Vector Calculus

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These notes are meant to be a support for the vector calculus module (MA2VC/MA3VC) taking place at the University of Reading in the Autumn term 2016.

The present document **does not substitute the notes taken in class**, where more examples and proofs are provided and where the content is discussed in greater detail.

These notes are self-contained and cover the material needed for the exam. The suggested textbook is [1] by R.A. Adams and C. Essex, which you may already have from the first year; several copies are available in the University library. We will cover only a part of the content of Chapters 10–16, more precise references to single sections will be given in the text and in the table in Appendix G. This book also contains numerous exercises (useful to prepare the exam) and applications of the theory to physical and engineering problems. Note that there exists another book by the same authors containing the worked solutions of the exercises in the textbook (however, you should better try hard to solve the exercises and, if unsuccessful, discuss them with your classmates). Several other good books on vector calculus and vector analysis are available and you are encouraged to find the book that suits you best. In particular, Serge Lang's book [5] is extremely clear. [7] is a collection of exercises, many of which with a worked solution (and it costs less than 10£). You can find other useful books on shelf 515.63 (and those nearby) in the University library. The lecture notes [2], the book [3] and the "Vector Calculus Primer" [6] are available online; on the web page [4] of O. Knill you can find plenty of exercises, lecture notes and graphs. Note that different books inevitably use different notation and conventions. In this notes we will take for granted what you learned in the previous classes, so the first year notes might be useful from time to time (in particular those for calculus, linear algebra and analysis).

Some of the figures in the text have been made with Matlab. The scripts used for generating these plots are available on Blackboard and on the web page[1] of the course; most of them can be run in Octave as well. You can use and modify them: playing with the different graphical representations of scalar and vector fields is a great way to familiarise with these concepts. On the same page you can find the file `VCplotter.m`, which you can use to visualise fields, curves and changes of variables in Matlab or Octave.

Warning 1: *The paragraphs and the proofs marked with a star "★" are addressed to students willing to deepen the subject and learn about some closely related topics. Some of these remarks try to relate the topic presented here to the content of other courses (e.g., analysis or physics); some others try to explain some important details that were glossed over in the text. These parts are* ***not requested for the exam*** *and can safely be skipped by students with modest ambitions.*

Warning 2: *These notes are not entirely mathematically rigorous, for example we usually assume (sometimes tacitly) that fields and domains are "smooth enough" without specifying in detail what we mean with this assumption; multidimensional analysis is treated in a rigorous fashion in other modules, e.g. "analysis in several variables". On the other hand, the (formal) proofs of vector identities and of some theorems are a fundamental part of the lectures, and* ***at the exam you will be asked to prove some simple results****. The purpose of this course is not only to learn how to compute integrals and divergences!*

Suggestion: *The content of this module can be seen as the extension to multiple variables and vector quantities of the calculus you learned in the first year. Hence, many results you will encounter in this course correspond to simpler similar facts, holding for real functions, you already know well from previous classes. Comparing the vector results and formulas you will learn here with the scalar ones you already know will greatly simplify the study and the understanding of the content of this course. (This also means that you need to know and remember well what you learned in the first year.)*

If you find typos or errors of any kind in these notes, please let me know at `a.moiola@reading.ac.uk` or in person during the office hours, before or after the lectures.

[1] http://www.personal.reading.ac.uk/~st904897/VC2016/VC2016.html

1 Fields and vector differential operators

For simplicity, in these notes we only consider the 3-dimensional Euclidean space $\mathbb{R}^3$, and, from time to time, the plane $\mathbb{R}^2$. However, all the results not involving neither the vector product nor the curl operator can be generalised to Euclidean spaces $\mathbb{R}^N$ of *any* dimensions $N \in \mathbb{N}$.

1.1 Review of vectors in 3-dimensional Euclidean space

We quickly recall some notions about vectors and vector operations known from previous modules; see Sections 10.2–10.3 of [1] and the first year calculus and linear algebra notes. We use the word "**scalar**" simply to denote any real number $x \in \mathbb{R}$.

We denote by $\mathbb{R}^3$ the three-dimensional Euclidean vector space. A **vector** is an element of $\mathbb{R}^3$.

Notation. In order to distinguish scalar from vector quantities, we denote vectors with boldface and a little arrow: $\vec{\mathbf{u}} \in \mathbb{R}^3$. Note that several books use underlined ($\underline{u}$) symbols. We use the hat symbol ($\hat{\ }$) to denote **unit vectors**, i.e. vectors of length 1.

You are probably used to write a vector $\vec{\mathbf{u}} \in \mathbb{R}^3$ in "matrix notation" as a "column vector"

$$\vec{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where the scalars u_1, u_2 and $u_3 \in \mathbb{R}$ are the **components**, or coordinates, of $\vec{\mathbf{u}}$. We will always use the equivalent notation

$$\boxed{\vec{\mathbf{u}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}},}$$

where $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are three fixed vectors that constitute the **canonical basis** of $\mathbb{R}^3$. When we draw vectors, we always assume that the canonical basis has a **right-handed orientation**, i.e. it is ordered according to the right-hand rule: closing the fingers of the right hand from the $\hat{\boldsymbol{\imath}}$ direction to the $\hat{\boldsymbol{\jmath}}$ direction, the thumb points towards the $\hat{\boldsymbol{k}}$ direction. You can think at $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ as the vectors $\hat{\boldsymbol{\imath}} = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $\hat{\boldsymbol{\jmath}} = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ and $\hat{\boldsymbol{k}} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$, so that the two notations above are consistent: $\vec{\mathbf{u}} = u_1\hat{\boldsymbol{\imath}} + u_2\hat{\boldsymbol{\jmath}} + u_3\hat{\boldsymbol{k}} = u_1\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) + u_2\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right) + u_3\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}u_1\\u_2\\u_3\end{smallmatrix}\right)$.

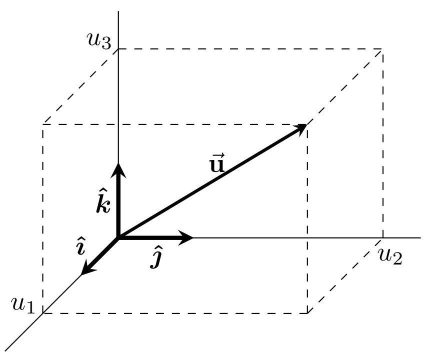


Figure 1: *The basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ and the components of the vector $\vec{\mathbf{u}}$.*

Definition 1.1. The **magnitude**, or **length**, or **norm**, of the vector $\vec{\mathbf{u}}$ is the scalar defined as [2]

$$\boxed{u := |\vec{\mathbf{u}}| := \sqrt{u_1^2 + u_2^2 + u_3^2}.}$$

The **direction** of a (non-zero) vector $\vec{\mathbf{u}}$ is the unit vector defined as

$$\boxed{\hat{\boldsymbol{u}} := \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}.}$$

Note that often the magnitude of a vector $\vec{\mathbf{u}}$ is written as $\|\vec{\mathbf{u}}\|$ (e.g. in your Linear Algebra lecture notes). We use the same notation $|\vec{\mathbf{u}}|$ for the magnitude of a vector and $|x|$ for the absolute value of a

[2] The notation "$A := B$" means "the object A is *defined* to be equal to the object B". A and B may be scalars, vectors, matrices, sets...

scalar, which can be thought as the magnitude of a one-dimensional vector; the meaning of the symbol should always be clear from the argument.

Every vector satisfies $\vec{\mathbf{u}} = |\vec{\mathbf{u}}|\hat{\boldsymbol{u}}$. Therefore **length and direction uniquely identify a vector**. The vector of length 0 (i.e. $\vec{\mathbf{0}} := 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$) does not have a specified direction. Note that, if we want to represent vectors with arrows, the point of application ("where" we draw the arrow) is not relevant: two arrows with the same direction and length represent the same vector; we can imagine that all the arrows have the application point in the origin $\vec{\mathbf{0}}$.

Physics provides many examples of vectors: e.g. velocity, acceleration, displacement, force, momentum.

Example 1.2 (The position vector)**.** The **position vector**

$$\boxed{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}$$

represents the position of a point in the three-dimensional Euclidean space relative to the origin. This is the only vector for which we do not use the notation $\vec{\mathbf{r}} = r_1\hat{\boldsymbol{\imath}} + r_2\hat{\boldsymbol{\jmath}} + r_3\hat{\boldsymbol{k}}$. We will use the position vector mainly for two purposes: (i) to describe subsets of $\mathbb{R}^3$ (for example $\{\vec{\mathbf{r}} \in \mathbb{R}^3, |\vec{\mathbf{r}}| < 1\}$ is the ball of radius 1 centred at the origin), and (ii) as argument of fields, which will be defined in Section 1.2.

★ **Remark 1.3** (Are vectors arrows, points, triples of numbers, or elements of a vector space?)**.** There are several different definitions of vectors, this fact may lead to some confusion. Vectors defined as geometric entities fully described by magnitude and direction are sometimes called "Euclidean vectors" or "geometric vectors". Note that, even though a vector is represented as an arrow, in order to be able to sum any two vectors the position of the arrow (the application point) has no importance.

One can use vectors to describe points, or positions, in the three-dimensional space, after an origin has been fixed. These are sometimes called "bound vectors". In this interpretation, the sum of two vectors does not make sense, while their difference is an Euclidean vector as described above. We use this interpretation to describe, for example, subsets of $\mathbb{R}^3$; the position vector $\vec{\mathbf{r}}$ is always understood as a bound vector.

Often, three-dimensional vectors are intended as triples of real numbers (the components). This is equivalent to the previous geometric definition once a canonical basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ is fixed. This approach is particularly helpful to manipulate vectors with a computer program (e.g. Matlab, Octave, Mathematica, Python...). We typically use this interpretation to write formulas and to do operations with vectors.

Vectors are usually rigorously defined as elements of an abstract "vector space" (who attended the linear algebra class should be familiar with this concept). This is an extremely general and powerful definition which immediately allows some algebraic manipulations (sums and multiplications with scalars) but in general does not provide the notions of magnitude, unit vector and direction. If the considered vector space is real, finite-dimensional and is provided with an inner product, then it is an Euclidean space (i.e., $\mathbb{R}^n$ for some natural number n). If a basis is fixed, then elements of $\mathbb{R}^n$ can be represented as n-tuples of real numbers (i.e., ordered sets of n real numbers).

See `http://en.wikipedia.org/wiki/Euclidean_vector` for a comparison of different definitions.

Several operations are possible with vectors. The most basic operations are the **addition** $\vec{\mathbf{u}} + \vec{\mathbf{w}}$ and the **multiplication** $\lambda\vec{\mathbf{u}}$ **with a scalar** $\lambda \in \mathbb{R}$:

$$\vec{\mathbf{u}} + \vec{\mathbf{w}} = (u_1 + w_1)\hat{\boldsymbol{\imath}} + (u_2 + w_2)\hat{\boldsymbol{\jmath}} + (u_3 + w_3)\hat{\boldsymbol{k}}, \qquad \lambda\vec{\mathbf{u}} = \lambda u_1\hat{\boldsymbol{\imath}} + \lambda u_2\hat{\boldsymbol{\jmath}} + \lambda u_3\hat{\boldsymbol{k}}.$$

These operations are defined in any vector space. In the following we briefly recall the definitions and the main properties of the scalar product, the vector product and the triple product. For more properties, examples and exercises we refer to [1, Sections 10.2–10.3].

> **Warning: frequent error 1.4.** There exist many operations involving scalar and vectors, but what is **never** possible is the addition of a scalar and a vector. Recall: for $\lambda \in \mathbb{R}$ and $\vec{\mathbf{u}} \in \mathbb{R}^3$ you can never write anything like "$\lambda + \vec{\mathbf{u}}$"! Writing a little arrow on each vector helps preventing mistakes.

★ **Remark 1.5.** The addition, the scalar multiplication and the scalar product are defined for Euclidean spaces of any dimension, while the vector product (thus also the triple product) is defined only in three dimensions.

1.1.1 Scalar product

Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$, the **scalar product** (also called **dot product** or **inner product**) gives in output a scalar denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$:

$$\boxed{\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} := u_1w_1 + u_2w_2 + u_3w_3} = |\vec{\mathbf{u}}||\vec{\mathbf{w}}|\cos\theta, \qquad (1)$$

where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$, as in Figure 2. The scalar product is commutative (i.e. $\vec{\mathbf{u}}\cdot\vec{\mathbf{w}} = \vec{\mathbf{w}}\cdot\vec{\mathbf{u}}$), is distributive with respect to the addition (i.e. $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}+\vec{\mathbf{w}}) = \vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}$) and can be used to compute the magnitude of a vector:

$$|\vec{\mathbf{u}}| = \sqrt{\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}}.$$

The scalar product can be used to evaluate the length of the **projection** of $\vec{\mathbf{u}}$ in the direction of $\vec{\mathbf{w}}$:

$$\text{projection} = |\vec{\mathbf{u}}|\cos\theta = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|} = \vec{\mathbf{u}}\cdot\hat{\boldsymbol{w}},$$

and the angle between two (non-zero) vectors:

$$\theta = \arccos\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}}{|\vec{\mathbf{u}}||\vec{\mathbf{w}}|}\right) = \arccos\big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{w}}\big).$$

The vector components can be computed as scalar products:

$$u_1 = \vec{\mathbf{u}}\cdot\hat{\boldsymbol{\imath}}, \quad u_2 = \vec{\mathbf{u}}\cdot\hat{\boldsymbol{\jmath}}, \quad u_3 = \vec{\mathbf{u}}\cdot\hat{\boldsymbol{k}}.$$

Two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are **orthogonal** or **perpendicular** if their scalar product is zero: $\vec{\mathbf{u}}\cdot\vec{\mathbf{w}} = 0$; they are **parallel** (or collinear) if $\vec{\mathbf{u}} = \alpha\vec{\mathbf{w}}$ for some scalar $\alpha \neq 0$ (in physics, if $\alpha < 0$ they are sometimes called "antiparallel").

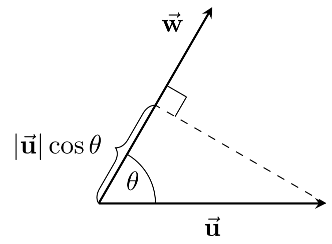


Figure 2: *The projection of $\vec{\mathbf{u}}$ in the direction of $\vec{\mathbf{w}}$.*

Exercise 1.6. ► Compute the scalar product between the elements of the canonical basis: $\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{k}}$ ...

Exercise 1.7 (Orthogonal decomposition)**.** ► Given two non-zero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$, prove that there exists a unique pair of vectors $\vec{\mathbf{u}}_\perp$ and $\vec{\mathbf{u}}_{||}$, such that: *(a)* $\vec{\mathbf{u}}_\perp$ is perpendicular to $\vec{\mathbf{w}}$, *(b)* $\vec{\mathbf{u}}_{||}$ is parallel to $\vec{\mathbf{w}}$ and *(c)* $\vec{\mathbf{u}} = \vec{\mathbf{u}}_\perp + \vec{\mathbf{u}}_{||}$.

Hint: in order to show the existence of $\vec{\mathbf{u}}_\perp$ and $\vec{\mathbf{u}}_{||}$ you can proceed in two ways. *(i)* You can use the condition "$\vec{\mathbf{u}}_{||}$ is parallel to $\vec{\mathbf{w}}$" to represent $\vec{\mathbf{u}}_\perp$ and $\vec{\mathbf{u}}_{||}$ in dependence of a parameter α, and use the condition "$\vec{\mathbf{u}}_{||}$ is perpendicular to $\vec{\mathbf{w}}$" to find an equation for α itself. *(ii)* You can use your geometric intuition to guess the expression of the two desired vectors and then verify that they satisfy all the required conditions. (Do not forget to prove uniqueness.)

1.1.2 Vector product

Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$, the **vector product** (or **cross product**) gives in output a vector denoted by $\vec{\mathbf{u}}\times\vec{\mathbf{w}}$ (sometimes written $\vec{\mathbf{u}}\wedge\vec{\mathbf{w}}$ to avoid confusion with the letter x on the board):

$$\boxed{\begin{aligned}\vec{\mathbf{u}}\times\vec{\mathbf{w}} :&= \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\end{vmatrix}\\ &= \hat{\boldsymbol{\imath}}\begin{vmatrix}u_2 & u_3\\ w_2 & w_3\end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix}u_1 & u_3\\ w_1 & w_3\end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix}u_1 & u_2\\ w_1 & w_2\end{vmatrix}\\ &= (u_2w_3 - u_3w_2)\hat{\boldsymbol{\imath}} + (u_3w_1 - u_1w_3)\hat{\boldsymbol{\jmath}} + (u_1w_2 - u_2w_1)\hat{\boldsymbol{k}},\end{aligned}} \qquad (2)$$

where $|::|$ denotes the matrix determinant. Note that the 3×3 determinant is a "formal determinant", as the matrix contains three vectors and six scalars: it is only a short form for the next expression containing three "true" 2×2 determinants.

The magnitude of the vector product

$$|\vec{\mathbf{u}}\times\vec{\mathbf{w}}| = |\vec{\mathbf{u}}||\vec{\mathbf{w}}|\sin\theta$$

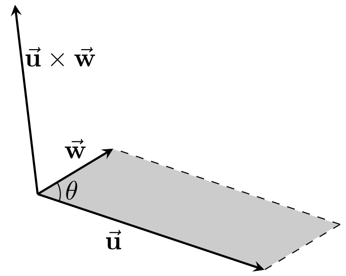


Figure 3: *The vector product $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$ has magnitude equal to the area of the grey parallelogram and is orthogonal to the plane that contains it.*

is equal to the **area** of the parallelogram defined by $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$. Its direction is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ and the triad $\vec{\mathbf{u}}, \vec{\mathbf{w}}, \vec{\mathbf{u}} \times \vec{\mathbf{w}}$ is right-handed. The vector product is distributive with respect to the sum (i.e. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$ and $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$) but is **not associative**: in general $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \neq (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \times \vec{\mathbf{w}}$.

Exercise 1.8. ▶ Prove what was claimed above: given $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}} \in \mathbb{R}^3$, their vector product $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$ is orthogonal to both of them.

Exercise 1.9. ▶ Show that the elements of the standard basis satisfy the following identities:

$$\begin{aligned} \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} &= \hat{\boldsymbol{k}}, & \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} &= \hat{\boldsymbol{\imath}}, & \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} &= \hat{\boldsymbol{\jmath}}, \\ \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} &= -\hat{\boldsymbol{k}}, & \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} &= -\hat{\boldsymbol{\imath}}, & \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} &= -\hat{\boldsymbol{\jmath}}, \end{aligned}$$
$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \vec{\mathbf{0}}.$$

Exercise 1.10. ▶ Show that the vector product is anticommutative, i.e.

$$\vec{\mathbf{w}} \times \vec{\mathbf{u}} = -\vec{\mathbf{u}} \times \vec{\mathbf{w}} \tag{3}$$

for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ in $\mathbb{R}^3$, and satisfies $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$.

Exercise 1.11. ▶ Prove that the vector product is not associative by showing that $(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) \times \hat{\boldsymbol{\jmath}} \neq \hat{\boldsymbol{\imath}} \times (\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}})$.

Exercise 1.12. ▶ Show that the following identity holds true:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}}(\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}) - \vec{\mathbf{w}}(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) \qquad \forall \vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3. \tag{4}$$

(Sometimes $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ is called "triple vector product".)

Exercise 1.13. ▶ Show that the following identities hold true for all $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}, \vec{\mathbf{p}} \in \mathbb{R}^3$:

$$\begin{aligned} &\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) + \vec{\mathbf{v}} \times (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) + \vec{\mathbf{w}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{0}} && \text{(Jacobi identity)}, \\ &(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{p}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{p}}) - (\vec{\mathbf{u}} \cdot \vec{\mathbf{p}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}) && \text{(Binet–Cauchy identity)}, \\ &|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|^2 + (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 = |\vec{\mathbf{u}}|^2 |\vec{\mathbf{v}}|^2 && \text{(Lagrange identity)}. \end{aligned}$$

Hint: to prove the first one you can either proceed componentwise (long and boring!) or use identity (4). For the second identity you can expand both sides and collect some terms appropriately. The last identity will follow easily from the previous one.

★ **Remark 1.14.** The vector product is used in physics to compute the angular momentum of a moving object, the torque of a force, the Lorentz force acting on a charge moving in a magnetic field.

Given three vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, their **triple product** is the scalar

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}).$$

Its absolute value is the **volume** of the parallelepiped P defined by the three vectors as in Figure 4. To see this, we define the unit vector orthogonal to the plane containing $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ as $\hat{\boldsymbol{n}} := \frac{\vec{\mathbf{v}} \times \vec{\mathbf{w}}}{|\vec{\mathbf{v}} \times \vec{\mathbf{w}}|}$. Then

$$\text{Volume } (P) := (\text{area of base}) \;\times\; \text{height} = |\vec{\mathbf{v}} \times \vec{\mathbf{w}}|\, |\vec{\mathbf{u}} \cdot \hat{\boldsymbol{n}}| = |\vec{\mathbf{v}} \times \vec{\mathbf{w}}| \left| \vec{\mathbf{u}} \cdot \frac{\vec{\mathbf{v}} \times \vec{\mathbf{w}}}{|\vec{\mathbf{v}} \times \vec{\mathbf{w}}|} \right| = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|\,.$$

From the definition (2) of vector product, we see that the triple product can be computed as the determinant of the matrix of the vector components:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{5}$$
$$= u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_1v_3w_2 - u_2v_1w_3 - u_3v_2w_1.$$

The triple product $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ is zero if the three vectors are linearly dependent, is positive if the triad $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ is right-handed, and is negative otherwise.

Exercise 1.15. ▶ Show the following identities (you may use the determinant representation (5))

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \qquad \forall \vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3.$$

What is the relation between $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ and $\vec{\mathbf{w}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}})$? What is $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{u}})$?

