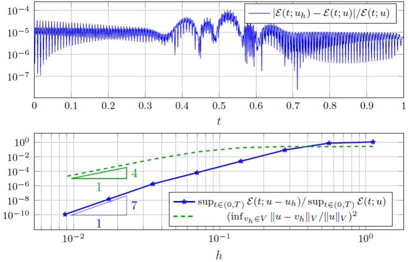
Numerical test: smooth solution, errors

▼ Relative errors: $L^2(Q)$, $H^1(Q)$, V
____ = Galerkin, _ _ _ = best approx





Galerkin energy error ▼



▲ Unconditional stability test, $h_t = 0.125$

Numerical test: singular solution

Singularity from incompatible boundary data:

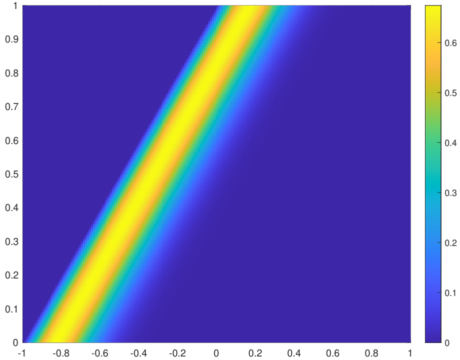
$f = g = 0$

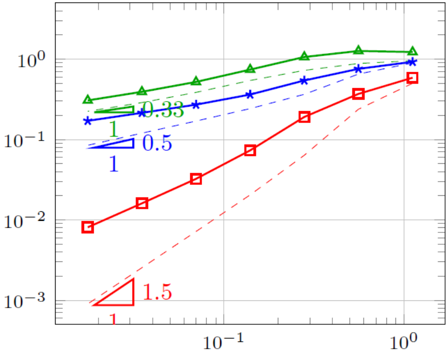
$u_0,\ u_1 = -u_0' \in C^\infty(\Omega)$

$Q = (-1,1) \times (0,1), \quad c = \theta = 1$

$u \in C^0(Q) \setminus C^1(Q) \quad — \quad u \in H^{3/2-\epsilon}(Q)$

$A_Q = 1$





A posteriori error estimator

With **Théophile Chaumont-Frelet**

Let u_N be Galerkin solution. Define residual error estimator $\eta^2 = \eta_Q^2 + \eta_{\Omega_0}^2 + \eta_\Sigma^2$ with

$$\eta_Q^2 = \left((\beta\nu)^2 + \frac{(\xi L)^2}{(cT)^2} + A_Q^2 \right) T^2 \| c(f - \mathcal{W} u_N) \|_Q^2$$

$$\eta_{\Omega_0}^2 = \left(2\frac{(\xi L)^2}{(cT)^2} + 2(\beta\nu)^2 \right) \left(T \| \nabla u_0 - \nabla u_N \|_{\Omega_0}^2 + T \| c^{-1}(u_1 - \partial_t u_N) \|_{\Omega_0}^2 \right) + A_{\Omega_0}^2 T^{-1} \| c^{-1} u_0 - u_N \|_{\Omega_0}^2$$

$$\eta_\Sigma^2 = \left(\xi^2 + (\beta\nu)^2 \frac{(cT)^2}{L^2} \right) L \| g_I - (\partial_{\mathbf{n}} u_N + (c\theta)^{-1} \partial_t u_N) \|_\Sigma^2$$

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Reliable and efficient: reliability equal to the coercivity constant

$$\min\left\{ \frac{\xi\delta}{4}, A_Q, A_{\Omega_0} \right\} \le \frac{\eta}{\|u - u_N\|_V} \le \max\{1, \theta^{-1}\} \max\left\{ \sqrt{2}, \frac{cT}{L} \right\} \left(\beta\nu + \frac{\xi L}{cT} + A_Q + A_{\Omega_0} \right)$$

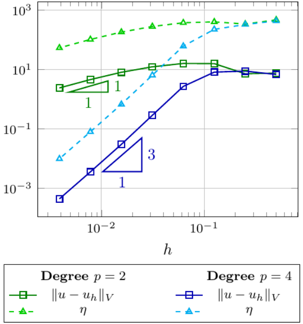
Fully explicit and computable

Localisable on mesh elements and faces: allows mesh adaptivity

A posteriori estimator and adaptivity: numerical results

Double-Gaussian example as before. Matlab+GeoPDEs, maximal regularity splines

Without adaptivity:



h	$\eta / \|u - u_h\|_V$	
	$p = 2$	$p = 4$
$5.0 \cdot 10^{-1}$	62.151	63.952
$2.5 \cdot 10^{-1}$	47.454	37.068
$1.3 \cdot 10^{-1}$	25.581	27.652
$6.3 \cdot 10^{-2}$	23.502	22.741
$3.1 \cdot 10^{-2}$	22.838	22.442
$1.6 \cdot 10^{-2}$	22.573	22.409
$7.8 \cdot 10^{-3}$	22.466	22.403
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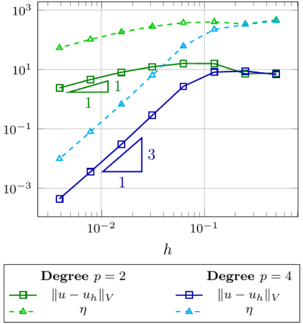

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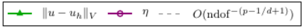
Adaptivity, Dörfler marking