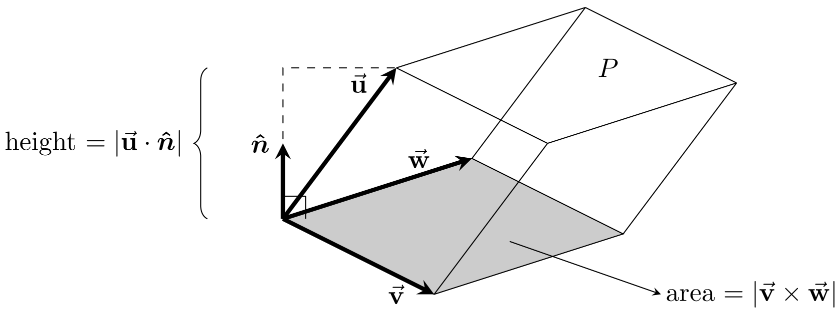


Figure 4: *The triple product* $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ *as the volume of a parallelepiped.*

Warning: frequent error 1.16 (Ambiguous expressions)**.** It is extremely important that all the formulas we write are unambiguous. For this reason, a correct use of brackets is fundamental, especially when dealing with vectors. For example, the expression "$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \times \vec{\mathbf{w}}$" is **ambiguous and not acceptable** because it does not make clear the order in which the products are performed: it might be interpreted either as $(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \times \vec{\mathbf{w}}$ or as $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$, which are in general not equal to each other; cf. Exercise 1.11.

On the other hand $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ might be written $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}}$, which is unambiguous and acceptable because there is only one possible way of bracketing this expression, i.e. $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$. The other option, "$(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) \times \vec{\mathbf{w}}$", makes no sense because it contains a vector product between a scalar and a vector. When in doubt, however, it is safer to write some extra brackets (in the correct positions) rather than skip them.

Recall also that by convention all kinds of products (scalar $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$, vector $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$, scalar-times-vector $\lambda\vec{\mathbf{u}}$) have priorities over sums. E.g. $\vec{\mathbf{u}} + \vec{\mathbf{v}} \times \vec{\mathbf{w}}$ means $\vec{\mathbf{u}} + (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ and is not equal to $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times \vec{\mathbf{w}}$.

Both scalar and vector products are *products*: each term is made of sums of products between a coefficient of the first vector and a coefficient of the second one. As such, they enjoy the distributive property with respect to addition: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})$ and $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) + (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$. This property involves addition, there is no such thing as distributive property between two products, e.g. $\lambda(\vec{\mathbf{u}} \times \vec{\mathbf{w}}) \neq (\lambda\vec{\mathbf{u}}) \times (\lambda\vec{\mathbf{w}})$ (a correct identity is $\lambda(\vec{\mathbf{u}} \times \vec{\mathbf{w}}) = (\lambda\vec{\mathbf{u}}) \times \vec{\mathbf{w}} = \vec{\mathbf{u}} \times (\lambda\vec{\mathbf{w}})$).

1.1.3 ★ Open and closed subsets of $\mathbb{R}^3$

Comparison with scalar calculus 1.17. In one dimension we call an interval "open" if it does not include its endpoints, e.g. $(0, 1) = \{t \in \mathbb{R}, 0 < t < 1\}$ and we call it "closed" if it includes them, e.g. $[0, 1] = \{t \in \mathbb{R}, 0 \le t \le 1\}$. More generally, you should remember from last year's Real Analysis lecture notes the definitions of open and closed subsets of $\mathbb{R}$ (see Definition 7.1 and Proposition 7.10 therein). Open sets are also related to the convergence of sequences. In this section we see how to extend these concepts to $\mathbb{R}^3$ (or $\mathbb{R}^n$ for any $n \in \mathbb{N}$).

Given a sequence of vectors $\{\vec{\mathbf{u}}_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^3$, i.e. a set of vectors indexed by a natural number $j \in \mathbb{N}$, we say that $\{\vec{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ has **limit** $\vec{\mathbf{u}} \in \mathbb{R}^3$ if $\lim_{j\to\infty} |\vec{\mathbf{u}}_j - \vec{\mathbf{u}}| = 0$. In this case, we also say that $\{\vec{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ converges to $\vec{\mathbf{u}}$ and we write $\vec{\mathbf{u}} = \lim_{j\to\infty} \vec{\mathbf{u}}_j$ and $\vec{\mathbf{u}}_j \to \vec{\mathbf{u}}$. (Note that, thanks to the use of the magnitude of the difference $\vec{\mathbf{u}}_j - \vec{\mathbf{u}}$, we have defined the limit of a sequence of vectors using the limit of a sequence of real numbers.)

Exercise 1.18. ▶ (i) Given a sequence of vectors $\{\vec{\mathbf{u}}_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^3$ and $\vec{\mathbf{u}} \in \mathbb{R}^3$, prove that $\lim_{j\to\infty} \vec{\mathbf{u}}_j = \vec{\mathbf{u}} \in \mathbb{R}^3$ if and only if each of the three real sequences of the three components of $\vec{\mathbf{u}}_j$ converges to the corresponding component of $\vec{\mathbf{u}}$. In formulas, you have to prove that

$$\lim_{j\to\infty} \vec{\mathbf{u}}_j = \vec{\mathbf{u}} \qquad \Longleftrightarrow \qquad \lim_{j\to\infty} (\vec{\mathbf{u}}_j)_1 = u_1, \quad \lim_{j\to\infty} (\vec{\mathbf{u}}_j)_2 = u_2, \quad \lim_{j\to\infty} (\vec{\mathbf{u}}_j)_3 = u_3.$$

(ii) Find a sequence of vectors $\{\vec{\mathbf{u}}_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^3$ and a vector $\vec{\mathbf{u}} \in \mathbb{R}^3$ such that $\lim_{j\to\infty} |\vec{\mathbf{u}}_j| - |\vec{\mathbf{u}}| = 0$ but $\vec{\mathbf{u}}_j$ does not converge to $\vec{\mathbf{u}}$.

A set $D \subset \mathbb{R}^3$ is called an **open set** if for every point $\vec{\mathbf{p}} \in D$, there exists $\epsilon > 0$ (depending on $\vec{\mathbf{p}}$) such that all points $\vec{\mathbf{q}}$ at distance smaller than ϵ from $\vec{\mathbf{p}}$ belong to D. In formulas:

$$D \text{ is open if } \quad \forall \vec{\mathbf{p}} \in D, \exists \epsilon > 0 \text{ s.t. } \{\vec{\mathbf{q}} \in \mathbb{R}^3, |\vec{\mathbf{q}} - \vec{\mathbf{p}}| < \epsilon\} \subset D.$$

The word **domain** is used as a synonym of open set (because the domain of definition of scalar and vector fields, described in the following, is usually chosen to be an open set). A set $C \subset \mathbb{R}^3$ is called a **closed set** if for all sequences contained in C and converging to a limit in $\mathbb{R}^3$, the limit belongs to C. In formulas:

$$C \text{ is closed if } \quad \forall \{\vec{\mathbf{p}}_j\}_{j\in\mathbb{N}} \subset C \text{ s.t. } \lim_{j\to\infty} \vec{\mathbf{p}}_j = \vec{\mathbf{p}} \in \mathbb{R}^3, \text{ we have } \vec{\mathbf{p}} \in C.$$

Examples of open sets are the open unit ball $\{\vec{\mathbf{r}} \in \mathbb{R}^3, |\vec{\mathbf{r}}| < 1\}$, the open unit cube $\{\vec{\mathbf{r}} \in \mathbb{R}^3, 0 < x, y, z < 1\}$, the open half space $\{\vec{\mathbf{r}} \in \mathbb{R}^3, x > 0\}$. Examples of closed sets are the closed unit ball $\{\vec{\mathbf{r}} \in \mathbb{R}^3, |\vec{\mathbf{r}}| \leq 1\}$, the closed unit cube $\{\vec{\mathbf{r}} \in \mathbb{R}^3, 0 \leq x, y, z \leq 1\}$, the closed half space $\{\vec{\mathbf{r}} \in \mathbb{R}^3, x \geq 0\}$, all the planes e.g. $\{\vec{\mathbf{r}} \in \mathbb{R}^3, x = 0\}$, the lines e.g. $\{\vec{\mathbf{r}} \in \mathbb{R}^3, x = y = 0\}$, the sets made by a single points e.g. $\{\vec{\mathbf{0}}\}$. Typically, the sets defined using strict inequalities (i.e. $>$ and $<$) are open, while those defined with non-strict inequalities or equalities (i.e. $\geq$, $\leq$ and $=$) are closed. Closed sets may be "thin", like planes and lines, while open sets are always "fat", as they contain little balls around each point. The empty set and $\mathbb{R}^3$ are the only two sets that are simultaneously open and closed.

Analogous definitions can be given for two dimensional sets, i.e. subsets of $\mathbb{R}^2$. Two-dimensional domains are also called **regions**.

Exercise 1.19. ▶ Prove that the complement of an open set is closed and vice versa. (This is not easy!)

Notation. In the following we will use the letters $D, E, \ldots$ to denote three-dimensional open sets, and the letters $P, Q, R, S, \ldots$ to denote two-dimensional open sets. S will also be used to name surfaces.

1.2 Scalar fields, vector fields and curves

The fundamental objects of first-year (scalar) calculus are "functions", in particular real functions of a real variable, i.e. rules that associate to every number $t \in \mathbb{R}$ a second number $f(t) \in \mathbb{R}$. In this section we begin the study of three different extensions of this concept to the vector case, i.e. we consider functions whose domains, or codomains, or both, are the three-dimensional Euclidean space $\mathbb{R}^3$ (or the plane $\mathbb{R}^2$), as opposed to the real line $\mathbb{R}$. Fields[3] are functions of position, described by the position vector $\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$, so their domain is $\mathbb{R}^3$ (or a subset of it). Depending on the kind of output, they are called either scalar fields or vector fields. Curves are vector-valued functions of a real variable. In Remark 1.27 we summarise the mapping properties of all these objects. Scalar fields, vector fields and curves are described in [1] in Sections 12.1, 15.1 and 11.1 respectively.

1.2.1 Scalar fields

A scalar field is a function $f : D \to \mathbb{R}$, where the domain D is an open subset of $\mathbb{R}^3$. The value of f at the point $\vec{\mathbf{r}}$ may be written as $f(\vec{\mathbf{r}})$ or $f(x, y, z)$. (This is because a scalar field can equivalently be interpreted as a function of one vector variable or as a function of three scalar variables.) Scalar field may also be called "multivariate functions" or "functions of several variables" (as in the first-year calculus modulus, see handout 5). Some examples of scalar fields are

$$f(\vec{\mathbf{r}}) = x^2 - y^2, \qquad g(\vec{\mathbf{r}}) = xy\mathrm{e}^z, \qquad h(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^4.$$

We will often consider *two-dimensional* scalar fields, namely functions $f : R \to \mathbb{R}$, where now R is a domain in $\mathbb{R}^2$, i.e. a region of the plane. Two-dimensional fields may also be thought as three-dimensional fields that do not depend on the third variable z (i.e. $f(x, y, z) = f(x, y)$ for all $z \in \mathbb{R}$).

[3] Note that the fields that are object of this section have nothing to do with the algebraic definition of fields as abstract sets provided with addition and multiplication (like $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$). The word "field" is commonly used for both meaning and can be a source of confusion; the use of "scalar field" and "vector field" is unambiguous.

Smooth scalar fields can be graphically represented using **level sets**, i.e. sets defined by the equations $\{f(\vec{\mathbf{r}})=\text{constant}\}$; see the left plot in Figure 5 for an example. The level sets of a two-dimensional field are (planar) **level curves** and can be easily drawn, while the level sets of a three-dimensional vector field are the **level surfaces**, which are harder to visualise. Level curves are also called **contour lines** or **isolines**, and level surfaces are called **isosurfaces**. Note that two level surfaces (or curves) never intersect each other, since at every point $\vec{\mathbf{r}}$ the field f takes only one value, but they might "self-intersect".

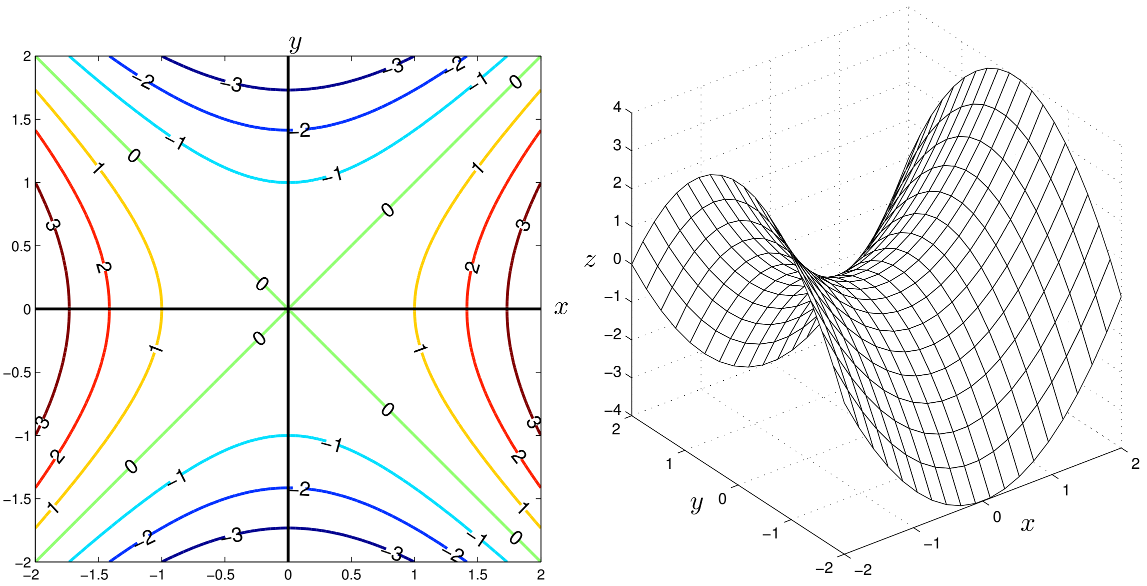


Figure 5: ***Left:*** *the level set representation of the scalar field* $f(\vec{\mathbf{r}}) = x^2 - y^2$*. Since* f *does not depend on the* z*-component of the position, we can think at it as a two-dimensional field and represent it with the level curves corresponding to a section in the* xy*-plane. Each colour represents the set of points* $\vec{\mathbf{r}}$ *such that* $f(\vec{\mathbf{r}})$ *is equal to a certain constant, e.g.* $f(\vec{\mathbf{r}}) = 0$ *along the green lines and* $f(\vec{\mathbf{r}}) = 2$ *along the red curves (see the .pdf file of these notes for the coloured version). This plot is obtained with Matlab's command* `contour`.
Right: *the same field* $f = x^2 - y^2$ *represented as the surface* $S_f = \{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } z = f(x, y)\}$*; see Remark 1.21. Surfaces can be drawn in Matlab with the commands* `mesh` *and* `surf`.

Example 1.20 (Different fields with the same level sets)**.** Consider the level surfaces of the scalar field $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$. Every surface corresponds to the set of points that are solutions of the quadratic equation $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2 = x^2 + y^2 + z^2 = C$ for some $C \in \mathbb{R}$ $(C \geq 0)$, thus they are the spheres centred at the origin.

Now consider the level surfaces of the scalar field $g(\vec{\mathbf{r}}) = \mathrm{e}^{-|\vec{\mathbf{r}}|^2}$. They correspond to the solutions of the equation $\mathrm{e}^{-x^2-y^2-z^2} = C$, or $(x^2 + y^2 + z^2) = -\log C$, for some $0 < C \leq 1$. Therefore also in this case they are the spheres centred at the origin (indeed the two fields are related by the identity $g(\vec{\mathbf{r}}) = \mathrm{e}^{-f(\vec{\mathbf{r}})}$).

We conclude that two different scalar fields may have the same level surfaces, associated with different field values (see Figure 6).

Comparison with scalar calculus 1.21 (Graph surfaces)**.** A real function of one variable $g : \mathbb{R} \to \mathbb{R}$ is usually represented with its **graph** $G_g = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } y = g(x)\}$, which is a subset of the plane. Exactly in the same way, a *two-dimensional* scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ can be visualised using its graph

$$S_f = \big\{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}} \in \mathbb{R}^3,\ z = f(x, y)\big\},$$

which is a **surface**, i.e. a two-dimensional set lying in $\mathbb{R}^3$; see the example in the right plot of Figure 6. (We study surfaces more in detail in Section 2.2.4.) The graph of a general three-dimensional scalar field is a **hypersurface** and we cannot easily visualise as it is a three-dimensional set living in a four-dimensional space.

Exercise: represent the graphs of the two fields defined in Example 1.20 (or their planar section at $z = 0$). Despite having the same level sets the surfaces representing their graphs are very different from each other.

★ **Remark 1.22.** A scalar field $f : D \to \mathbb{R}$, where D is an open set, is **continuous** at $\vec{\mathbf{r}}_0 \in D$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\vec{\mathbf{r}} \in D$ with $|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0| < \delta$ then $|f(\vec{\mathbf{r}}) - f(\vec{\mathbf{r}}_0)| < \epsilon$. The field f is continuous in D if it is continuous at each point of D. Equivalently, f is continuous at $\vec{\mathbf{r}}_0 \in D$ if for all sequences of points $\{\vec{\mathbf{r}}_j\}_{j\in\mathbb{N}} \subset D$ with $\lim_{j\to\infty} \vec{\mathbf{r}}_j = \vec{\mathbf{r}}_0$, it follows that $\lim_{j\to\infty} f(\vec{\mathbf{r}}_j) = f(\vec{\mathbf{r}}_0)$. A continuous field is often called "of class C^0".

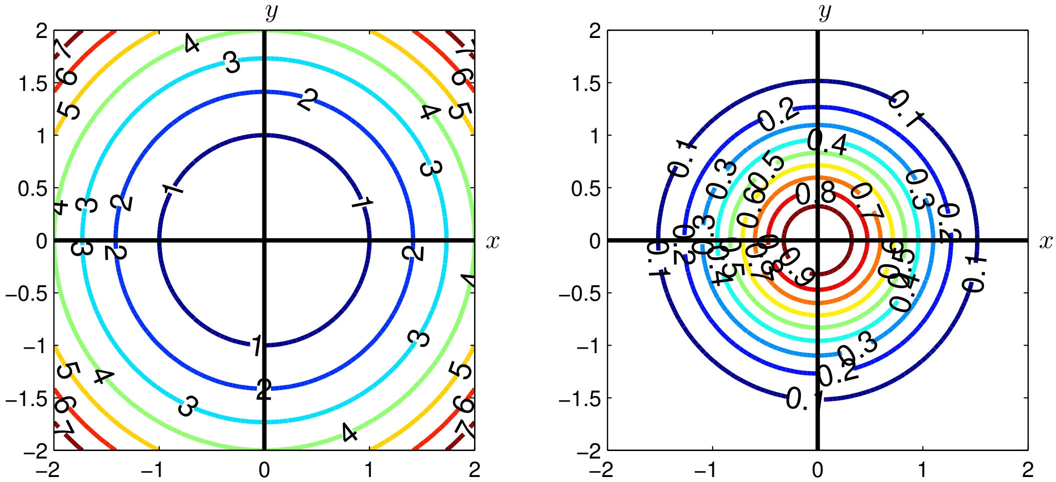


Figure 6: *The level sets of the two scalar fields described in Example 1.20; only the plane $z = 0$ is represented here. In the left plot, the field $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$ (level sets for the values 1, 2,..., 7); in the right plot, the field $g(\vec{\mathbf{r}}) = \mathrm{e}^{-|\vec{\mathbf{r}}|^2}$ (level sets for the values 0.1, 0.2,..., 0.9). Each level set of f is also a level set of g (do not forget that the plots represent only few level sets).*

Note that studying the continuity of a field can be much more tricky than for a real function. For example, it is clear that the field $1/|\vec{\mathbf{r}}|$ is discontinuous at the origin, $1/xyz$ is discontinuous on the three coordinate planes and $sign(\sin x)\mathrm{e}^y$ has jump discontinuities on the planes $\{x = \pi n\}$, $n \in \mathbb{N}$. However, in other cases the discontinuity is somewhat hidden: can you see why the two-dimensional fields $f(\vec{\mathbf{r}}) = \frac{2xy}{x^2+y^2}$ and $g(\vec{\mathbf{r}}) = \frac{2x^2y}{x^4+y^2}$ are both discontinuous at the origin? (You may learn more on this in the "analysis in several variables" module; if you cannot wait take a look at Chapter 12 of [1].)

★ **Remark 1.23** (Admissible smoothness of fields)**.** In this notes we always assume that scalar and vector fields we use are "smooth enough" to be able to take all the derivatives we need. We will never be very precise on this important point, and you can safely ignore it for this module (only!). When we require a field to be "smooth", we usually mean that it is described by a C^∞ function, i.e. that it is continuous and all its partial derivatives (of any order) are continuous. However, in most cases C^2 regularity will be enough for our purposes (i.e. we need the continuity of the field and its derivatives up to second order only). For example, whenever we write an identity involving the derivatives of a scalar field f, we will be able to consider $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$, which enjoys C^∞ regularity, but not $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|$ whose partial derivatives are not well-defined at the origin (it is not continuously differentiable, i.e. it is not of class C^1). See also Remarks 1.22, 1.31 and 1.42 for the definition of the regularity classes C^k.

1.2.2 Vector fields

A **vector field** $\vec{\mathbf{F}}$ is a function of position whose output is a vector, namely it is a function

$$\vec{\mathbf{F}} : D \to \mathbb{R}^3, \qquad \vec{\mathbf{F}}(\vec{\mathbf{r}}) = F_1(\vec{\mathbf{r}})\,\hat{\boldsymbol{\imath}} + F_2(\vec{\mathbf{r}})\,\hat{\boldsymbol{\jmath}} + F_3(\vec{\mathbf{r}})\,\hat{\boldsymbol{k}},$$

where the domain D is a subset of $\mathbb{R}^3$. The three scalar fields F_1, F_2 and F_3 are the components of $\vec{\mathbf{F}}$. Some examples of vector fields are

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{G}}(\vec{\mathbf{r}}) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} + \cos y\,\hat{\boldsymbol{k}}.$$

Figure 7 shows two common graphical representations of vector fields.

Two-dimensional (or planar) vector fields are functions $\vec{\mathbf{F}} : R \to \mathbb{R}^2$, with R a region in $\mathbb{R}^2$. Two-dimensional vector fields may also be considered as three-dimensional fields that do not depend on the third variable z, and whose third component $\vec{\mathbf{F}}_3$ is identically zero, i.e. $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = F_1(x,y)\hat{\boldsymbol{\imath}} + F_2(x,y)\hat{\boldsymbol{\jmath}}$. The field depicted in Figure 7 is two dimensional.

1.2.3 Curves

Curves are vector-valued functions of a real variable $\vec{\mathbf{a}} : I \to \mathbb{R}^3$, where I is either the real line $I = \mathbb{R}$ or an interval $I \subset \mathbb{R}$. Since the value $\vec{\mathbf{a}}(t)$ at each $t \in I$ is a vector, we expand a curve as:

$$\vec{\mathbf{a}}(t) = a_1(t)\hat{\boldsymbol{\imath}} + a_2(t)\hat{\boldsymbol{\jmath}} + a_3(t)\hat{\boldsymbol{k}},$$

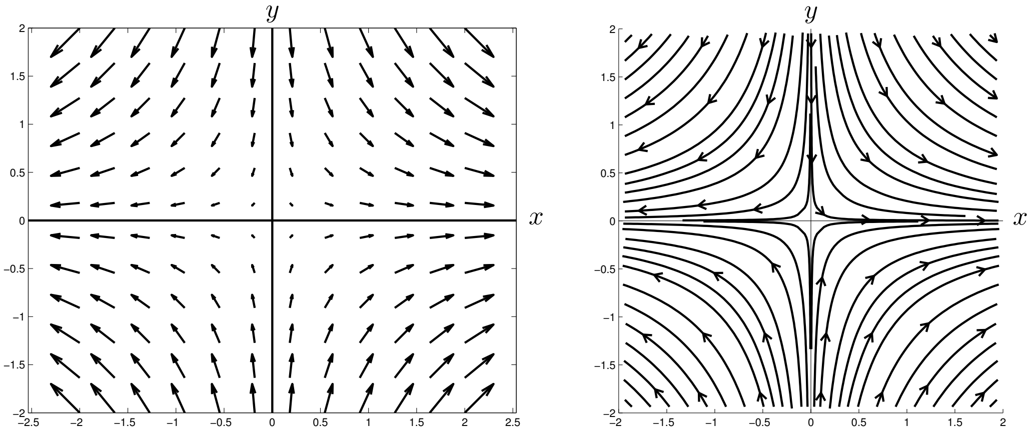


Figure 7: *Two visualisations of the two-dimensional vector field* $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}$*. In the left plot, the direction of the arrow positioned in* $\vec{\mathbf{r}}$ *indicated the direction of the field in that point and its length is proportional to the magnitude of* $\vec{\mathbf{F}}(\vec{\mathbf{r}})$*. In the right plot the lines are tangent to the field* $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ *in every point; these curves are called* ***field lines*** *or* ***streamlines****. Note that the streamline representation does not give any information about the magnitude of the vector field. These plots are obtained with Matlab's commands* `quiver` *and* `streamslice` *(unfortunately the latter command is not available in Octave).*

where a_1, a_2 and $a_3 : I \to \mathbb{R}$ (the components of $\vec{\mathbf{a}}$) are real functions.

We say that $\vec{\mathbf{a}}$ is continuous if its three components are continuous real functions. If the interval I is closed and bounded, namely $I = [t_I, t_F] \subset \mathbb{R}$, and $\vec{\mathbf{a}} : I \to \mathbb{R}^3$ is a curve satisfying $\vec{\mathbf{a}}(t_I) = \vec{\mathbf{a}}(t_F)$ (i.e. it starts and ends in the same point), then $\vec{\mathbf{a}}$ is called **loop**.

We denote by $\Gamma := \{\vec{\mathbf{r}} \in \mathbb{R}^3, \text{ s.t. } \vec{\mathbf{r}} = \vec{\mathbf{a}}(t), \text{ for some } t \in I\} \subset \mathbb{R}^3$ the image of the curve $\vec{\mathbf{a}}$, i.e. the set of the points covered by $\vec{\mathbf{a}}$ for the various values of $t \in I$; we call Γ the **path** (or trace, or support) of $\vec{\mathbf{a}}$. We say that the path Γ is **parametrised** by the curve $\vec{\mathbf{a}}$, or vice versa that $\vec{\mathbf{a}}$ **parametrises** Γ (i.e. it associates to every "parameter" $t \in I$ a point on the path Γ). Some examples of paths of curves are shown in Figure 8.

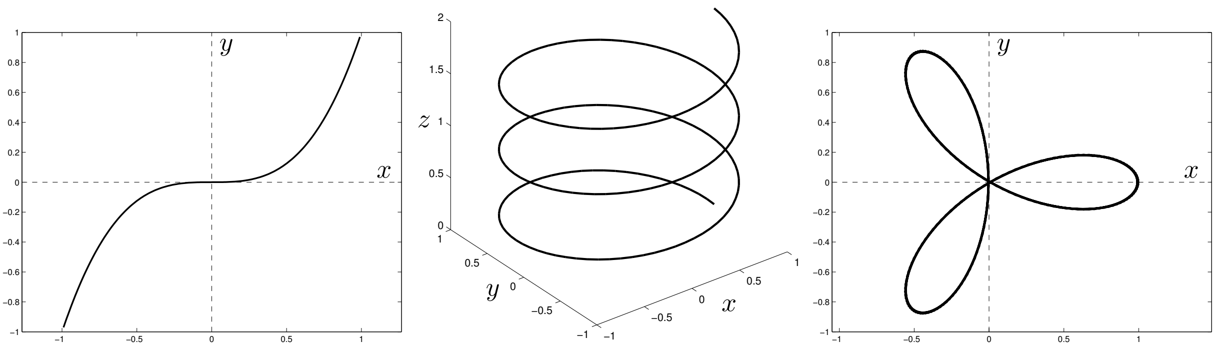


Figure 8: *Three examples of paths defined by curves. In the left plot the cubic* $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + t^3\hat{\boldsymbol{\jmath}}$*, plotted for the interval* $[-1, 1]$*. In the centre plot the helix* $\vec{\mathbf{b}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 0.1t\hat{\boldsymbol{k}}$*, for* $t \in [0, 6\pi]$*. In the right plot the "clover"* $\vec{\mathbf{c}}(t) = (\cos 3t)(\cos t)\hat{\boldsymbol{\imath}} + (\cos 3t)(\sin t)\hat{\boldsymbol{\jmath}}$ *for* $t \in [0, \pi]$*. Note that* $\vec{\mathbf{c}}$ *is not injective as it intersects itself in the origin (*$\vec{\mathbf{c}}(\pi/6) = \vec{\mathbf{c}}(\pi/2) = \vec{\mathbf{c}}(5\pi/6) = \vec{\mathbf{0}}$*). As* $\vec{\mathbf{c}}(0) = \vec{\mathbf{c}}(\pi)$*, the curve* $\vec{\mathbf{c}}$ *is a loop. The curves are drawn with the Matlab commands* `plot` *(planar curves) and* `plot3` *(three-dimensional curves).*

If we interpret the variable t as time, a curve may represent the trajectory of a point moving in space. Following this interpretation, curves are "oriented", i.e. they possess a preferred direction of travelling, while paths are not.

We stress that, as opposed to the common non-mathematical meaning, the word "curve" indicates a function $\vec{\mathbf{a}}$, whose image (its path) is a subset of $\mathbb{R}^3$, and not the image itself. Indeed, different curves may define the same path. For example

$$\vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} \quad t \in [0, 2\pi) \qquad \text{and} \qquad \vec{\mathbf{b}}(\tau) = \cos 2\tau\,\hat{\boldsymbol{\imath}} + \sin 2\tau\,\hat{\boldsymbol{\jmath}} \quad \tau \in [0, \pi)$$

are two *different curves* which have the *same path*: the unit circle in the xy-plane.

In the textbook [1] and in several other books, the general curves and their components are denoted $\vec{\mathbf{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$.

Remark 1.24 (How to parametrise a path)**.** Given a path (or a curve), how can we find its curve (or its path)?

In one direction the procedure is straightforward. If we have the analytic expression of a curve $\vec{\mathbf{a}} : I \to \mathbb{R}^3$, its path can be drawn (approximately) simply by evaluating $\vec{\mathbf{a}}(t_j)$ for many $t_j \in I$ and "connecting the points". This is exactly what is done automatically by Matlab or Octave to make e.g. Figure 8.

The opposite operation is sometimes very difficult: since every path corresponds to infinitely many curves, there is no universal procedure to construct $\vec{\mathbf{a}}$ from Γ. Here we only consider a few simple, but very important, examples. Try to draw all the paths by computing some points via the parametrisation.

- Γ is the **segment** with vertices $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$. A parametrisation of Γ is

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{p}} + t(\vec{\mathbf{q}} - \vec{\mathbf{p}}) = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, \quad \text{for } t \in [0,1].$$

  (The components are $a_1(t) = (1-t)p_1 + tq_1$, $a_2(t) = (1-t)p_2 + tq_2$ and $a_3(t) = (1-t)p_3 + tq_3$.)

- Γ is part of the **graph** of a real function $g : I \to \mathbb{R}$ laying in the xy-plane (where $I \subset \mathbb{R}$ is an interval). Then we can take $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + g(t)\hat{\boldsymbol{\jmath}}$, for $t \in I$. (The components are $a_1(t) = t$, $a_2(t) = g(t)$ and $a_3(t) = 0$.)

- Γ is the **circumference** with centre $\vec{\mathbf{p}} = p_1\hat{\boldsymbol{\imath}} + p_2\hat{\boldsymbol{\jmath}}$ and radius R lying in the xy-plane. Then we can take $\vec{\mathbf{a}}(t) = \vec{\mathbf{p}} + R\cos t\hat{\boldsymbol{\imath}} + R\sin t\hat{\boldsymbol{\jmath}}$ for $t \in [0, 2\pi)$. Indeed, this circumference has equation $\Gamma = \{\vec{\mathbf{r}},\, (x-p_1)^2 + (y-p_2)^2 = R^2,\, z = 0\}$, which is satisfied by the curve above (verify this).

- What can we do if the path is defined by an equation? This might require some guesswork. For example, consider the planar ellipse $\Gamma = \{\vec{\mathbf{r}},\, x^2/9 + y^2/4 = 1, z = 0\}$. Since it lies in the plane $z = 0$, we fix $a_3(t) = 0$ and we look for $a_1(t)$ and $a_2(t)$. This must be functions of t that satisfy $a_1(t)^2/9 + a_2(t)^2/4 = 1$. Since the ellipse is an affine deformation of the unit circumference, we may expect to use trigonometric functions, and indeed we see that $\vec{\mathbf{a}}(t) = 3\cos t\hat{\boldsymbol{\imath}} + 2\sin t\hat{\boldsymbol{\jmath}}$ satisfies the desired equation.

Try to understand well these simple examples (segments, graphs, circumferences): they will be used very often during the course and the solution of many exercises will require their use.

Remark 1.25 (How to change parametrisation)**.** Sometimes we have a curve $\vec{\mathbf{a}} : I \to \mathbb{R}^3$, and we want to find a different curve $\vec{\mathbf{b}} : J \to \mathbb{R}^3$ with the same path Γ, but defined on a different interval $J \subset \mathbb{R}$. To this purpose, it is enough to find a function $g : J \to I$ that is bijective and continuous and define $\vec{\mathbf{b}}(\tau) := \vec{\mathbf{a}}(g(\tau))$ for $\tau \in J$. Two parametrisations $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ of the same path are always related one another by a function g of this kind. If g is increasing, then the two parametrisations have the same orientation (they run the path in the same direction); if g is decreasing, then the two parametrisations have the opposite orientation (they run the path in opposite directions). We see a few examples.

- In the example above, the unit circle is parametrised by $\vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ for $t \in I = [0, 2\pi)$ and $\vec{\mathbf{b}}(\tau) = \cos 2\tau\,\hat{\boldsymbol{\imath}} + \sin 2\tau$ for $\tau \in J = [0, \pi)$, so we have $g(\tau) = 2\tau$.

- Consider the unit half circle centred at the origin and located in the half plane $\{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\, y \geq 0\}$, which can be defined by either of the two parametrisations

$$\vec{\mathbf{a}} : [0,\pi] \to \mathbb{R}^3, \quad \vec{\mathbf{a}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{b}} : [-1,1] \to \mathbb{R}^3, \quad \vec{\mathbf{b}}(\tau) = \tau\hat{\boldsymbol{\imath}} + \sqrt{1-\tau^2}\hat{\boldsymbol{\jmath}}.$$

  Here we use $g = \arccos : [-1,1] \to [0,\pi]$ (which is decreasing: $\vec{\mathbf{a}}$ runs anti-clockwise, $\vec{\mathbf{b}}$ clockwise).

- We can use a change of parametrisation to invert the orientation of a curve. If $\vec{\mathbf{a}}$ is defined on the interval $I = [0, t_F]$, choosing $g(\tau) = t_F - \tau$ we obtain $\vec{\mathbf{b}}(\tau) = \vec{\mathbf{a}}(t_F - \tau)$ which maps $I \to \Gamma$ (as $\vec{\mathbf{a}}$ does) but with opposite orientation.

  In the special case of a segment, from $\vec{\mathbf{a}}(t) = \vec{\mathbf{p}} + t(\vec{\mathbf{q}} - \vec{\mathbf{p}}) = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}$ for $t \in [0,1]$ we obtain $\vec{\mathbf{b}}(\tau) = \vec{\mathbf{q}} + \tau(\vec{\mathbf{p}} - \vec{\mathbf{q}}) = \tau\vec{\mathbf{p}} + (1-\tau)\vec{\mathbf{q}}$ for $\tau \in [0,1]$. $\vec{\mathbf{a}}$ runs from $\vec{\mathbf{p}}$ to $\vec{\mathbf{q}}$, while $\vec{\mathbf{b}}$ runs from $\vec{\mathbf{q}}$ to $\vec{\mathbf{p}}$.

⋆ **Remark 1.26** (Fields in physics)**.** Fields are important in all branches of science and model many **physical quantities**. For example, consider a domain D that models a portion of Earth's atmosphere. One can associate to every point $\vec{\mathbf{r}} \in D$ several numerical quantities representing temperature, density, air pressure, concentration of water vapour or some pollutant (at a given instant): each of these physical quantities can be mathematically represented by a scalar field (a scalar quantity is associated to each point in space). Other physical quantities involve magnitude and direction (which can both vary in different points in space), thus they can be represented as vector fields: for example the gravitational force, the wind velocity, the magnetic field (pointing to the magnetic north pole). The plots you commonly see in weather forecast are representations of some of these fields (e.g. level sets of pressure at a given altitude). (Note that all these fields may vary in time, so they are actually functions of four scalar variables: three spacial and one temporal.) Also curves are used in physics, for instance to describe trajectories of electrons, bullets, aircraft, planets and any other kind of body.

Warning: frequent error 1.27. At this point it is extremely important not to mix up the different definitions of scalar fields, vector fields and curves. **Treating vectors as scalars or scalars as vectors is one of the main sources of mistakes in vector calculus exams!** We recall that scalar fields take as input a vector and return a real number ($\vec{\mathbf{r}} \mapsto f(\vec{\mathbf{r}})$), vector fields take as input a vector and return a vector ($\vec{\mathbf{r}} \mapsto \vec{\mathbf{F}}(\vec{\mathbf{r}})$), curves take as input a real number and return a vector ($t \mapsto \vec{\mathbf{a}}(t)$):

Real functions (of real variable)	$f:\mathbb{R}\to\mathbb{R}$	$t\mapsto f(t)$,	Increasing complexity ↓
Curves	$\vec{\mathbf{a}}:\mathbb{R}\to\mathbb{R}^3$	$t\mapsto \vec{\mathbf{a}}(t)$,	
Scalar fields	$f:\mathbb{R}^3\to\mathbb{R}$	$\vec{\mathbf{r}}\mapsto f(\vec{\mathbf{r}})$,	
Vector fields	$\vec{\mathbf{F}}:\mathbb{R}^3\to\mathbb{R}^3$	$\vec{\mathbf{r}}\mapsto \vec{\mathbf{F}}(\vec{\mathbf{r}})$.	

Vector fields might be thought as combinations of three scalar fields (the components) and curves as combinations of three real functions.

1.3 Vector differential operators

We learned in the calculus class what is the derivative of a smooth real function of one variable $f:\mathbb{R}\to\mathbb{R}$. We now study the derivatives of scalar and vector fields. It turns out that there are several important "differential operators", which generalise the concept of derivative. In this section we introduce them, while in the next one we study some relations between them. The fundamental differential operators for fields are the partial derivatives, described in Section 1.3.1; all the vector differential operators described in the rest of this section are defined starting from them.

As before, for simplicity we consider here the three-dimensional case only; all the operators, with the relevant exception of the curl operator, can immediately be defined in $\mathbb{R}^n$ for any dimension $n\in\mathbb{N}$.

The textbook [1] describes partial derivatives in Sections 12.3–5, the Jacobian matrix in 12.6, the gradient and directional derivatives in 12.7, divergence and curl operator in 16.1 and the Laplacian in 16.2.

⋆ **Remark 1.28** (What is an operator?)**.** An **operator** is anything that operates on functions or fields, i.e. a "function of functions" or "function of fields". For example, we can define the "doubling operator" T that, given a real function $f:\mathbb{R}\to\mathbb{R}$, returns its double, i.e. Tf is the function $Tf:\mathbb{R}\to\mathbb{R}$, such that $Tf(x)=2f(x)$.

A **differential operator** is an operator that involves some differentiation. For example, the derivative $\frac{d}{dx}$ maps $f:\mathbb{R}\to\mathbb{R}$ in $\frac{d}{dx}f=f':\mathbb{R}\to\mathbb{R}$.

Operators and differential operators are rigorously defined using the concept of "function space", which is an infinite-dimensional vector space whose elements are functions. This is well beyond the scope of this class and is one of the topics studied by functional analysis.

1.3.1 Partial derivatives

Consider a smooth scalar field $f:D\to\mathbb{R}$. The **partial derivatives** of f in the point $\vec{\mathbf{r}}=x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\mathbf{k}}\in D$ are defined as limits of difference quotients, when these limits exist:

$$\begin{aligned}\frac{\partial f}{\partial x}(\vec{\mathbf{r}}) &:= \lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h},\\ \frac{\partial f}{\partial y}(\vec{\mathbf{r}}) &:= \lim_{h\to 0}\frac{f(x,y+h,z)-f(x,y,z)}{h},\\ \frac{\partial f}{\partial z}(\vec{\mathbf{r}}) &:= \lim_{h\to 0}\frac{f(x,y,z+h)-f(x,y,z)}{h}.\end{aligned} \tag{6}$$

In other words, $\frac{\partial f}{\partial x}(\vec{\mathbf{r}})$ can be understood as the derivative of the real function $x\mapsto f(x,y,z)$ which "freezes" the y- and z-variables. If the partial derivatives are defined for all points in D, then they constitute three scalar fields, denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. We can also write

$$\frac{\partial f}{\partial x}(\vec{\mathbf{r}})=\lim_{h\to0}\frac{f(\vec{\mathbf{r}}+h\hat{\boldsymbol{\imath}})-f(\vec{\mathbf{r}})}{h},\qquad \frac{\partial f}{\partial y}(\vec{\mathbf{r}})=\lim_{h\to0}\frac{f(\vec{\mathbf{r}}+h\hat{\boldsymbol{\jmath}})-f(\vec{\mathbf{r}})}{h},\qquad \frac{\partial f}{\partial z}(\vec{\mathbf{r}})=\lim_{h\to0}\frac{f(\vec{\mathbf{r}}+h\hat{\mathbf{k}})-f(\vec{\mathbf{r}})}{h}.$$

We also use the notation $\frac{\partial\vec{\mathbf{F}}}{\partial x}:=\frac{\partial F_1}{\partial x}\hat{\boldsymbol{\imath}}+\frac{\partial F_2}{\partial x}\hat{\boldsymbol{\jmath}}+\frac{\partial F_3}{\partial x}\hat{\mathbf{k}}$ to denote the vector field whose components are the partial derivatives with respect to x of the components of a vector field $\vec{\mathbf{F}}$ (and similarly $\frac{\partial\vec{\mathbf{F}}}{\partial y}$ and $\frac{\partial\vec{\mathbf{F}}}{\partial z}$).