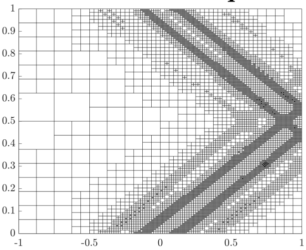
Truncated hierarchical B-splines (THB)

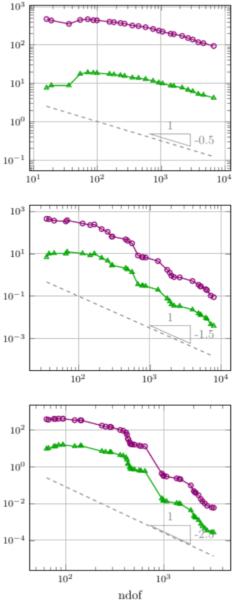
$p=2,4,6$ ▶

$$(\#DOFs)^{-\frac{p-1}{d+1}}$$



Mesh obtained for $p=2$:





XT C^0-interior penalty formulation

With **Théophile Chaumont-Frelet**

C^1-discretisations are cumbersome, want formulation that allows any C^0-FEM space!

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- Add terms with jumps $[\![\bullet]\!]$ and averages $\{\!\{\bullet\}\!\}$ on inner faces to bilinear form $b(u, v)$:

$$\int_{\mathcal{F}_h^{\text{inner}}} -\mathbf{n}^\top \mathbf{W}\mathbf{n}\Big(\{\!\{\mathcal{M}u\}\!\}[\![\partial_\mathbf{n} v]\!] + [\![\partial_\mathbf{n} u]\!]\{\!\{\mathcal{M}v\}\!\}\Big) - \xi\mathbf{x}\cdot\mathbf{n}\big((\nabla_T u)^\top [\![\mathbf{W}]\!]\nabla_T v\big)$$

$$+ A_{\text{IP}}\frac{L^2}{h}(\mathbf{n}^\top|\mathbf{W}|\mathbf{n})^{3/2}[\![\partial_\mathbf{n} u]\!][\![\partial_\mathbf{n} v]\!] \qquad \mathbf{W} = \operatorname{diag}(-1, \dots, -1, c^{-2}) \in \mathbb{R}^{d+1\times d+1}$$

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if $A_{\text{IP}} \geq \xi + \frac{C_{\text{inv}}}{\xi}(\xi + \beta\nu\frac{cT}{L})^2$ — use standard inverse estimate

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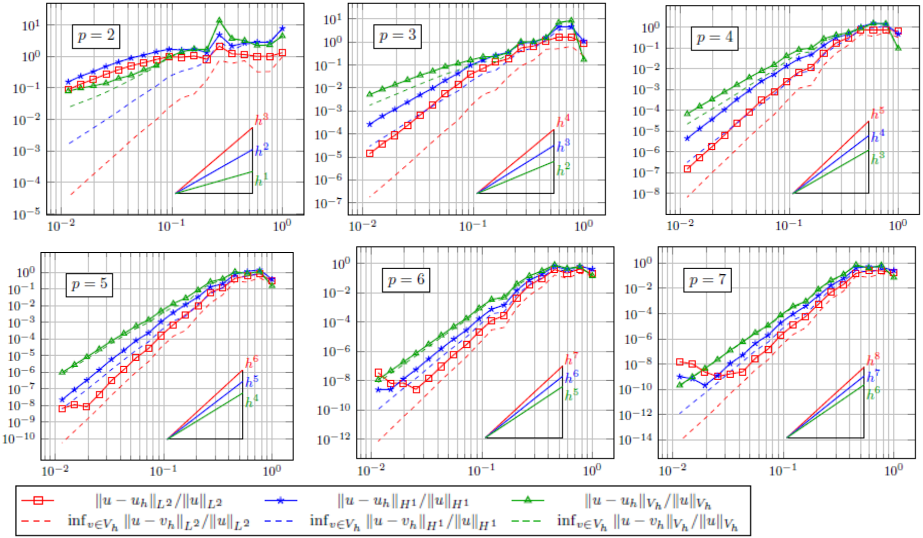
if $A_{\text{IP}} \geq \xi + \frac{C_{\text{inv}}}{\xi}(\xi + \beta\nu\frac{cT}{L})^2$ use standard inverse estimate
- Slightly modified quasi-optimality $(1 + \frac{\text{cont.}}{\text{coerc.}})$, estimator, a posteriori analysis. . .

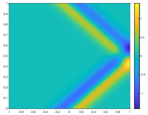
XT C^0-IP: numerics

NGSolve

Same double-Gaussian u as before

Quasi-uniform unstructured triangular meshes, h-convergence for $p = 2, \ldots, 7$:





$L^2(Q) \sim h^{p+1}$

$H^1(Q) \sim h^p$

$V \sim h^{p-1}$

That's all!

Coercive & continuous space–time variational formulation for impedance wave IBVPs

Explicit analysis based on Morawetz multipliers

Requires star-shaped domains and either C^1-conforming discrete spaces or C^0-IP

Extensions & improvements

- Wider class of non-trapping non star-shaped domains, non-reflecting BCs
- Sparsification, solvers, preconditioners, optimal rates in $H^1(Q), L^2(Q)$
- Maxwell, elasticity
- Clarify function space questions: $V = W$? Minimal data regularity ensuring $u \in V$

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adaptive→ `https://github.com/pbignardi/SplinesTests-XTWave/`

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