Exercise 1.29. ▶ Compute all the partial derivatives of the following scalar fields:

$$f(\vec{\mathbf{r}})=xye^z,\quad g(\vec{\mathbf{r}})=\frac{xy}{y+z},\quad h(\vec{\mathbf{r}})=\log(1+z^2e^{yz}),\quad \ell(\vec{\mathbf{r}})=\sqrt{x^2+y^4+z^6},\quad m(\vec{\mathbf{r}})=x^y,\quad p(\vec{\mathbf{r}})=\frac{|\vec{\mathbf{r}}|^2}{x^2}.$$

Since partial derivative are nothing else than usual derivatives for the functions obtained by freezing all variables except one, the usual rules for derivatives apply. For f and g scalar fields, λ and $\mu \in \mathbb{R}$, and a real function $G : \mathbb{R} \to \mathbb{R}$, we have the following identities: **linearity**

$$\boxed{\frac{\partial(\lambda f + \mu g)}{\partial x} = \lambda \frac{\partial f}{\partial x} + \mu \frac{\partial g}{\partial x}, \qquad \frac{\partial(\lambda f + \mu g)}{\partial y} = \lambda \frac{\partial f}{\partial y} + \mu \frac{\partial g}{\partial y}, \qquad \frac{\partial(\lambda f + \mu g)}{\partial z} = \lambda \frac{\partial f}{\partial z} + \mu \frac{\partial g}{\partial z};} \tag{7}$$

the **product rule** (extending the well-known formula $(FG)' = F'G + FG'$ for real functions)

$$\boxed{\frac{\partial(fg)}{\partial x} = g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, \qquad \frac{\partial(fg)}{\partial y} = g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, \qquad \frac{\partial(fg)}{\partial z} = g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z};} \tag{8}$$

and the **chain rule**

$$\boxed{\frac{\partial\big(G(f)\big)}{\partial x} = G'(f)\frac{\partial f}{\partial x}, \qquad \frac{\partial\big(G(f)\big)}{\partial y} = G'(f)\frac{\partial f}{\partial y}, \qquad \frac{\partial\big(G(f)\big)}{\partial z} = G'(f)\frac{\partial f}{\partial z}.} \tag{9}$$

Note that the product rule (8) must be used when we compute a partial derivative of a *product*; the chain rule (9) when we compute a partial derivative of a *composition*.[4] In other words, in (8) the two scalar fields f and g are multiplied to each other, in (9) the function G is evaluated in $f(\vec{\mathbf{r}})$, which is a scalar: $\frac{\partial(G(f))}{\partial x}(\vec{\mathbf{r}}) = G'(f(\vec{\mathbf{r}}))\frac{\partial f}{\partial x}(\vec{\mathbf{r}})$.

★ **Remark 1.30.** Note that to be able to define the partial derivatives of a field f in a point $\vec{\mathbf{r}}$ we need to be able to take the limits in (6), so to evaluate f "nearby" $\vec{\mathbf{r}}$. This is possible if f is defined in a open set D, as defined in Section 1.1.3, which ensures that all its points are completely surrounded by points of the same set.

For example, the "open half space" $D = \{x > 0\}$ is open, while the "closed half space" $E = \{x \geq 0\}$ is not. If a field f is defined in E, we can not evaluate $f(h, y, z)$ for negative h, so the limit $\lim_{h\to 0} \frac{f(h,y,z)-f(0,y,z)}{h}$ in (6) is not defined and we cannot compute $\frac{\partial f}{\partial x}$ in all the points $y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \in E$ (those with $x = 0$).

★ **Remark 1.31.** A scalar field $f : D \to \mathbb{R}$ is called **differentiable** or "differentiable of class C^1" if it is continuous (see Remark 1.22) and all its first-order partial derivatives exist and are continuous. A vector field is differentiable (of class C^1) if its three components are differentiable.

1.3.2 The gradient

Let f be a *scalar field*. The *vector field* whose three components are the three partial derivatives of f (in the usual order) is called **gradient** of f and denoted by

$$\boxed{\vec{\nabla} f(\vec{\mathbf{r}}) := \operatorname{grad} f(\vec{\mathbf{r}}) := \hat{\boldsymbol{\imath}}\frac{\partial f}{\partial x}(\vec{\mathbf{r}}) + \hat{\boldsymbol{\jmath}}\frac{\partial f}{\partial y}(\vec{\mathbf{r}}) + \hat{\boldsymbol{k}}\frac{\partial f}{\partial z}(\vec{\mathbf{r}}).} \tag{10}$$

The symbol

$$\vec{\nabla} := \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} \tag{11}$$

is called "**nabla**" operator, or "**del**", and is usually denoted simply by ∇.

Given a smooth scalar field f and a unit vector $\hat{\boldsymbol{u}}$, the **directional derivative** of f in direction $\hat{\boldsymbol{u}}$ is defined as the scalar product $\frac{\partial f}{\partial \hat{\boldsymbol{u}}}(\vec{\mathbf{r}}) := \hat{\boldsymbol{u}} \cdot \vec{\nabla} f(\vec{\mathbf{r}})$. If $\hat{\boldsymbol{n}}$ is the unit vector orthogonal to a surface, $\frac{\partial f}{\partial n} := \frac{\partial f}{\partial \hat{\boldsymbol{n}}}$ is called **normal derivative**.

Example 1.32. The gradient of $f(\vec{\mathbf{r}}) = x^2 - y^2$ is

$$\vec{\nabla} f(\vec{\mathbf{r}}) = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x}(x^2 - y^2) + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y}(x^2 - y^2) + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}(x^2 - y^2) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}};$$

thus the vector field in Figure 7 is the gradient of the scalar field depicted in Figure 5.

[4] We recall the notion of **composition** of functions: for any three sets A, B, C, and two functions $F : A \to B$ and $G : B \to C$, the composition of G with F is the function $(G \circ F) : A \to C$ obtained by applying F and then G to the obtained output. In formulas: $(G \circ F)(x) := G(F(x))$ for all $x \in A$. If $A = B = C = \mathbb{R}$ (or they are appropriate subsets of $\mathbb{R}$) and F and G are differentiable, then from basic calculus we know the derivative of the composition: $(G \circ F)'(x) = G'(F(x))F'(x)$ (this is the most basic example of chain rule).

Proposition 1.33 (Properties of the gradient)**.** Given two smooth scalar fields $f, g : D \to \mathbb{R}$, their gradients satisfy the following properties:

1. for any constant $\lambda, \mu \in \mathbb{R}$, the following identity holds (linearity)
$$\vec{\nabla}(\lambda f + \mu g) = \lambda \vec{\nabla} f + \mu \vec{\nabla} g; \tag{12}$$
2. the following identity holds (product rule or Leibniz rule)
$$\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g; \tag{13}$$
3. for any differentiable real function $G : \mathbb{R} \to \mathbb{R}$ the chain rule holds:
$$\vec{\nabla}(G \circ f)(\vec{\mathbf{r}}) = \vec{\nabla} G\big(f(\vec{\mathbf{r}})\big) = G'\big(f(\vec{\mathbf{r}})\big)\vec{\nabla} f(\vec{\mathbf{r}}), \tag{14}$$
where $G'(f(\vec{\mathbf{r}}))$ is the derivative of G evaluated in $f(\vec{\mathbf{r}})$ and $G \circ f$ denotes the composition of G with f;
4. $\vec{\nabla} f(\vec{\mathbf{r}})$ is perpendicular to the level surface of f passing through $\vec{\mathbf{r}}$ (i.e. to $\{\vec{\mathbf{r}}' \in D \text{ s.t. } f(\vec{\mathbf{r}}') = f(\vec{\mathbf{r}})\}$);
5. $\vec{\nabla} f(\vec{\mathbf{r}})$ points in the direction of maximal increase of f.

Proof of 1., 2. and 3. We use the definition of the gradient (10), the linearity (7), the product rule (8) and the chain rule for partial derivatives (9):

$$\begin{aligned}
\vec{\nabla}(\lambda f + \mu g) &= \hat{\boldsymbol{\imath}}\frac{\partial(\lambda f + \mu g)}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial(\lambda f + \mu g)}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial(\lambda f + \mu g)}{\partial z} \\
&\overset{(7)}{=} \hat{\boldsymbol{\imath}}\lambda\frac{\partial f}{\partial x} + \hat{\boldsymbol{\imath}}\mu\frac{\partial g}{\partial x} + \hat{\boldsymbol{\jmath}}\lambda\frac{\partial f}{\partial y} + \hat{\boldsymbol{\jmath}}\mu\frac{\partial g}{\partial y} + \hat{\boldsymbol{k}}\lambda\frac{\partial f}{\partial z} + \hat{\boldsymbol{k}}\mu\frac{\partial g}{\partial z} \qquad = \lambda\vec{\nabla} f + \mu\vec{\nabla} g; \\
\vec{\nabla}(fg) &= \hat{\boldsymbol{\imath}}\frac{\partial(fg)}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial(fg)}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial(fg)}{\partial z} \\
&\overset{(8)}{=} \hat{\boldsymbol{\imath}}\left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right) + \hat{\boldsymbol{\jmath}}\left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right) + \hat{\boldsymbol{k}}\left(g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right) \qquad = g\vec{\nabla} f + f\vec{\nabla} g, \\
\vec{\nabla}(G \circ f)(\vec{\mathbf{r}}) &= \hat{\boldsymbol{\imath}}\frac{\partial(G \circ f)}{\partial x}(\vec{\mathbf{r}}) + \hat{\boldsymbol{\jmath}}\frac{\partial(G \circ f)}{\partial y}(\vec{\mathbf{r}}) + \hat{\boldsymbol{k}}\frac{\partial(G \circ f)}{\partial z}(\vec{\mathbf{r}}) \\
&\overset{(9)}{=} \hat{\boldsymbol{\imath}}G'\big(f(\vec{\mathbf{r}})\big)\frac{\partial f}{\partial x}(\vec{\mathbf{r}}) + \hat{\boldsymbol{\jmath}}G'\big(f(\vec{\mathbf{r}})\big)\frac{\partial f}{\partial y}(\vec{\mathbf{r}}) + \hat{\boldsymbol{k}}G'\big(f(\vec{\mathbf{r}})\big)\frac{\partial f}{\partial z}(\vec{\mathbf{r}}) \qquad = G'\big(f(\vec{\mathbf{r}})\big)\,\vec{\nabla} f(\vec{\mathbf{r}}).
\end{aligned}$$

Parts 4. and 5. will be proved in Remarks 1.88 and 1.89. □

Exercise 1.34. ▶ Write the gradients of the scalar fields in Exercise 1.29.

Exercise 1.35. ▶ Verify that the gradients of the "magnitude scalar field" $m(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|$ and of its square $s(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$ satisfy the identities

$$\vec{\nabla} s(\vec{\mathbf{r}}) = \vec{\nabla}(|\vec{\mathbf{r}}|^2) = 2\vec{\mathbf{r}}, \qquad \vec{\nabla} m(\vec{\mathbf{r}}) = \vec{\nabla}(|\vec{\mathbf{r}}|) = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} \qquad \forall \vec{\mathbf{r}} \in \mathbb{R},\ \vec{\mathbf{r}} \neq \vec{\mathbf{0}}. \tag{15}$$

Represent the two gradients as in Figure 7. Can you compute $\vec{\nabla}(|\vec{\mathbf{r}}|^\alpha)$ for a general $\alpha \in \mathbb{R}$?

Comparison with scalar calculus 1.36. We know from first year calculus that we can infer information on a function of real variable from its derivative. Many of the relations between functions and their derivatives carry over to scalar fields and their gradients, we summarise some of them in the following table. (Recall that "affine" is equivalent to "polynomial of degree at most one".)

$F : \mathbb{R} \to \mathbb{R}$ is a function of real variable:	$f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field:
$F' = 0 \iff F$ is constant ($F(t) = \lambda, \forall t \in \mathbb{R}$),	$\vec{\nabla} f = \vec{\mathbf{0}} \iff f$ is constant ($f(\vec{\mathbf{r}}) = \lambda, \forall \vec{\mathbf{r}} \in \mathbb{R}^3$),
$F' = \lambda \iff F$ is affine ($F(t) = t\lambda + \mu$),	$\vec{\nabla} f = \vec{\mathbf{u}} \iff f$ is affine ($f(\vec{\mathbf{r}}) = \vec{\mathbf{r}} \cdot \vec{\mathbf{u}} + \mu$),
F' is a polynomial of degree p $\iff F$ is a polynomial of degree $p+1$,	the components of $\vec{\nabla} f$ are polynomials of degree p $\iff f$ is a polynomial of degree $p+1$ in x, y, z,
$F' > 0$ $\iff F$ is increasing ($F(s) < F(t), s < t$),	$\hat{\boldsymbol{n}} \cdot \vec{\nabla} f > 0 \iff f$ is increasing in direction $\hat{\boldsymbol{n}}$ ($f(\vec{\mathbf{r}} + h\hat{\boldsymbol{n}}) > f(\vec{\mathbf{r}}), h > 0$, here $\hat{\boldsymbol{n}}$ is a unit vector),
t_0 is a local maximum/minumum for F $\Rightarrow F'(t_0) = 0$,	$\vec{\mathbf{r}}_0$ is a local maximum/minumum for f $\Rightarrow \vec{\nabla} f(\vec{\mathbf{r}}_0) = \vec{\mathbf{0}}$.

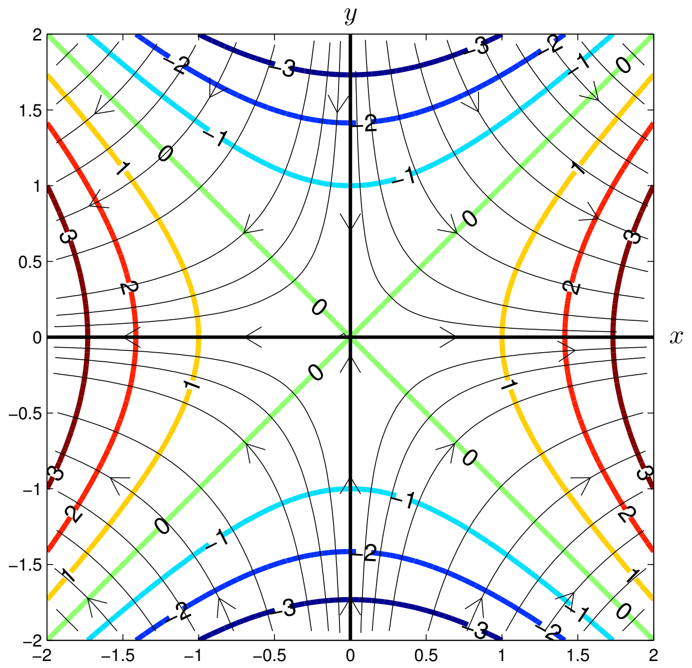


Figure 9: *A combined representation of the two-dimensional scalar field* $f(\vec{\mathbf{r}}) = x^2 - y^2$ *(coloured level curves) and its gradient* $\vec{\nabla} f(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}$ *(thin black streamlines). Note that the streamlines are perpendicular to the level lines in every point and they point from areas with lower values of* f *(blue) to areas with higher values (red), compare with Proposition 1.33. (See the coloured plot in the .pdf file of this notes.)*

Exercise 1.37. ▶ Consider the scalar field $f = (2x + y)^3$ and the vector field $\vec{\mathbf{G}} = 2e^z\hat{\boldsymbol{\imath}} + e^z\hat{\boldsymbol{\jmath}}$. Verify that, at every point $\vec{\mathbf{r}} \in \mathbb{R}^3$, the level set of f is orthogonal to $\vec{\mathbf{G}}$, but $\vec{\mathbf{G}}$ is not the gradient of f. Can you find a formula relating $\vec{\mathbf{G}}$ to $\vec{\nabla} f$?

Hint: figure out the exact shape of the level sets of f, compute two (linearly independent) tangent vectors to the level set in each point, and verify that both these vectors are orthogonal to $\vec{\mathbf{G}}$. Alternatively, use the relation between $\vec{\mathbf{G}}$ and $\vec{\nabla} f$.

Example 1.38 (Finding the unit normal to a surface)**.** Consider the surface

$$S = \left\{ \vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } y = \sqrt{1 + x^2} \right\}$$

and compute the field $\hat{\boldsymbol{n}} : S \to \mathbb{R}^3$ composed of the unit vectors perpendicular to S and pointing upwards as in the left plot of Figure 10.

From Proposition 1.33(4), we know that the gradient of a field that admits S as level set indicates the direction perpendicular to S itself. First, we find a scalar field f such that S is given as $f(\vec{\mathbf{r}})$ =constant: we can choose $f(\vec{\mathbf{r}}) = x^2 - y^2$, which satisfies $f(\vec{\mathbf{r}}) = -1$ for all $\vec{\mathbf{r}} \in S$. Taking the partial derivatives of f we immediately see that its gradient is

$$\vec{\nabla} f(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \Rightarrow \qquad |\vec{\nabla} f| = 2\sqrt{x^2 + y^2}.$$

Thus for every $\vec{\mathbf{r}} \in S$, $\hat{\boldsymbol{n}}(\vec{\mathbf{r}})$ is one of the two unit vectors parallel to $\vec{\nabla} f(\vec{\mathbf{r}})$, i.e.

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \pm \frac{\vec{\nabla} f(\vec{\mathbf{r}})}{|\vec{\nabla} f(\vec{\mathbf{r}})|} = \pm \frac{x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}}}{\sqrt{x^2 + y^2}}.$$

From the left plot in Figure 10 we see that we want to choose the sign such that $\vec{\mathbf{n}}(0, 1, 0) = \hat{\boldsymbol{\jmath}}$, thus we choose the minus sign in the last equation above and conclude

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \frac{-x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{\sqrt{x^2 + y^2}}.$$

Exercise 1.39. ▶ Let $f(\vec{\mathbf{r}}) = e^x - \sin y + xz^2$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = x\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ be a scalar and a vector field defined in $\mathbb{R}^3$. Compute the directional derivative of f in the direction of $\vec{\mathbf{F}}$. (This is a directional derivative whose direction may be different at every point).

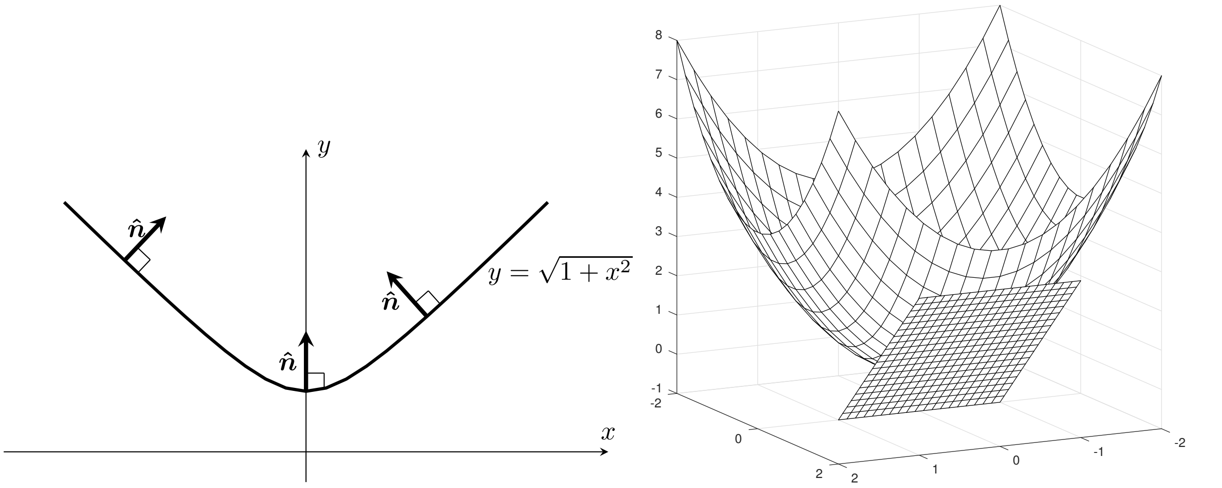


Figure 10: ***Left:*** *The section in the xy-plane ($z = 0$) of the surface S of Example 1.38 and its unit normal vectors.*
Right: *The surface graph of the planar scalar field $f = x^2+y^2$ and its tangent plane at $\vec{\mathbf{r}}_0+f(\vec{\mathbf{r}}_0)\hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}$ for $\vec{\mathbf{r}}_0 = \hat{\boldsymbol{\jmath}}$. The tangent plane has equation $z = f(\vec{\mathbf{r}}_0)+\vec{\nabla} f(\vec{\mathbf{r}}_0)\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = 1+(2\hat{\boldsymbol{\jmath}})\cdot(x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{\jmath}}) = 2y-1$.*

⋆ **Remark 1.40.** The gradient of a scalar field f at a given position $\vec{\mathbf{r}}_0$ can be defined as the best linear approximation of f near $\vec{\mathbf{r}}_0$. More precisely, for a differentiable scalar field $f : D \to \mathbb{R}$ and a point $\vec{\mathbf{r}}_0 \in D$, the gradient $\vec{\nabla} f(\vec{\mathbf{r}}_0)$ is the unique vector $\vec{\mathbf{L}}$ such that

$$f(\vec{\mathbf{r}}) = f(\vec{\mathbf{r}}_0) + \vec{\mathbf{L}}\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) + |\vec{\mathbf{r}}-\vec{\mathbf{r}}_0|g(\vec{\mathbf{r}}) \qquad \text{where } \lim_{\vec{\mathbf{r}}\to\vec{\mathbf{r}}_0} g(\vec{\mathbf{r}}) = 0 \text{ and } \vec{\mathbf{r}} \in D.$$

We know from first-year calculus that, given a differentiable real function $F : \mathbb{R} \to \mathbb{R}$, the tangent line to its graph at a point $(t_0, F(t_0))$ is $t \mapsto F(t_0) + F'(t_0)(t - t_0)$ and the derivative $F'(t_0)$ represents its slope. The tangent line can be defined as the line that best approximates F near t_0. Similarly, the affine (i.e. "flat") field that best approximates f near $\vec{\mathbf{r}}_0$ is $\vec{\mathbf{r}} \mapsto f(\vec{\mathbf{r}}_0) + \vec{\nabla} f(\vec{\mathbf{r}}_0)\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0)$. If the field f is two-dimensional, $\{z = f(\vec{\mathbf{r}}_0) + \vec{\nabla} f(\vec{\mathbf{r}}_0)\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0)\}$ is the tangent plane to the surface graph of f (see Remark 1.21), an example is in the right plot in Figure 10. If the field is three-dimensional, this is the tangent space to the graph of f, which now lives in a four-dimensional space, so it is hard to visualise.

1.3.3 The Jacobian matrix

We have seen that a scalar field has three first-order partial derivatives that can be collected in a vector field (the gradient). A vector field has nine partial derivatives, three for each component (which is a scalar field). To represent them compactly we collect them in a matrix.

Consider a smooth vector field $\vec{\mathbf{F}}$, the **Jacobian matrix** $J\vec{\mathbf{F}}$ (or simply the Jacobian, named after Carl Gustav Jacob Jacobi 1804–1851) of $\vec{\mathbf{F}}$ is

$$\boxed{J\vec{\mathbf{F}} := \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix}}. \tag{16}$$

This is a field whose values at each point are 3×3 matrices. Note that sometimes the noun "Jacobian" is used to indicate the determinant of the Jacobian matrix.

Exercise 1.41. ▶ Compute the Jacobian of the following vector fields

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{G}}(\vec{\mathbf{r}}) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} + \cos y\hat{\boldsymbol{k}}.$$

1.3.4 Second-order partial derivatives, the Laplacian and the Hessian

The second-order partial derivatives of a scalar field f are the partial derivatives of the partial derivatives of f. When partial derivatives are taken twice with respect to the same component of the position, they are denoted by

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial f}{\partial x},\qquad \frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\qquad\text{and}\qquad \frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\frac{\partial f}{\partial z}.$$

When they are taken with respect to two different components, they are called “mixed derivatives” and denoted by

$$\begin{aligned}&\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}, &&\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial}{\partial x}\frac{\partial f}{\partial z}, &&\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\frac{\partial f}{\partial x},\\ &\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial}{\partial y}\frac{\partial f}{\partial z}, &&\frac{\partial^2 f}{\partial z\partial x}=\frac{\partial}{\partial z}\frac{\partial f}{\partial x}, \quad\text{and} &&\frac{\partial^2 f}{\partial z\partial y}=\frac{\partial}{\partial z}\frac{\partial f}{\partial y}.\end{aligned}$$

If f is smooth, the order of differentiation is not relevant (Clairault’s or Schwarz theorem):

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x},\qquad \frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial z\partial x}\qquad\text{and}\qquad \frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}. \tag{17}$$

★ **Remark 1.42.** It is clear that we can define partial derivatives of higher order, e.g. $\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$. A scalar field $f:D\to\mathbb{R}$ is called “differentiable of class C^k” with $k\in\mathbb{N}$, if it is continuous (see Remark 1.22), all its partial derivatives of order up to k exist and are continuous. It is called “differentiable of class C^∞” if all its partial derivatives of any order exist and are continuous.

The **Laplacian** Δf (or Laplace operator, sometimes denoted by $\nabla^2 f$, named after Pierre-Simon Laplace 1749–1827)[5] of a smooth scalar field f is the scalar field obtained as sum of the pure second partial derivatives of f:

$$\boxed{\Delta f:=\nabla^2 f:=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.} \tag{18}$$

A scalar field whose Laplacian vanishes everywhere is called **harmonic function**.

★ **Remark 1.43** (Applications of the Laplacian)**.** The Laplacian is used in some of the most studied partial differential equations (PDEs) such as the Poisson equation $\Delta u=f$ and the Helmholtz equation $-\Delta u-k^2u=f$. Therefore, it is ubiquitous and extremely important in physics and engineering, e.g. in the models of diffusion of heat or fluids, electromagnetism, acoustics, wave propagation, quantum mechanics. In the words of T. Tao[6]: “The Laplacian of a function u at a point x measures the average extent to which the value of u at x deviates from the value of u at nearby points to x (cf. the mean value theorem for harmonic functions). As such, it naturally occurs in any system in which some quantity at a point is influenced by the value of the same quantity at nearby points.”

The **Hessian** Hf (or Hessian matrix, named after Ludwig Otto Hesse 1811–1874) is the 3×3 matrix field of the second derivatives of the scalar field f:

$$\boxed{Hf:=J(\vec{\nabla}f)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial z\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\partial y}\\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2}\end{pmatrix}.} \tag{19}$$

Note that, thanks to Clairault’s theorem (17), (if the field f is smooth enough) the Hessian is a symmetric matrix in every point. The Laplacian equals the trace of the Hessian, i.e. the sum of its diagonal terms:

$$\Delta f=\mathrm{Tr}(Hf). \tag{20}$$

The **vector Laplacian** $\vec{\boldsymbol{\Delta}}$ is the Laplace operator applied componentwise to vector fields:

$$\begin{aligned}&\boxed{\vec{\boldsymbol{\Delta}}\vec{\mathbf{F}}:=(\Delta F_1)\hat{\boldsymbol{\imath}}+(\Delta F_2)\hat{\boldsymbol{\jmath}}+(\Delta F_3)\hat{\boldsymbol{k}}} \\ &=\Big(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\Big)\hat{\boldsymbol{\imath}}+\Big(\frac{\partial^2 F_2}{\partial x^2}+\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_2}{\partial z^2}\Big)\hat{\boldsymbol{\jmath}}+\Big(\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}+\frac{\partial^2 F_3}{\partial z^2}\Big)\hat{\boldsymbol{k}}.\end{aligned} \tag{21}$$

[5] The reason for the use of the notation ∇^2 will be slightly more clear after equation (24); this symbol is mainly used by physicists and engineers.

[6] http://mathoverflow.net/questions/54986

Exercise 1.44. ▶ Compute the Laplacian and the Hessian of the following scalar fields:

$$f(\vec{\mathbf{r}}) = x^2 - y^2, \qquad g(\vec{\mathbf{r}}) = xye^z, \qquad h(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^4.$$

Warning: frequent error 1.45. Once again, we remind that symbols cannot be moved around freely in mathematical expressions. For example, $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$, $\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$, $\frac{\partial^2 (fg)}{\partial x \partial y}$ and $g\frac{\partial^2 f}{\partial x \partial y}$, are all different fields and must not be confused with each other. Equation (17) only says that we can swap the order of derivation for the *same* field, not between two fields, i.e. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ but $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \neq \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. Moreover $\frac{\partial fg}{\partial x}$ does not mean anything: is it supposed to be $\frac{\partial (fg)}{\partial x}$ or $(\frac{\partial f}{\partial x})g$? All these errors appears too often in exams, watch out!

1.3.5 The divergence operator

Given a differentiable vector field $\vec{\mathbf{F}}$, its divergence is defined as the the trace of its Jacobian matrix:

$$\vec{\nabla} \cdot \vec{\mathbf{F}} := \operatorname{div} \vec{\mathbf{F}} := \operatorname{Tr}(J\vec{\mathbf{F}}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{22}$$

The notation $\vec{\nabla} \cdot \vec{\mathbf{F}}$ is justified by the fact that the divergence can be written as the *formal* scalar product between the nabla symbol $\vec{\nabla}$ and the field $\vec{\mathbf{F}}$:

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = \Big(\hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\Big) \cdot \Big(F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}} + F_3\hat{\boldsymbol{k}}\Big).$$

Exercise 1.46. ▶ Compute the divergence of the following vector fields:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{G}}(\vec{\mathbf{r}}) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} + \cos y\hat{\boldsymbol{k}}.$$

What is the intuitive meaning of the divergence of a vector field $\vec{\mathbf{F}}$? Very roughly speaking, the divergence measures the "spreading" of $\vec{\mathbf{F}}$ around a point $\vec{\mathbf{r}}$, i.e. the difference between the amount of $\vec{\mathbf{F}}$ exiting from an infinitesimally small ball centred at $\vec{\mathbf{r}}$ and the amount entering it. If $\vec{\nabla} \cdot \vec{\mathbf{F}}$ is positive in $\vec{\mathbf{r}}$ we may expect $\vec{\mathbf{F}}$ to spread or expand near $\vec{\mathbf{r}}$, while if it is negative $\vec{\mathbf{F}}$ will shrink. Of course this is explanation is extremely hand-waving: we will prove a precise characterisation of the meaning of the divergence in Remark 3.33.

Example 1.47 (Positive and negative divergence)**.** The field $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ has positive divergence $\vec{\nabla} \cdot \vec{\mathbf{F}} = 1 + 1 + 0 = 2 > 0$, thus we see in the left plot of Figure 11 that it is somewhat spreading. The field $\vec{\mathbf{G}} = (-x - 2y)\hat{\boldsymbol{\imath}} + (2x - y)\hat{\boldsymbol{\jmath}}$ has negative divergence $\vec{\nabla} \cdot \vec{\mathbf{G}} = -1 - 1 + 0 = -2 < 0$, and we see from the right plot that the field is converging.

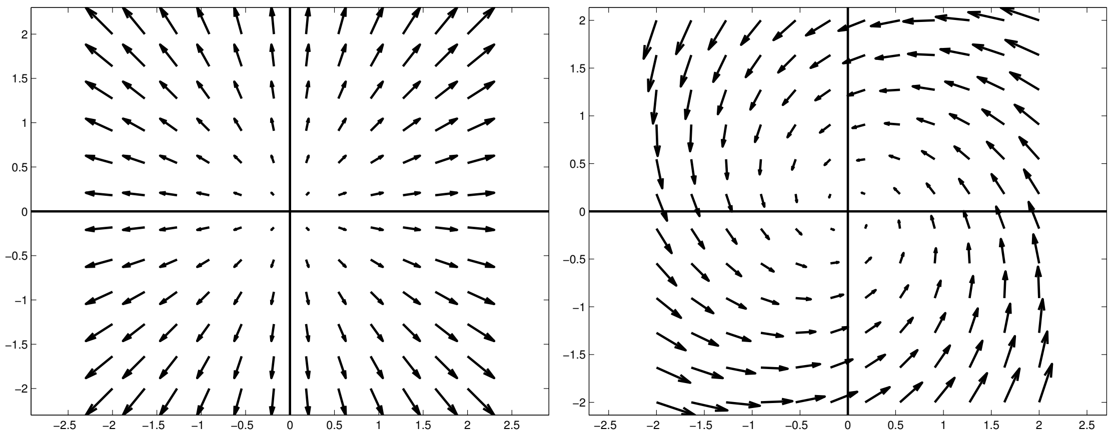


Figure 11: *A representation of the fields $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (left) and $\vec{\mathbf{G}} = (-x - 2y)\hat{\boldsymbol{\imath}} + (2x - y)\hat{\boldsymbol{\jmath}}$ (right) from Example 1.47. $\vec{\mathbf{F}}$ has positive divergence and $\vec{\mathbf{G}}$ negative.*

★ **Remark 1.48.** From the previous example it seems to be possible to deduce the sign of the divergence of a vector field from its plot, observing whether the arrows "converge" or "diverge". However, this can be misleading, as also the magnitude (the length of the arrows) matters. For example, the fields $\vec{\mathbf{F}}_a = |\vec{\mathbf{r}}|^{-a}\vec{\mathbf{r}}$ defined in $\mathbb{R}^3 \setminus \{\vec{\mathbf{0}}\}$, where a is a real positive parameter, have similar plots but they have positive divergence if $a < 3$ and negative if $a > 3$. (Can you show this fact? It is not easy!)

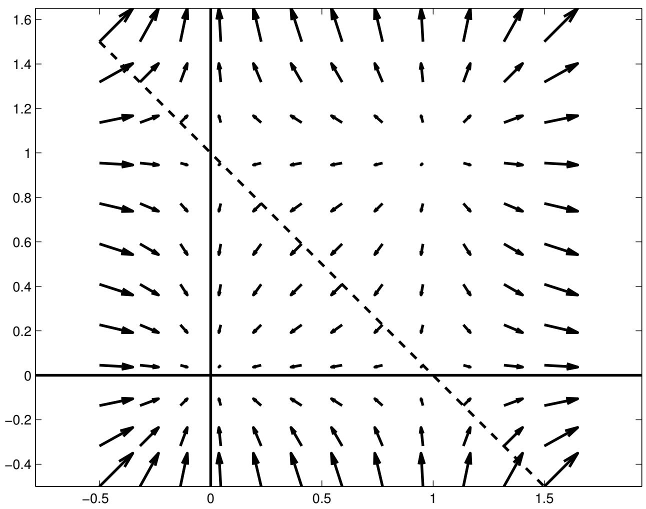


Figure 12: *The field* $\vec{\mathbf{F}} = (x^2 - x)\hat{\boldsymbol{\imath}} + (y^2 - y)\hat{\boldsymbol{\jmath}}$ *has divergence* $\vec{\nabla} \cdot \vec{\mathbf{F}} = 2(x + y - 1)$ *which is positive above the straight line* $x + y = 1$ *and negative below. If* $\vec{\mathbf{F}}$ *represents the velocity of a fluid, the upper part of the space acts like a source and the lower part as a sink.*

Warning: frequent error 1.49. Since strictly speaking $\vec{\nabla}$ is not a vector (11), the symbol $\vec{\nabla} \cdot \vec{\mathbf{F}}$ does not represent a scalar product and[7]

$$\vec{\nabla} \cdot \vec{\mathbf{F}} \neq \vec{\mathbf{F}} \cdot \vec{\nabla}.$$

Recall: you can not move around the nabla symbol $\vec{\nabla}$ in a mathematical expression!

1.3.6 The curl operator

The last differential operator we define is the **curl operator** $\vec{\nabla}\times$ (often denoted "curl" and more rarely "rot" and called rotor or rotational), which maps vector fields to vector fields:

$$\vec{\nabla} \times \vec{\mathbf{F}} := \operatorname{curl} \vec{\mathbf{F}} := \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\boldsymbol{k}}. \tag{23}$$

As in the definition of the vector product (2), the matrix determinant is purely formal, since it contains vectors, differential operators and scalar fields. Again, $\vec{\nabla} \times \vec{\mathbf{F}} \neq -\vec{\mathbf{F}} \times \vec{\nabla}$, as the left-hand side is a vector field while the right-hand side is a differential operator.

Among all the differential operators introduced so far, the curl operator is the only one which can be defined *only* in three dimensions, since it is related to the vector product.

Exercise 1.50. ▶ Compute the curl of the following vector fields:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{G}}(\vec{\mathbf{r}}) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} + \cos y\hat{\boldsymbol{k}}.$$

How can we interpret the curl of a field? The curl is in some way a measure of the "rotation" of the field. If we imagine to place a microscopic paddle-wheel at a point $\vec{\mathbf{r}}$ in a fluid moving with velocity $\vec{\mathbf{F}}$, it will rotate with angular velocity proportional to $|\vec{\nabla} \times \vec{\mathbf{F}}(\vec{\mathbf{r}})|$ and rotation axis parallel to $\vec{\nabla} \times \vec{\mathbf{F}}(\vec{\mathbf{r}})$. If the fluid motion is planar, i.e. $F_3 = 0$, and anti-clockwise, the curl will point out of page, if the motion is clockwise it will point into the page (as the motion of a usual screw). See also Figure 13.

[7] Indeed $\vec{\nabla} \cdot \vec{\mathbf{F}}$ and $\vec{\mathbf{F}} \cdot \vec{\nabla}$ are two very different mathematical objects: $\vec{\nabla} \cdot \vec{\mathbf{F}}$ is a *scalar field*, while $\vec{\mathbf{F}} \cdot \vec{\nabla}$ is an *operator* that maps scalar fields to scalar fields

$$(\vec{\mathbf{F}} \cdot \vec{\nabla})f = \Big(F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}\Big)f = F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}.$$

For example if $\vec{\mathbf{G}} = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}$ and $g = xye^z$, then

$$(\vec{\mathbf{G}} \cdot \vec{\nabla})g = \Big(yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}\Big)g = yz\,ye^z + xz\,xe^z + xy\,xye^z = (x^2z + y^2z + x^2y^2)e^z.$$

The operator $\vec{\mathbf{F}} \cdot \vec{\nabla}$ can also be applied componentwise to a vector field (and in this case it returns a vector field, see (35)). $\vec{\mathbf{F}} \cdot \vec{\nabla}$ is often called **advection operator**; when $|\vec{\mathbf{F}}| = 1$, then it corresponds to the directional derivative we have already encountered in Section 1.3.2. We will use the advection operator *only* in Remark 1.56; in any other situation in this module you should not use the symbol $(\vec{\mathbf{F}} \cdot \vec{\nabla})$.

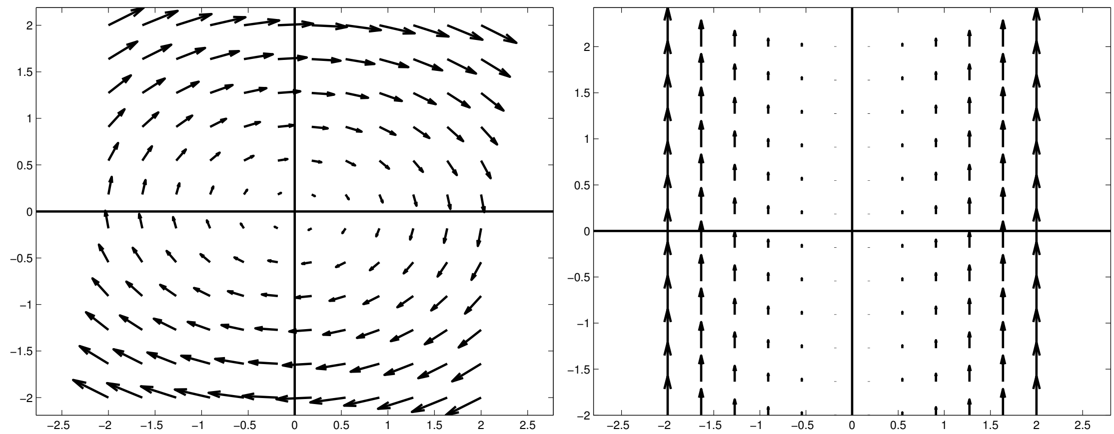


Figure 13: *The planar field* $\vec{\mathbf{F}} = 2y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}$ *depicted in the left plot has curl equal to* $\vec{\nabla} \times \vec{\mathbf{F}} = -3\hat{\boldsymbol{k}}$ *which "points into the page", in agreement with the clockwise rotation of* $\vec{\mathbf{F}}$*. The planar field* $\vec{\mathbf{G}} = x^2\hat{\boldsymbol{\jmath}}$ *depicted in the right plot has curl equal to* $\vec{\nabla} \times \vec{\mathbf{G}} = 2x\hat{\boldsymbol{k}}$ *which points "into the page" for* $x < 0$ *and "out of the page" for* $x > 0$*. Indeed, if we imagine to immerse some paddle-wheels in the field, due to the differences in the magnitude of the field on their two sides, they will rotate clockwise in the negative-x half-plane and anti-clockwise in the positive-x half-plane.*

★ **Remark 1.51** (Linearity of differential operators)**.** All operators introduced in this section (partial derivatives, gradient, Jacobian, Laplacian, Hessian, divergence, curl) are **linear** operators. This means that, denoting $\mathcal{D}$ any of these operators, for all $\lambda, \mu \in \mathbb{R}$ and for any two suitable fields F and G (either both scalar or both vector fields, depending on which operator is considered), we have the identity $\mathcal{D}(\lambda F + \mu G) = \lambda\mathcal{D}(F) + \mu\mathcal{D}(G)$.

1.4 Vector differential identities

In the following two propositions we prove several identities involving scalar and vector fields and the differential operators defined so far. It is fundamental to apply each operator to a field of the appropriate kind, for example we can not compute the divergence of a scalar field because we have not given any meaning to this object. For this reason, we summarise in Table 1 which types of fields are taken as input and returned as output by the various operators.

Operator name	symbol	field taken as input	field returned as output	order
Partial derivative	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$	scalar	scalar	1st
Gradient	$\vec{\nabla}$	scalar	vector	1st
Jacobian	J	vector	matrix	1st
Laplacian	Δ	scalar	scalar	2nd
Hessian	H	scalar	matrix	2nd
Vector Laplacian	$\vec{\boldsymbol{\Delta}}$	vector	vector	2nd
Divergence	$\vec{\nabla}\cdot$ (or div)	vector	scalar	1st
Curl (only 3D)	$\vec{\nabla}\times$ (or curl)	vector	vector	1st

Table 1: *A summary of the differential operators defined in Section 1.3. The last column shows the order of the partial derivatives involved in each operator.*

1.4.1 Second-order vector differential identities

Next proposition shows the result of taking the divergence or the curl of gradients and curls.

Proposition 1.52 (Vector differential identities for a single field)**.** Let f be a scalar field and $\vec{\mathbf{F}}$ be a vector field, both defined in a domain $D \subset \mathbb{R}^3$. Then the following identities hold true:

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \Delta f, \tag{24}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0, \tag{25}$$

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{\mathbf{0}}, \tag{26}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{F}}) - \vec{\boldsymbol{\Delta}}\vec{\mathbf{F}}. \tag{27}$$

Proof. The two scalar identities (24) and (25) can be proved using only the definitions of the differential operators involved (note that we write $(\vec{\nabla} f)_1$ and $(\vec{\nabla}\times\vec{\mathbf{F}})_1$ to denote the x-components of the corresponding vector fields, and similarly for y and z):

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{\nabla} f) &\overset{(22)}{=} \frac{\partial\big((\vec{\nabla} f)_1\big)}{\partial x}+\frac{\partial\big((\vec{\nabla} f)_2\big)}{\partial y}+\frac{\partial\big((\vec{\nabla} f)_3\big)}{\partial z}\\
&\overset{(10)}{=} \frac{\partial}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial}{\partial y}\frac{\partial f}{\partial y}+\frac{\partial}{\partial z}\frac{\partial f}{\partial z}\\
&\overset{(18)}{=} \Delta f,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathbf{F}}) &\overset{(22)}{=} \frac{\partial\big((\vec{\nabla}\times\vec{\mathbf{F}})_1\big)}{\partial x}+\frac{\partial\big((\vec{\nabla}\times\vec{\mathbf{F}})_2\big)}{\partial y}+\frac{\partial\big((\vec{\nabla}\times\vec{\mathbf{F}})_3\big)}{\partial z}\\
&\overset{(23)}{=} \frac{\partial\Big(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\Big)}{\partial x}+\frac{\partial\Big(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\Big)}{\partial y}+\frac{\partial\Big(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\Big)}{\partial z}\\
&= \frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_1}{\partial z\partial y}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_3}{\partial y\partial x}\\
&\overset{(17)}{=} 0.
\end{aligned}$$

An alternative proof of (24) uses the facts that the divergence is the trace of the Jacobian matrix (22), the Jacobian matrix of a gradient is the Hessian (19), and the trace of the Hessian is the Laplacian (20):

$$\vec{\nabla}\cdot(\vec{\nabla} f) \overset{(22)}{=} \operatorname{Tr}(J\vec{\nabla} f) \overset{(19)}{=} \operatorname{Tr}(Hf) \overset{(20)}{=} \Delta f.$$

The two remaining vector identities can be proved componentwise, i.e. by showing that each of the three components of the left-hand side is equal to the corresponding component of the right-hand side:

$$\begin{aligned}
\big(\vec{\nabla}\times(\vec{\nabla} f)\big)_1 &\overset{(23)}{=} \frac{\partial\big((\vec{\nabla} f)_3\big)}{\partial y}-\frac{\partial\big((\vec{\nabla} f)_2\big)}{\partial z}\\
&\overset{(10)}{=} \frac{\partial}{\partial y}\frac{\partial f}{\partial z}-\frac{\partial}{\partial z}\frac{\partial f}{\partial y}\\
&\overset{(17)}{=} 0,\\
\big(\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{F}})\big)_1 &\overset{(23)}{=} \frac{\partial\big((\vec{\nabla}\times\vec{\mathbf{F}})_3\big)}{\partial y}-\frac{\partial\big((\vec{\nabla}\times\vec{\mathbf{F}})_2\big)}{\partial z}\\
&\overset{(23)}{=} \frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\\
&\overset{(17)}{=} \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}\\
&= \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\frac{\partial^2 F_1}{\partial x^2}-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}\\
&\overset{(22),(18)}{=} \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{\mathbf{F}})-\Delta F_1\\
&\overset{(10),(21)}{=} \Big(\vec{\nabla}(\vec{\nabla}\cdot\vec{\mathbf{F}})-\vec{\boldsymbol{\Delta}}\vec{\mathbf{F}}\Big)_1,
\end{aligned}$$

and similarly for the y- and z-components. □

Note that, when we prove a certain vector identity, even if we consider only the first component of the equality, this may involve *all components* of the fields involved and partial derivatives with respect to *all variables*; see for instance the proofs of (26) and (27).

★ **Remark 1.53.** Since in all the identities of Proposition 1.52 we take second derivatives, the precise assumptions for this proposition are that f and (the components of) $\vec{\mathbf{F}}$ are differentiable of class C^2.

1.4.2 Vector product rules

Comparison with scalar calculus 1.54. When we compute the derivative of the product of two functions of real variable we apply the well-known **product rule** $(fg)' = f'g + g'f$. When we calculate partial derivatives or the gradient of the product of two scalar fields, similar rules apply, see Equations (8) and (13). But scalar and vector fields may be multiplied in several different ways (scalar product, vector product and scalar-vector product) and other differential operators may be applied to these products (divergence, curl, Laplacian,...). In these cases, the product rules obtained are slightly more complicated; they are described in next Proposition 1.55 (and Remark 1.56). Note the common structure of all these identities: a differential operator applied to a product of two fields (at the left-hand side) is equal to the sum of some terms containing the derivative of the first field only and some terms containing derivatives of the second field only (at the right-hand side).

A special identity is (32): since the Laplacian contains second-order partial derivatives, all terms at the right-hand side contain two derivatives, exactly as in the 1D product rule for second derivatives $(fg)'' = f''g + 2f'g' + fg''$.

Proposition 1.55 (Vector differential identities for two fields, product rules for differential operators)**.** Let f and g be scalar fields, $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be vector fields, all of them defined in the same domain $D \subset \mathbb{R}^3$. Then the following identities hold true[8]:

$$\vec\nabla(fg) = f\vec\nabla g + g\vec\nabla f, \tag{28}$$

$$\vec\nabla\cdot(f\vec{\mathbf{G}}) = (\vec\nabla f)\cdot\vec{\mathbf{G}} + f\vec\nabla\cdot\vec{\mathbf{G}}, \tag{29}$$

$$\vec\nabla\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = (\vec\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec\nabla\times\vec{\mathbf{G}}), \tag{30}$$

$$\vec\nabla\times(f\vec{\mathbf{G}}) = (\vec\nabla f)\times\vec{\mathbf{G}} + f\vec\nabla\times\vec{\mathbf{G}}, \tag{31}$$

$$\Delta(fg) = (\Delta f)g + 2\vec\nabla f\cdot\vec\nabla g + f(\Delta g). \tag{32}$$

Proof. Identity (28) has already been proven in Proposition 1.33. We show here the proof of identities (30) and (32) only; identities (29) and (31) can be proven with a similar technique. All the proofs use only the definitions of the differential operators and of the vector operations (scalar and vector product), the product rule (8) for partial derivatives, together with some smart rearrangements of the terms. In some cases, it is convenient to start the proof from the expression at the right-hand side, or even to expand both sides and match the results obtained. The vector identity (31) is proven componentwise (see the similar proof of (33) below).

Identity (30) can be proved as follows:

$$\begin{aligned}
\vec\nabla\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) &\overset{(22)}{=} \frac{\partial\big((\vec{\mathbf{F}}\times\vec{\mathbf{G}})_1\big)}{\partial x} + \frac{\partial\big((\vec{\mathbf{F}}\times\vec{\mathbf{G}})_2\big)}{\partial y} + \frac{\partial\big((\vec{\mathbf{F}}\times\vec{\mathbf{G}})_3\big)}{\partial z}\\
&\overset{(2)}{=} \frac{\partial\big(F_2G_3 - F_3G_2\big)}{\partial x} + \frac{\partial\big(F_3G_1 - F_1G_3\big)}{\partial y} + \frac{\partial\big(F_1G_2 - F_2G_1\big)}{\partial z}\\
&\overset{(8)}{=} \frac{\partial F_2}{\partial x}G_3 + F_2\frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x}G_2 - F_3\frac{\partial G_2}{\partial x} + \frac{\partial F_3}{\partial y}G_1 + F_3\frac{\partial G_1}{\partial y} - \frac{\partial F_1}{\partial y}G_3 - F_1\frac{\partial G_3}{\partial y}\\
&\qquad + \frac{\partial F_1}{\partial z}G_2 + F_1\frac{\partial G_2}{\partial z} - \frac{\partial F_2}{\partial z}G_1 - F_2\frac{\partial G_1}{\partial z} \qquad\qquad \text{(now collect non-derivative terms)}\\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)G_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)G_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)G_3\\
&\qquad - F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\\
&\overset{(23)}{=} (\vec\nabla\times\vec{\mathbf{F}})_1G_1 + (\vec\nabla\times\vec{\mathbf{F}})_2G_2 + (\vec\nabla\times\vec{\mathbf{F}})_3G_3 - F_1(\vec\nabla\times\vec{\mathbf{G}})_1 - F_2(\vec\nabla\times\vec{\mathbf{G}})_2 - F_3(\vec\nabla\times\vec{\mathbf{G}})_3\\
&\overset{(1)}{=} (\vec\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec\nabla\times\vec{\mathbf{G}}).
\end{aligned}$$

Identity (32) follows from a repeated application of the product rule (8):

$$\begin{aligned}
\Delta(fg) &\overset{(18)}{=} \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}\\
&\overset{(8)}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right)\\
&\overset{(8)}{=} \left(\frac{\partial^2 f}{\partial x^2}g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial^2 f}{\partial y^2}g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial^2 f}{\partial z^2}g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right)\\
&\overset{(18),(10),(1)}{=} (\Delta f)g + 2\vec\nabla f\cdot\vec\nabla g + f(\Delta g).
\end{aligned}$$

[8] You do not need to remember these identities by hearth, even though are simple, but you need to be able to use them and to derive them.
★ Here the precise assumptions for the proposition are that all fields are differentiable of class C^1, except for identity (32) where class C^2 is required.

The following is an alternative proof of identity (32) that uses the vector identities previously proved:

$$\begin{aligned}\Delta(fg) &\overset{(24)}{=} \vec\nabla\cdot(\vec\nabla(fg))\\ &\overset{(28)}{=} \vec\nabla\cdot(f\vec\nabla g+g\vec\nabla f)\\ &\overset{(12)}{=} \vec\nabla\cdot(f\vec\nabla g)+\vec\nabla\cdot(g\vec\nabla f)\\ &\overset{(29)}{=} (\vec\nabla f)\cdot(\vec\nabla g)+f\vec\nabla\cdot(\vec\nabla g)+(\vec\nabla g)\cdot(\vec\nabla f)+g\vec\nabla\cdot(\vec\nabla f)\\ &\overset{(24)}{=} (\Delta f)g+2\vec\nabla f\cdot\vec\nabla g+f(\Delta g).\end{aligned}$$

□

★ **Remark 1.56.** In Proposition 1.55 we found expression (28) for the gradient of the product of two scalar fields. What about the gradient of the scalar product of two vector fields? We also computed the divergence (29) and the curl (31) of the product of a vector and a scalar field, and the divergence of the vector product between two vector fields (30). What about the curl of a vector product? The answer two these two questions is given by the following two vector product rules:

$$\vec\nabla(\vec{\mathbf F}\cdot\vec{\mathbf G})=(\vec{\mathbf F}\cdot\vec\nabla)\vec{\mathbf G}+(\vec{\mathbf G}\cdot\vec\nabla)\vec{\mathbf F}+\vec{\mathbf G}\times(\vec\nabla\times\vec{\mathbf F})+\vec{\mathbf F}\times(\vec\nabla\times\vec{\mathbf G}), \tag{33}$$

$$\vec\nabla\times(\vec{\mathbf F}\times\vec{\mathbf G})=(\vec\nabla\cdot\vec{\mathbf G})\vec{\mathbf F}-(\vec\nabla\cdot\vec{\mathbf F})\vec{\mathbf G}+(\vec{\mathbf G}\cdot\vec\nabla)\vec{\mathbf F}-(\vec{\mathbf F}\cdot\vec\nabla)\vec{\mathbf G}. \tag{34}$$

In identities (33) and (34) we use the **advection operator** (which was mentioned in Remark 1.49) applied to a vector field:

$$\begin{aligned}&(\vec{\mathbf G}\cdot\vec\nabla)\vec{\mathbf F}=(\vec{\mathbf G}\cdot\vec\nabla)F_1\hat{\boldsymbol\imath}+(\vec{\mathbf G}\cdot\vec\nabla)F_2\hat{\boldsymbol\jmath}+(\vec{\mathbf G}\cdot\vec\nabla)F_3\hat{\mathbf k} \qquad (35)\\ &=\Big(G_1\frac{\partial F_1}{\partial x}+G_2\frac{\partial F_1}{\partial y}+G_3\frac{\partial F_1}{\partial z}\Big)\hat{\boldsymbol\imath}+\Big(G_1\frac{\partial F_2}{\partial x}+G_2\frac{\partial F_2}{\partial y}+G_3\frac{\partial F_2}{\partial z}\Big)\hat{\boldsymbol\jmath}+\Big(G_1\frac{\partial F_3}{\partial x}+G_2\frac{\partial F_3}{\partial y}+G_3\frac{\partial F_3}{\partial z}\Big)\hat{\mathbf k}.\end{aligned}$$

Proof of (33). We prove (33) componentwise, starting from the first component of the expression at the right-hand side:

$$\begin{aligned}&\Big((\vec{\mathbf F}\cdot\vec\nabla)\vec{\mathbf G}+(\vec{\mathbf G}\cdot\vec\nabla)\vec{\mathbf F}+\vec{\mathbf G}\times(\vec\nabla\times\vec{\mathbf F})+\vec{\mathbf F}\times(\vec\nabla\times\vec{\mathbf G})\Big)_1\\ &\overset{(2)}{=}(\vec{\mathbf F}\cdot\vec\nabla)G_1+(\vec{\mathbf G}\cdot\vec\nabla)F_1+G_2(\vec\nabla\times\vec{\mathbf F})_3-G_3(\vec\nabla\times\vec{\mathbf F})_2+F_2(\vec\nabla\times\vec{\mathbf G})_3-F_3(\vec\nabla\times\vec{\mathbf G})_2\\ &\overset{(23)}{=}\Big(F_1\frac{\partial G_1}{\partial x}+F_2\frac{\partial G_1}{\partial y}+F_3\frac{\partial G_1}{\partial z}\Big)+\Big(G_1\frac{\partial F_1}{\partial x}+G_2\frac{\partial F_1}{\partial y}+G_3\frac{\partial F_1}{\partial z}\Big)\\ &\qquad+G_2\Big(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\Big)-G_3\Big(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\Big)+F_2\Big(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\Big)-F_3\Big(\frac{\partial G_1}{\partial z}-\frac{\partial G_3}{\partial x}\Big)\\ &\qquad\qquad\text{(collect the six non-derivative terms } F_1,\ F_2,\dots)\\ &=F_1\frac{\partial G_1}{\partial x}+F_2\frac{\partial G_2}{\partial x}+F_3\frac{\partial G_3}{\partial x}+G_1\frac{\partial F_1}{\partial x}+G_2\frac{\partial F_2}{\partial x}+G_3\frac{\partial F_3}{\partial x}\qquad\text{(collect terms pairwise)}\\ &=\Big(F_1\frac{\partial G_1}{\partial x}+G_1\frac{\partial F_1}{\partial x}\Big)+\Big(F_2\frac{\partial G_2}{\partial x}+G_2\frac{\partial F_2}{\partial x}\Big)+\Big(F_3\frac{\partial G_3}{\partial x}+G_3\frac{\partial F_3}{\partial x}\Big)\\ &\overset{(8)}{=}\frac{\partial(F_1G_1)}{\partial x}+\frac{\partial(F_2G_2)}{\partial x}+\frac{\partial(F_3G_3)}{\partial x}\\ &\overset{(1)}{=}\frac{\partial(\vec{\mathbf F}\cdot\vec{\mathbf G})}{\partial x}\\ &\overset{(10)}{=}\big(\vec\nabla(\vec{\mathbf F}\cdot\vec{\mathbf G})\big)_1,\end{aligned}$$

and similarly for the y- and z-components. □

Exercise 1.57. Prove identities (29), (31) and (34) .

Remark 1.58 (Why use vector identities?)**.** Despite looking complicated at a first glance, the identities presented in this section are very useful tools to simplify computations involving differential operators. If you learn how to use them—rather than reinventing the wheel each time expanding all expressions in components and the operators in partial derivatives—many exercises and theoretical results in the rest of this module and in future courses will definitely turn out easier.

Consider, for example the following exercise, which was part of the exam in May 2014.

Let f and g be two smooth scalar fields. Prove the following identity: $\vec\nabla\times(f\vec\nabla g)+\vec\nabla\times(g\vec\nabla f)=\vec{\mathbf 0}$.

Using two vector identities and the linearity of curl, the solution takes no more than one line[9]:

$$\vec{\nabla}\times(f\vec{\nabla}g)+\vec{\nabla}\times(g\vec{\nabla}f) \quad=\quad \vec{\nabla}\times(f\vec{\nabla}g+g\vec{\nabla}f) \quad\overset{(28)}{=}\quad \vec{\nabla}\times\big(\vec{\nabla}(fg)\big) \quad\overset{(26)}{=}\quad \vec{\mathbf{0}}.$$

If you do not use the vector identities as above, but prove the result componentwise, the solution is longer and definitely much more subject to mistakes: for the first component we have

$$\begin{aligned}
\big(\vec{\nabla}\times(f\vec{\nabla}g)+\vec{\nabla}\times(g\vec{\nabla}f)\big)_1 &\overset{(23)}{=} \frac{\partial}{\partial y}(f\vec{\nabla}g)_3-\frac{\partial}{\partial z}(f\vec{\nabla}g)_2+\frac{\partial}{\partial y}(g\vec{\nabla}f)_3-\frac{\partial}{\partial z}(g\vec{\nabla}f)_2\\
&\overset{(10)}{=} \frac{\partial}{\partial y}\Big(f\frac{\partial g}{\partial z}\Big)-\frac{\partial}{\partial z}\Big(f\frac{\partial g}{\partial y}\Big)+\frac{\partial}{\partial y}\Big(g\frac{\partial f}{\partial z}\Big)-\frac{\partial}{\partial z}\Big(g\frac{\partial f}{\partial y}\Big)\\
&\overset{(8)}{=} \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial y\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial z\partial y}+\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}+g\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial g}{\partial z}\frac{\partial f}{\partial y}+g\frac{\partial^2 f}{\partial z\partial y}\\
&\overset{(17)}{=} 0,
\end{aligned}$$

and similarly for the second and third components.

★ **Remark 1.59.** As a further example of the convenience of the use of vector notation consider the following example. The so-called "Maxwell equations in time-harmonic regime" describe the propagation of an electric field $\vec{\mathbf{E}}$ (a vector field) in a homogeneous materials under certain physical conditions. These partial differential equations in vector form simply read as:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{E}})-k^2\vec{\mathbf{E}}=\vec{\mathbf{0}},$$

where k is a positive scalar called wavenumber. The same equations written in coordinates have a more complicated and obscure expression (compare with the proof of (27)):

$$\begin{cases}
\dfrac{\partial^2 E_2}{\partial x\partial y}+\dfrac{\partial^2 E_3}{\partial x\partial z}-\dfrac{\partial^2 E_1}{\partial y^2}-\dfrac{\partial^2 E_1}{\partial z^2}-k^2E_1=0,\\
\dfrac{\partial^2 E_1}{\partial x\partial y}+\dfrac{\partial^2 E_3}{\partial y\partial z}-\dfrac{\partial^2 E_2}{\partial x^2}-\dfrac{\partial^2 E_2}{\partial z^2}-k^2E_2=0,\\
\dfrac{\partial^2 E_1}{\partial x\partial z}+\dfrac{\partial^2 E_2}{\partial y\partial z}-\dfrac{\partial^2 E_3}{\partial x^2}-\dfrac{\partial^2 E_3}{\partial y^2}-k^2E_3=0.
\end{cases}$$

Example 1.60. We demonstrate some of the results of Proposition 1.52 for the vector field $\vec{\mathbf{F}}=xy\hat{\boldsymbol{\imath}}+e^y\cos x\,\hat{\boldsymbol{\jmath}}+z^2\hat{\boldsymbol{k}}$:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{\mathbf{F}} &= y+e^y\cos x+2z,\\
\vec{\nabla}\times\vec{\mathbf{F}} &= (0-0)\hat{\boldsymbol{\imath}}+(0-0)\hat{\boldsymbol{\jmath}}+(e^y\sin x-x)\hat{\boldsymbol{k}}=(e^y\sin x-x)\hat{\boldsymbol{k}},\\
\vec{\nabla}(\vec{\nabla}\cdot\vec{\mathbf{F}}) &= -e^y\sin x\,\hat{\boldsymbol{\imath}}+(1+e^y\cos x)\hat{\boldsymbol{\jmath}}+2\hat{\boldsymbol{k}},\\
\vec{\nabla}\times\vec{\nabla}\times\vec{\mathbf{F}} &= e^y\sin x\hat{\boldsymbol{\imath}}-(e^y\cos x-1)\hat{\boldsymbol{\jmath}},\\
\vec{\boldsymbol{\Delta}}\vec{\mathbf{F}} &= \Delta(xy)\hat{\boldsymbol{\imath}}+\Delta(e^y\cos x)\hat{\boldsymbol{\jmath}}+\Delta(z^2)\hat{\boldsymbol{k}}=0+(-e^y\cos x+e^y\cos x)\hat{\boldsymbol{\jmath}}+2\hat{\boldsymbol{k}}=2\hat{\boldsymbol{k}}
\end{aligned}$$

and from the last three formulas we immediately demonstrate (27). Since $\vec{\nabla}\times\vec{\mathbf{F}}$ is parallel to $\hat{\boldsymbol{k}}$ and independent of z, its divergence vanishes, thus also the identity (25) is verified.

Exercise 1.61. ▶ Demonstrate identity (30) for the fields $\vec{\mathbf{F}}=z\hat{\boldsymbol{\imath}}+x^2\hat{\boldsymbol{\jmath}}+y^3\hat{\boldsymbol{k}}$ and $\vec{\mathbf{G}}=e^{xy}\hat{\boldsymbol{k}}$.

Exercise 1.62. ▶ Let $\alpha\in\mathbb{R}$ be a constant. Prove that (Hint: recall Exercise 1.35.)

$$\vec{\nabla}\cdot(|\vec{\mathbf{r}}|^\alpha\vec{\mathbf{r}})=(3+\alpha)|\vec{\mathbf{r}}|^\alpha \qquad\text{and}\qquad \vec{\nabla}\times(|\vec{\mathbf{r}}|^\alpha\vec{\mathbf{r}})=\vec{\mathbf{0}}.$$

Exercise 1.63. ▶ *(i)* Prove that $\vec{\nabla}\times(f\vec{\nabla}f)=\vec{\mathbf{0}}$ for all smooth scalar fields f.

(ii) Prove that, for all natural numbers $n,\ell\in\mathbb{N}$, the more general identity $\vec{\nabla}\times(f^n\vec{\nabla}(f^\ell))=\vec{\mathbf{0}}$ holds true. Here f^n and f^ℓ simply denote the nth and ℓth powers of f.

Hint: use an appropriate version of the chain rule to compute the gradients of f^ℓ and f^n.

★ **Remark 1.64.** The following identity involving the Jacobian matrix holds true:

$$(\vec{\nabla}\times\vec{\mathbf{F}})\times\vec{\mathbf{G}}=\big(J\vec{\mathbf{F}}-(J\vec{\mathbf{F}})^T\big)\vec{\mathbf{G}},$$

where the symbol ^{T} denotes matrix transposition. Try to prove it for exercise.

[9] Another possible solution using two vector identities and the anti-commutativity of the vector product is the following:
$$\vec{\nabla}\times(f\vec{\nabla}g)+\vec{\nabla}\times(g\vec{\nabla}f)\overset{(31)}{=}(\vec{\nabla}f)\times(\vec{\nabla}g)+f\underbrace{\vec{\nabla}\times(\vec{\nabla}g)}_{=\vec{\mathbf{0}},\ (26)}+(\vec{\nabla}g)\times(\vec{\nabla}f)+g\underbrace{\vec{\nabla}\times(\vec{\nabla}f)}_{=\vec{\mathbf{0}},\ (26)}\overset{(3)}{=}(\vec{\nabla}f)\times(\vec{\nabla}g)-(\vec{\nabla}f)\times(\vec{\nabla}g)=\vec{\mathbf{0}}.$$

1.5 Special vector fields and potentials

Definition 1.65. Consider a vector field $\vec{\mathbf{F}}$ defined on a domain $D \subset \mathbb{R}^3$.
If $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is called **irrotational** (or curl-free).
If $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is called **solenoidal** (or divergence-free, or incompressible).
If $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ for some scalar field φ, then $\vec{\mathbf{F}}$ is called **conservative** and φ is called **scalar potential** of $\vec{\mathbf{F}}$.
If $\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{A}}$ for some vector field $\vec{\mathbf{A}}$, then $\vec{\mathbf{A}}$ is called **vector potential** of $\vec{\mathbf{F}}$.

[10]From the definition of the curl operator, we note that $\vec{\mathbf{F}}$ is irrotational precisely when

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \qquad \text{and} \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$

i.e. when the Jacobian matrix $J\vec{\mathbf{F}}$ is symmetric. If $\vec{\mathbf{F}}$ admits a scalar or a vector potential, then **the potential is not unique**: for all constant scalars $\lambda \in \mathbb{R}$ and for all scalar fields g (recall (26))

$$\text{if} \quad \vec{\mathbf{F}} = \vec{\nabla}\varphi \quad \text{then} \quad \vec{\mathbf{F}} = \vec{\nabla}(\varphi + \lambda); \qquad \text{if} \quad \vec{\mathbf{G}} = \vec{\nabla} \times \vec{\mathbf{A}} \quad \text{then} \quad \vec{\mathbf{G}} = \vec{\nabla} \times (\vec{\mathbf{A}} + \vec{\nabla} g);$$

therefore $\widetilde{\varphi} := \varphi + \lambda$ and $\widetilde{\vec{\mathbf{A}}} := \vec{\mathbf{A}} + \vec{\nabla} g$ are alternative potentials for $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$.

Example 1.66. Consider the following vector fields:

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{G}}(\vec{\mathbf{r}}) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} + \cos y\hat{\boldsymbol{k}}.$$

In Exercises 1.46 and 1.50 we showed that $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are both irrotational and solenoidal while $\vec{\mathbf{H}}$ is neither irrotational nor solenoidal. Moreover, $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ admit both scalar and vector potentials (thus in particular they are conservative):

$$\vec{\mathbf{F}} = \vec{\nabla}(x^2 - y^2) = \vec{\nabla} \times (2xy\vec{\mathbf{k}}), \qquad \vec{\mathbf{G}} = \vec{\nabla}(xyz) = \vec{\nabla} \times \left(\frac{z^2x\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}} + y^2z\hat{\boldsymbol{k}}}{2}\right).$$

(For the technique used to compute these potentials see Example 1.67 below.) Of course there exist fields which are irrotational but not solenoidal and the other way round: verify this fact for the following fields

$$\vec{\mathbf{F}}_A = 2x\hat{\boldsymbol{\imath}} + 3y^2\hat{\boldsymbol{\jmath}} + e^z\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}}_B = e^{-x^2}\hat{\boldsymbol{\imath}} + \frac{1}{1+z^2}\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}}_C = x^3y^4\hat{\boldsymbol{\imath}} + x^4y^3\hat{\boldsymbol{\jmath}} + z^2\hat{\boldsymbol{k}},$$

$$\vec{\mathbf{F}}_D = (x+y)\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{F}}_E = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} - xy\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}}_F = (x^2 + \cos z)\hat{\boldsymbol{\imath}} - 2xy\hat{\boldsymbol{\jmath}}.$$

Example 1.67 (Computation of potentials)**.** Given a field, how can we compute a scalar or vector potential?
We consider as example the irrotational and solenoidal field

$$\vec{\mathbf{F}} = z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{k}}.$$

If φ is a scalar potential of $\vec{\mathbf{F}}$, it must satisfy $\vec{\mathbf{F}} = \vec{\nabla}\varphi$, thus, acting componentwise,

$$\frac{\partial \varphi}{\partial x} = z \quad \Rightarrow \quad \varphi = xz + f(y,z) \quad \text{for some two-dimensional scalar field } f,$$

$$\frac{\partial \varphi}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial\big(xz + f(y,z)\big)}{\partial y} = \frac{\partial f(y,z)}{\partial y} = 0 \quad \Rightarrow \quad \varphi = xz + g(z) \quad \text{for some real function } g,$$

$$\frac{\partial \varphi}{\partial z} = x \quad \Rightarrow \quad \frac{\partial\big(xz + g(z)\big)}{\partial z} = x + \frac{\partial g(z)}{\partial z} = x \quad \Rightarrow \quad \varphi = xz + \lambda.$$

Thus, for every real constant λ, the fields $xz + \lambda$ are scalar potentials of $\vec{\mathbf{F}}$.
A vector potential $\vec{\mathbf{A}}$ must satisfy $\vec{\nabla} \times \vec{\mathbf{A}} = \vec{\mathbf{F}}$ thus

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = z, \qquad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = 0, \qquad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = x.$$

Many choices for $\vec{\mathbf{A}}$ are possible, the simplest one is to set $A_1 = A_3 = 0$ and $A_2(\vec{\mathbf{r}}) = a(x,z)$ for some two-dimensional scalar field $a(x,z)$ which now must satisfy

$$\frac{\partial a}{\partial x} = x, \qquad \frac{\partial a}{\partial z} = -z.$$

[10]In physics, the scalar potential is often defined to satisfy $\vec{\mathbf{F}} = -\vec{\nabla}\varphi$, this can be a big source of confusion!

We integrate and obtain

$$a=\frac{x^2}{2}+b(z) \qquad\Rightarrow\qquad \frac{\partial\big(\frac12 x^2+b(z)\big)}{\partial z}=\frac{\partial b(z)}{\partial z}=-z \qquad\Rightarrow\qquad b(z)=\frac{-z^2}{2},$$

and finally $\vec{\mathbf{A}}=\frac12(x^2-z^2)\hat{\boldsymbol{\jmath}}$ is a vector potential of $\vec{\mathbf{F}}$ (and many others are possible).

As exercise consider the fields defined in Example 1.66 and compute a scalar potential of $\vec{\mathbf{F}}_A$ and $\vec{\mathbf{F}}_C$ and a vector potential of $\vec{\mathbf{F}}_D$.

You can practice more in the computation of potentials with Exercises 1–10 of [1, Section 15.2, p. 857].

Note that what we are actually doing in Example 1.67 is computing indefinite integrals, so in most practical cases a closed form for the potentials is not available.

Exercise 1.68. ▶ Verify that the position field $\vec{\mathbf{r}}$ is irrotational, conservative and not solenoidal. Compute a scalar potential for $\vec{\mathbf{r}}$.

Exercise 1.69. ▶ Use two irrotational vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ to construct a solenoidal vector field $\vec{\mathbf{H}}$. Hint: use the identities proved in the previous section.

Because of identities (26) and (25) ($\vec\nabla\times(\vec\nabla\varphi)=\vec{\mathbf{0}}$, $\vec\nabla\cdot(\vec\nabla\times\vec{\mathbf{A}})=0$) **all conservative fields are irrotational** and **all fields admitting a vector potential are solenoidal**.

★ **Remark 1.70** (Existence of potentials)**.** Under some assumptions on the domain D of definition of $\vec{\mathbf{F}}$ the converse of the statements in the box is true. If D is $\mathbb{R}^3$ or a ball or a parallelepiped, for example, then all irrotational fields are conservatives and all solenoidal fields admit a vector potential.

More general assumptions on the shape of D can be made. If the domain D is "simply connected", which means, informally, that it has no holes or tunnels passing through it, then the converse of the first statement above is true: every irrotational field on D is conservative. If the domain D "has no holes in its interior", the converse of the second statement is true: every solenoidal field on D admits a vector potential $\vec{\mathbf{A}}$. E.g. a doughnut is not simply connected and the complement of a ball does not satisfies the second condition. We will prove the first statement for a simpler class of domains in Theorem 2.18.

★ **Remark 1.71** (Relation between potentials and domains)**.** The fact that a vector field $\vec{\mathbf{F}}$ defined in D is conservative may depend on the choice of the domain D. For instance, consider the irrotational (planar) field

$$\vec{\mathbf{F}}=\frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}}+\frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

(exercise: show that $\vec{\mathbf{F}}$ is irrotational); see also [5, VII Section 3]. In the half space $D=\{\vec{\mathbf{r}}\in\mathbb{R}^3 \text{ s.t. } x>0\}$, which is simply connected, it admits the scalar potential $\varphi=\arctan\frac{y}{x}$ (exercise: show that $\vec\nabla\varphi=\vec{\mathbf{F}}$ in D, using $\frac{\partial}{\partial t}\arctan t=\frac{1}{1+t^2}$). However, $\vec{\mathbf{F}}$ may be defined on the larger domain $E=\{\vec{\mathbf{r}}\in\mathbb{R}^3 \text{ s.t. } x^2+y^2>0\}$, i.e. the whole space without the z-axis; E is not simply connected. The potential φ can not be extended to the whole E and it is not possible to define any other scalar potential for $\vec{\mathbf{F}}$ in the whole of E; this will follow from some properties of conservative fields we will see in Section 2.1.3.

To summarise: $\vec{\mathbf{F}}$ is irrotational in E, conservative in the subset D but not conservative in the whole of E.

★ **Remark 1.72** (Helmholtz decomposition)**.** Any smooth vector field $\vec{\mathbf{F}}$ defined in a bounded domain D can be expressed as the sum of a conservative field and a field admitting a vector potential: $\vec{\mathbf{F}}=\vec\nabla\varphi+\vec\nabla\times\vec{\mathbf{A}}$ for some scalar and vector field φ and $\vec{\mathbf{A}}$, respectively. The potentials are not unique. This decomposition is termed **Helmholtz decomposition** after Hermann von Helmholtz (1821–1894).

Exercise 1.73. ▶ Show that the divergence of a conservative field equals the Laplacian of its scalar potential.

Exercise 1.74. ▶ Find all smooth scalar fields in $\mathbb{R}^3$ whose gradient admits a conservative vector potential.

Exercise 1.75. ▶ Assume $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are conservative fields with scalar potentials φ and ψ, respectively. Use the identities of Section 1.4 to show that the product $\vec{\mathbf{F}}\times\vec{\mathbf{G}}$ is solenoidal and find a vector potential for it.

1.6 Total derivatives of a curve

So far we have applied various differential operators to scalar and vector fields but we have not considered the derivatives of the curves introduced in Section 1.2.3. These are indeed easier to deal with than the derivative of fields, since they can be computed as the "usual derivatives" of the components (which are real-valued functions of a single variable) as learned in the first calculus class:

$$\frac{d\vec{\mathbf{a}}}{dt}(t):=\vec{\mathbf{a}}'(t):=\frac{da_1}{dt}(t)\hat{\boldsymbol{\imath}}+\frac{da_2}{dt}(t)\hat{\boldsymbol{\jmath}}+\frac{da_3}{dt}(t)\hat{\mathbf{k}}.$$

No partial derivatives are involved here. Note that we used the **total derivative** symbol $\frac{d}{dt}$ instead of the *partial derivative* one $\frac{\partial}{\partial t}$ because $\vec{\mathbf{a}}$ and its three components depend on one variable only (i.e. on t). The derivative of a curve is a curve.

Remark 1.76 (Difference between partial and total derivatives)**.** For $f(\vec{\mathbf{r}}) = f(x, y, z)$, the partial derivative $\frac{\partial f}{\partial x}$ is the derivative of f with respect to x when all the other variables are kept fixed. The total derivative $\frac{df}{dx}$ takes into account also the possible dependence of the other variables (y and z) on x. We will see some examples later on, e.g. in Section 1.7.

Given two curves $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and two scalar constants λ, μ, linearity and product rule hold:

$$\begin{aligned} \frac{d(\lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}})}{dt} &= \lambda\frac{d\vec{\mathbf{a}}}{dt} + \mu\frac{d\vec{\mathbf{b}}}{dt}, \\ (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})' = \frac{d(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}{dt} &= \frac{d\vec{\mathbf{a}}}{dt}\cdot\vec{\mathbf{b}} + \vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{b}}}{dt}, \\ \frac{d(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}{dt} &= \frac{d\vec{\mathbf{a}}}{dt}\times\vec{\mathbf{b}} + \vec{\mathbf{a}}\times\frac{d\vec{\mathbf{b}}}{dt}. \end{aligned} \tag{36}$$

Note that $t \mapsto (\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{b}}(t))$ is simply a real function of a real variable, thus we write its derivative as $(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})'$.

★ **Remark 1.77** (Arc length)**.** A curve $\vec{\mathbf{a}} : I \to \mathbb{R}^3$ is called smooth if its three components are smooth functions and $|\frac{d\vec{\mathbf{a}}}{dt}| \neq 0$ for all $t \in I$. If a curve $\vec{\mathbf{a}} : I \to \mathbb{R}^3$ with path Γ satisfies $|\frac{d\vec{\mathbf{a}}}{dt}| = 1$ for all $t \in I$, then t is called **arc length** of Γ and $\vec{\mathbf{a}}$ is called intrinsic parametrisation of Γ. Every smooth path has exactly two intrinsic parametrisations, proceeding in opposite directions from each other.

If you wonder why a curve to be smooth needs to satisfy $|\frac{d\vec{\mathbf{a}}}{dt}| \neq 0$, draw the curve $\vec{\mathbf{a}}(t) = t^3\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}$, whose components are smooth functions (polynomials). Recall that the path of a curve is its image, not its graph.

Exercise 1.78. ▶ Compute the total derivatives of the two parametrisations of the unit circle seen in Section 1.2.3: $\vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ with $t \in [0, 2\pi)$ and $\vec{\mathbf{b}}(t) = \cos 2\tau\,\hat{\boldsymbol{\imath}} + \sin 2\tau\,\hat{\boldsymbol{\jmath}}$ with $\tau \in [0, \pi)$.

The paths of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ coincide, is the same true for the paths of $\frac{d\vec{\mathbf{a}}}{dt}$ and $\frac{d\vec{\mathbf{b}}}{dt}$?

Can you find a curve $\vec{\mathbf{c}}$ with path different from $\vec{\mathbf{a}}$ whose derivative $\frac{d\vec{\mathbf{c}}}{dt}$ has the same path of $\frac{d\vec{\mathbf{a}}}{dt}$?

Exercise 1.79. ▶ Demonstrate the two product rules in equation (36) (for scalar and vector product) for the two parametrisations of the unit circle in Exercise 1.78: $\vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ and $\vec{\mathbf{b}}(t) = \cos 2t\,\hat{\boldsymbol{\imath}} + \sin 2t\,\hat{\boldsymbol{\jmath}}$.

Exercise 1.80. ▶ Prove the product rules for scalar and vector products of curves in equation (36).

Exercise 1.81. ▶ Given a vector $\vec{\mathbf{u}}$, find a curve $\vec{\mathbf{a}}(t)$ defined for $t \geq 0$ that is equal to its own total derivative and such that $\vec{\mathbf{a}}(0) = \vec{\mathbf{u}}$. Is this curve unique? What is the path of $\vec{\mathbf{a}}$?

1.7 Chain rule for fields and curves

Comparison with scalar calculus 1.82. The chain rule is the formula that allows to compute derivatives of the **composition** of functions, i.e. the application of a function to the output of a second one. Given two differentiable real functions $F, G : \mathbb{R} \to \mathbb{R}$, you know well the corresponding chain rule:

$$(G \circ F)'(t) = \big(G(F(t))\big)' = G'\big(F(t)\big)F'(t),$$

(recall footnote 4). See also page 11 of Handout 2 of last year's Calculus lecture notes. In this section we study some extensions of this formula to curves and fields. We have seen in (9) and (14) how to compute partial derivatives and the gradient of the composition of a real function with a scalar field ($\vec{\mathbf{r}} \mapsto f(\vec{\mathbf{r}}) \mapsto G(f(\vec{\mathbf{r}}))$).

Note the special structure of the chain rule formula above. The derivative of the composition equals the derivative of the "external" function G evaluated in the output of F times the derivative of the "internal" function F. All versions of the chain rule share the same structure (see (9), (14), (37) and (38)).

Imagine we have a scalar field f evaluated on a smooth curve $\vec{\mathbf{a}}$, i.e. $f(\vec{\mathbf{a}})$. This is a real function of real variable $t \mapsto (f \circ \vec{\mathbf{a}})(t) = f(\vec{\mathbf{a}}(t))$. Its derivative in t can be computed with the **chain rule**:

$$\boxed{\begin{aligned} \frac{d\big(f(\vec{\mathbf{a}})\big)}{dt}(t) &= \frac{\partial f}{\partial x}\big(\vec{\mathbf{a}}(t)\big)\frac{da_1}{dt}(t) + \frac{\partial f}{\partial y}\big(\vec{\mathbf{a}}(t)\big)\frac{da_2}{dt}(t) + \frac{\partial f}{\partial z}\big(\vec{\mathbf{a}}(t)\big)\frac{da_3}{dt}(t) \\ &= \vec{\nabla} f\big(\vec{\mathbf{a}}(t)\big)\cdot\frac{d\vec{\mathbf{a}}}{dt}(t). \end{aligned}} \tag{37}$$

When no ambiguity can arise, we will write $\frac{df}{dt}$ for $\frac{d(f(\vec{\mathbf{a}}))}{dt}(t)$ and $\frac{\partial f}{\partial x}$ for $\frac{\partial f}{\partial x}(\vec{\mathbf{a}}(t))$ omitting the argument.

Exercise 1.83. ► Consider the scalar field $f = xye^z$ and the curve $\vec{\mathbf{a}} = t\hat{\boldsymbol{\imath}} + t^3\hat{\boldsymbol{\jmath}}$ (see Figure 8). Compute the total derivative of $f(\vec{\mathbf{a}})$ using either the chain rule or explicitly computing the composition $f(\vec{\mathbf{a}})$.

We have seen different chain rules for the composition of real functions with scalar fields in (9) and (14) ($\vec{\mathbf{r}} \mapsto f(\vec{\mathbf{r}}) \mapsto G(f(\vec{\mathbf{r}}))$) and for the composition of scalar fields with curves in (37) ($t \mapsto \vec{\mathbf{a}}(t) \mapsto f(\vec{\mathbf{a}}(t))$). What about the partial derivatives of the composition of a scalar field with a vector field? Let g be a scalar field and $\vec{\mathbf{F}}$ a vector field, and define the scalar field $\phi(\vec{\mathbf{r}}) := g(\vec{\mathbf{F}}(\vec{\mathbf{r}})) = (g \circ \vec{\mathbf{F}})(\vec{\mathbf{r}})$. Then the partial derivatives and the gradient of ϕ can be computed as

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \vec{\nabla} g \cdot \frac{\partial \vec{\mathbf{F}}}{\partial x} = (\vec{\nabla} g)\big(\vec{\mathbf{F}}(\vec{\mathbf{r}})\big) \cdot \frac{\partial \vec{\mathbf{F}}}{\partial x}(\vec{\mathbf{r}}) = \vec{\nabla} g \cdot \Big(\frac{\partial F_1}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial F_2}{\partial x}\hat{\boldsymbol{\jmath}} + \frac{\partial F_3}{\partial x}\hat{\boldsymbol{k}}\Big),\\
\frac{\partial\phi}{\partial y} &= \vec{\nabla} g \cdot \frac{\partial \vec{\mathbf{F}}}{\partial y} = (\vec{\nabla} g)\big(\vec{\mathbf{F}}(\vec{\mathbf{r}})\big) \cdot \frac{\partial \vec{\mathbf{F}}}{\partial y}(\vec{\mathbf{r}}) = \vec{\nabla} g \cdot \Big(\frac{\partial F_1}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial F_2}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial F_3}{\partial y}\hat{\boldsymbol{k}}\Big), \qquad \boxed{\vec{\nabla}\phi = (J\vec{\mathbf{F}})^T\,(\vec{\nabla} g).} \qquad (38)\\
\frac{\partial\phi}{\partial x} &= \vec{\nabla} g \cdot \frac{\partial \vec{\mathbf{F}}}{\partial z} = (\vec{\nabla} g)\big(\vec{\mathbf{F}}(\vec{\mathbf{r}})\big) \cdot \frac{\partial \vec{\mathbf{F}}}{\partial z}(\vec{\mathbf{r}}) = \vec{\nabla} g \cdot \Big(\frac{\partial F_1}{\partial z}\hat{\boldsymbol{\imath}} + \frac{\partial F_2}{\partial z}\hat{\boldsymbol{\jmath}} + \frac{\partial F_3}{\partial z}\hat{\boldsymbol{k}}\Big),
\end{aligned}$$

In (38), $(J\vec{\mathbf{F}})^T(\vec{\nabla} g)$ denotes the matrix–vector multiplication between the transpose of the Jacobian matrix of $\vec{\mathbf{F}}$ and the gradient of g. Note that in this formula $\vec{\nabla} g$ is always evaluated in $\vec{\mathbf{F}}(\vec{\mathbf{r}})$, even when we do not write it explicitly. Similar formulas hold for two-dimensional fields.

★ **Remark 1.84.** All instances of the chain rule (9), (14), (37) and (38) are special cases of a more general result for functions between Euclidean spaces of arbitrary dimensions, see e.g. [1, pages 708–709] or [5, XVI, Section 3].

Example 1.85 (Derivative of a field constrained to a surface)**.** In some problems, the coordinates of $\vec{\mathbf{r}}$ depend on each other. In this case the partial derivatives of a field $g(\vec{\mathbf{r}})$ do not give the desired rate of change of g. For example, imagine that the point $\vec{\mathbf{r}}$ is constrained to the graph surface of $h : \mathbb{R}^2 \to \mathbb{R}$

$$S_h = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ \text{s.t. } \vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + h(x,y)\hat{\boldsymbol{k}}\big\}.$$

In this case, the component z depends on x and y through the two-dimensional field h. Thus also the value of g on S_h only depends on x and y. We define $\widetilde{g}(x,y) = g(x,y,h(x,y))$ to be the evaluation of g on S_h; $\widetilde{g}$ is a field depending on two variables only. We can compute the derivative of g along x or y using the chain rule:

$$\begin{aligned}
\frac{\partial \widetilde{g}}{\partial x}(x,y) &= \frac{\partial\Big(g\big(x,y,h(x,y)\big)\Big)}{\partial x} = \frac{\partial g}{\partial x}(x,y,h(x,y)) + \Big(\frac{\partial g}{\partial z}(x,y,h(x,y))\Big)\Big(\frac{\partial h}{\partial x}(x,y)\Big),\\
\frac{\partial \widetilde{g}}{\partial y}(x,y) &= \frac{\partial\Big(g\big(x,y,h(x,y)\big)\Big)}{\partial y} = \frac{\partial g}{\partial y}(x,y,h(x,y)) + \Big(\frac{\partial g}{\partial z}(x,y,h(x,y))\Big)\Big(\frac{\partial h}{\partial y}(x,y)\Big).
\end{aligned}$$

Here $\frac{\partial g}{\partial x}(x,y,h(x,y))$ (similarly, $\frac{\partial g}{\partial y}(x,y,h(x,y))$ and $\frac{\partial g}{\partial z}(x,y,h(x,y))$) indicates the derivative of g with respect to its first (second and third, respectively) scalar variable; the field is evaluated in $(x,y,h(x,y))$ *after* the derivative has been taken. This formula can be seen as a special case of the chain rule (38) for the vector field $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + h(x,y)\hat{\boldsymbol{k}}$ (independent of z), whose Jacobian reads

$$J\vec{\mathbf{F}} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & 0 \end{pmatrix}.$$

Let us consider a concrete example. Let $h(x,y) = x^2 - y^2$, so that the surface $S_h = \{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (x^2 - y^2)\hat{\boldsymbol{k}}\}$ is that depicted in Figure 14 (compare also with the right plot in Figure 5). Let $g = x\sin z$. Then, from the formulas above, we compute the derivatives of g with respect to x and y:

$$\begin{aligned}
\frac{\partial\Big(g\big(x,y,h(x,y)\big)\Big)}{\partial x} &= \sin z + (x\cos z)(2x) = \sin z + 2x^2\cos z = \sin(x^2-y^2) + 2x^2\cos(x^2-y^2),\\
\frac{\partial\Big(g\big(x,y,h(x,y)\big)\Big)}{\partial y} &= 0 + (x\cos z)(-2y) = -2xy\cos z = -2xy\cos(x^2-y^2).
\end{aligned}$$

Note that even if g is independent of the y-coordinate, its derivative along the surface defined by h (which does depend on y) is non-zero. We could compute these derivatives also by first computing $\widetilde{g}(x,y) = g(x,y,h(x,y)) = x\sin(x^2-y^2)$ and then deriving.

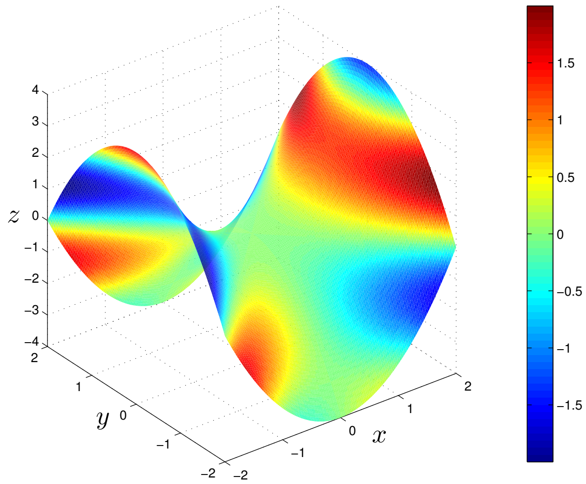


Figure 14: *The surface* $S_h = \{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (x^2 - y^2)\hat{\mathbf{k}}\}$ *described in Example 1.85. The colours represent the values of the field* $g = x\sin z$*, which can be written as* $g = x\sin(x^2 - y^2)$ *when restricted to* S_h*. Note that* g *varies in the* y *coordinate if we move along* S_h *keeping* x *fixed, while it is constant in that direction if we move in free space (see the .pdf file of these notes for the coloured version).*

Example 1.86 (Derivative of a field constrained to a curve)**.** Compute the total derivative

$$\frac{d}{dz} g\big(a(z), b(z), z\big)$$

where $g = \mathrm{e}^{xz} y^4$, $a(z) = \cos z$, $b(z) = \sin z$.

This is again a derivative along the curve $a(t)\hat{\boldsymbol{\imath}} + b(t)\hat{\boldsymbol{\jmath}} + t\hat{\mathbf{k}}$ (compare with the centre plot of Figure 8) with t and z identified with each other, so formula (37) applies. We combine the partial derivatives of g and the total derivatives of a and b:

$$\begin{aligned}\frac{d}{dz} g\big(a(z), b(z), z\big) = \frac{\partial g}{\partial x}\frac{da}{dz} + \frac{\partial g}{\partial y}\frac{db}{dz} + \frac{\partial g}{\partial z} &= -z\mathrm{e}^{xz} y^4 \sin z + 4\mathrm{e}^{xz} y^3 \cos z + x\mathrm{e}^{xz} y^4 \\ &= \mathrm{e}^{z\cos z}\big(- z\sin^5 z + 4\sin^3 z\cos z + \cos z\sin^4 z\big).\end{aligned}$$

Exercise 1.87. ▶ Given the curve $\vec{\mathbf{a}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}}$, the scalar field $g = x^2 - y^2$ and the vector field $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} - y\hat{\mathbf{k}}$, compute the partial derivatives of $g \circ \vec{\mathbf{F}}$ and the total derivatives of $g \circ \vec{\mathbf{a}}$ and of $g \circ \vec{\mathbf{F}} \circ \vec{\mathbf{a}}$. Compute them twice: (1) by calculating the compositions and then deriving them, and (2) by using the vector formulas introduced in this section.

Remark 1.88. In Section 1.3.2 we defined the directional derivative $\frac{\partial f}{\partial \hat{\mathbf{u}}}$ of a scalar field f in the direction of the unit vector $\hat{\mathbf{u}}$ as $\frac{\partial f}{\partial \hat{\mathbf{u}}} = \vec{\nabla} f \cdot \hat{\mathbf{u}}$. Using the chain rule (37) it is immediate to verify that it corresponds to the rate of increase of f in the direction $\hat{\mathbf{u}}$:

$$\frac{\partial f}{\partial \hat{\mathbf{u}}}(\vec{\mathbf{r}}) = \vec{\nabla} f(\vec{\mathbf{r}}) \cdot \hat{\mathbf{u}} = \frac{d\big(f(\vec{\mathbf{r}} + t\hat{\mathbf{u}})\big)}{dt}\Big|_{t=0}.$$

From Cauchy–Schwarz inequality (see Proposition 3.5 in last year's Linear Algebra lecture notes, handout 2, or Proposition 13.4 in two years ago's notes, handout 4), we have $|\frac{\partial f}{\partial \hat{\mathbf{u}}}| = |\vec{\nabla} f \cdot \hat{\mathbf{u}}| \le |\vec{\nabla} f|\,|\hat{\mathbf{u}}| = |\vec{\nabla} f|$. This means that the magnitude of the gradient is at least as large as the rate of increase of f in any other direction. Since $|\hat{\mathbf{u}} \cdot \hat{\mathbf{w}}| = |\hat{\mathbf{u}}||\hat{\mathbf{w}}|$ if and only if $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are parallel, the direction of the gradient is the *unique* direction of maximal increase for f at $\vec{\mathbf{r}}$ (when $\vec{\nabla} f(\vec{\mathbf{r}}) \neq \vec{\mathbf{0}}$). We have proved part 5 of Proposition 1.33.

★ **Remark 1.89.** We can use the chain rule to prove part 4 of Proposition 1.33. Assume f is a smooth scalar field defined in an open set D, fix $\vec{\mathbf{r}}_0 \in D$ and denote the corresponding level set of f as $L = \{\vec{\mathbf{r}} \in D$ such that $f(\vec{\mathbf{r}}) = f(\vec{\mathbf{r}}_0)\}$. Consider a smooth curve $\vec{\mathbf{a}} : I \to L$ with $\vec{\mathbf{a}}(t_0) = \vec{\mathbf{r}}_0$ for some $t_0 \in I$, where I is an interval. Then, by the definition of the level set, the real function $t \mapsto f(\vec{\mathbf{a}}(t))$ is constant, so its derivative $\frac{d(f(\vec{\mathbf{a}}))}{dt}$ vanishes. By the chain rule (37), $\vec{\nabla} f(\vec{\mathbf{r}}_0) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t_0) = \frac{d(f(\vec{\mathbf{a}}))}{dt}(t_0) = 0$. This means that the gradient of f in $\vec{\mathbf{r}}_0$ is perpendicular to the tangent vectors $\frac{d\vec{\mathbf{a}}}{dt}$ to all smooth curves $\vec{\mathbf{a}}$ passing through $\vec{\mathbf{r}}_0$ and lying in the level set L. This is the same as saying that the gradient is perpendicular to the level set itself (note that so far we have not defined perpendicularity between vectors and sets).

★ **Remark 1.90.** (Differentials) As in one-dimensional calculus, we can consider "differentials", infinitesimal changes in the considered quantities. The differential

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

of the scalar field $f(\vec{\mathbf{r}})$ represents the change in $f(\vec{\mathbf{r}})$ due to an infinitesimal change in $\vec{\mathbf{r}}$. If $\vec{\mathbf{r}}$ depends on t as $\vec{\mathbf{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$, we can express its vector differential as

$$\mathrm{d}\vec{\mathbf{r}} = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} + \mathrm{d}z\hat{\boldsymbol{k}} = \Big(\frac{dx}{dt}\,\mathrm{d}t\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{dy}{dt}\,\mathrm{d}t\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{dz}{dt}\,\mathrm{d}t\Big)\hat{\boldsymbol{k}} = \Big(\frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}}\Big)\,\mathrm{d}t.$$

Thus, if f is evaluated in $\vec{\mathbf{r}}(t)$, its differential $\mathrm{d}f$ can be expressed in terms of $\mathrm{d}t$:

$$\mathrm{d}f = \vec{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} = \frac{\partial f}{\partial x}\frac{dx}{dt}\,\mathrm{d}t + \frac{\partial f}{\partial y}\frac{dy}{dt}\,\mathrm{d}t + \frac{\partial f}{\partial z}\frac{dz}{dt}\,\mathrm{d}t.$$

Differentials can be used to estimate (compute approximately) the values of a field. Consider for example the field $f = \sqrt{x^2 + y^3}$. We want to estimate its value at the point $\vec{\mathbf{r}} = 1.03\hat{\boldsymbol{\imath}} + 1.99\hat{\boldsymbol{\jmath}}$ knowing that f and its gradient in the "nearby" point $\vec{\mathbf{r}}_0 = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ take values

$$f(\vec{\mathbf{r}}_0) = \sqrt{1+8} = 3, \qquad \vec{\nabla} f(\vec{\mathbf{r}}_0) = \frac{2x_0\hat{\boldsymbol{\imath}} + 3y_0^2\hat{\boldsymbol{\jmath}}}{2\sqrt{x_0^2 + y_0^3}} = \frac{1}{3}\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}.$$

We approximate the desired value as

$$f(\vec{\mathbf{r}}) \approx f(\vec{\mathbf{r}}_0) + \vec{\delta} \cdot \vec{\nabla} f(\vec{\mathbf{r}}_0) = 3 + (0.03\hat{\boldsymbol{\imath}} - 0.01\hat{\boldsymbol{\jmath}}) \cdot \Big(\frac{1}{3}\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}\Big) = 2.99,$$

where $\vec{\delta} = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0$ is the discrete increment. (The exact value is $f(\vec{\mathbf{r}}) = 2.9902339\ldots$)

1.8 Review exercises for Section 1

Consider the following scalar and vector fields:

$$\begin{aligned}
f &= xy^2z^3, & \vec{\mathbf{F}} &= (y+z)\hat{\boldsymbol{\imath}} + (x+z)\hat{\boldsymbol{\jmath}} + (x+y)\hat{\boldsymbol{k}},\\
g &= \cos x + \sin(2y + x), & \vec{\mathbf{G}} &= \mathrm{e}^{-z^2}\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}},\\
h &= \mathrm{e}^x \cos y, & \vec{\mathbf{H}} &= \vec{\mathbf{r}} \times \hat{\boldsymbol{\imath}},\\
\ell &= x|\vec{\mathbf{r}}|^2 - y^3, & \vec{\mathbf{L}} &= |\vec{\mathbf{r}}|^2\,\vec{\mathbf{r}},\\
m &= x^y \qquad (x > 0), & \vec{\mathbf{M}} &= yz^2(yz\hat{\boldsymbol{\imath}} + 2xz\hat{\boldsymbol{\jmath}} + 3xy\hat{\boldsymbol{k}}).
\end{aligned} \tag{39}$$

Consider also the three following curves, defined in the interval $-1 < t < 1$:

$$\vec{\mathbf{a}} = (t^3 - t)\hat{\boldsymbol{\imath}} + (1 - t^2)\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{b}} = t^3\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}, \qquad \vec{\mathbf{c}} = \mathrm{e}^t\cos(2\pi t)\hat{\boldsymbol{\imath}} + \mathrm{e}^t\sin(2\pi t)\hat{\boldsymbol{\jmath}}.$$

Answer the questions, trying to be smart and to avoid brute force computations whenever possible.

1. Compute gradient, Hessian and Laplacian of the five scalar fields.
2. Compute Jacobian, divergence, curl and vector Laplacian of the five vector fields.
3. Which of the fields are solenoidal and which are irrotational? Which are harmonic?
4. Compute a scalar and a vector potential for $\vec{\mathbf{F}}$, a vector potential for $\vec{\mathbf{H}}$ (can (31) help you?) and a scalar potential for $\vec{\mathbf{M}}$. Can you guess a scalar potential for $\vec{\mathbf{L}}$?
5. Show that $\vec{\mathbf{G}}$ does not admit neither a scalar nor a vector potential.
6. Show that $\vec{\mathbf{H}}(\vec{\mathbf{r}})$ and $\vec{\mathbf{L}}(\vec{\mathbf{r}})$ are orthogonal to each other at every point $\vec{\mathbf{r}} \in \mathbb{R}^3$.
7. Try to graphically represent the fields in (39). E.g. you can draw a qualitative plot like those of Section 1.2.2 for $\vec{\mathbf{G}}$ on the plane $y = 0$, for $\vec{\mathbf{F}}$, $\vec{\mathbf{H}}$ and $\vec{\mathbf{L}}$ on the plane $z = 0$.
8. Demonstrate identities (25) and (27) for $\vec{\mathbf{G}}$; (26) for f; (28) for f and h; (33) for $\vec{\mathbf{G}}$ and $\vec{\mathbf{H}}$; (29) and (31) for h and $\vec{\mathbf{H}}$. (You can also demonstrate other identities of Propositions 1.52 and 1.55 for the various fields in (39).)
9. Compute the (total) derivatives of the curves (i.e. $\frac{d\vec{\mathbf{a}}}{dt}$, $\frac{d\vec{\mathbf{b}}}{dt}$ and $\frac{d\vec{\mathbf{c}}}{dt}$) and try to draw them.
10. Compute the following total derivatives of the scalar fields along the curves: $\frac{dh(\vec{\mathbf{a}})}{dt}$, $\frac{df(\vec{\mathbf{b}})}{dt}$ and $\frac{d\ell(\vec{\mathbf{c}})}{dt}$. (You can either use the chain rule (37) or first compute the composition.)

2 Vector integration

So far we have extended the concept of differentiation from real functions to vectors and fields (*differential* vector calculus); in this section we consider the extension of the idea of integration for the same objects (*integral* vector calculus).

In vector calculus we do not consider "indefinite integrals", so we usually do not understand integration as "inverse of derivation".[11] On the contrary, every integral defined in this section must be understood as a cumulative measure of a quantity (named **integrand**, which is a scalar or vector field) distributed on a geometric object, the **domain of integration**. This can be a one-, two- or three-dimensional subset of $\mathbb{R}^3$, namely the path of a curve, a surface or a domain, respectively. This is reflected in the notation: the number of integral signs used represents the dimension of the domain of integration, which is written at subscript. Integrals along a path Γ use the symbol $\int_\Gamma$ and are called "line integrals", those over a surface S are denoted by $\iint_S$ and called "surface integrals", those on a domain D are written as $\iiint_D$ and called "volume integrals".

A good way to have a physical understanding of the integrals of (positive) scalar fields, is to think at the domain as a physical object (in the cases of paths or surfaces, as wires or plates of infinitesimal diameter or thickness, respectively) and at the scalar field as the **density** of mass of the object, which can vary in different points if the material is heterogeneous. The integral is then the total mass of the object. If the field changes sign, it can be thought as electric **charge** density, and its integral as the total charge. The integrals of vector fields, representing e.g. the work of a force or the flux of a substance through an aperture, are slightly more complicated and involve concepts such as the orientation of a path or a surface, and will be described in the following sections.

All integrals of scalar fields will have to satisfy some fundamental properties[12]:

1. **Linearity**: the integral of $f + g$ equals the sum of the integral of f and the integral of g on the same domain; the integral of λf equals λ times the integral of f (here f and g are scalar fields and $\lambda \in \mathbb{R}$).
2. **Additivity**: the integral over the union of two disjoint (i.e. with empty intersection) domains of integration (may be two paths, two surfaces or two domains) equals the sum of the two integrals of the same field over each domain.
3. The integral of the **constant** field $f(\vec{\mathbf{r}}) = 1$ equals the **measure** of the domain of integration: the length of a path, the area of a surface, the volume of a domain.

(40)

Items 1.–2.–3. above extend the following well-known facts about integrals of functions of a real variable:

$$\begin{aligned}
&1. \qquad \int_a^b (\lambda f + \mu g)\,\mathrm{d}t = \lambda \int_a^b f\,\mathrm{d}t + \mu \int_a^b g\,\mathrm{d}t,\\
&2. \qquad \int_a^c f\,\mathrm{d}t = \int_a^b f\,\mathrm{d}t + \int_b^c f\,\mathrm{d}t,\\
&3. \qquad \int_a^b 1\,\mathrm{d}t = b - a \qquad\qquad \text{for all } a < b < c,\ \lambda,\mu \in \mathbb{R},\ f,g : [a,c] \to \mathbb{R}.
\end{aligned}$$

Other natural properties of the integrals follow: the integral of positive scalar field is a positive number; a dilation of the domain by a factor $\lambda > 0$ corresponds to a multiplication of the integral by a factor λ if the domain is a curve, λ^2 if it is a surface, and λ^3 if it is a volume.

2.1 Line integrals

Here we study how to integrate scalar and vector fields along curves. See also Sections 15.3 and 15.4 of [1], which contain several useful exercises.

2.1.1 Line integrals of scalar fields

Consider a curve $\vec{\mathbf{a}} : [t_I, t_F] \to \mathbb{R}^3$ (where $[t_I, t_F] \subset \mathbb{R}$ is an interval) and a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$. We recall that $\Gamma = \vec{\mathbf{a}}([t_I, t_F]) \subset \mathbb{R}^3$ is the path of the curve $\vec{\mathbf{a}}$. We want to define the integral of f over Γ, which will be denoted by $\int_\Gamma f\,\mathrm{d}s$, satisfying the conditions 1.–2.–3. stipulated in (40).

[11] In Sections 2.1.3 and 3, however, we study some integrals of differential operators, generalising the fundamental theorem of calculus, but we do not use this idea to define the integrals themselves.

[12] We stress again a fundamental concept. All integrals requires two inputs: an integrand (a function or field), and a domain (a set). Linearity concerns integrals of different integrand on the same domain, while additivity concerns integrals of an integrand over different domains.

Intuitively, what does this integral represent? In the special case of a planar curve, i.e. $\vec{\mathbf{a}}(t) = a_1(t)\hat{\boldsymbol{\imath}} + a_2(t)\hat{\boldsymbol{\jmath}}$, and a two-dimensional positive scalar field $f(x, y) > 0$ we can have a simple geometric representation of the integral. We know well that the one-dimensional integral of a function of a real variable represents the (signed) area between its graph and its projection on the x-axis, which is the domain of integration (see the left plot in Figure 15). Similarly, the line integral will represent the area of the (curvilinear!) surface between the path of the curve in the xy-plane (the domain of integration) and the graph of f, represented as a surface as we have seen in Remark 1.21. The example depicted in Figure 15 is probably more helpful than the description. Note that two surfaces are involved here: the graph of the scalar field f (the upper mesh in the plot), and the vertical surface between the graph and xy-plane (the grey one in the plot).

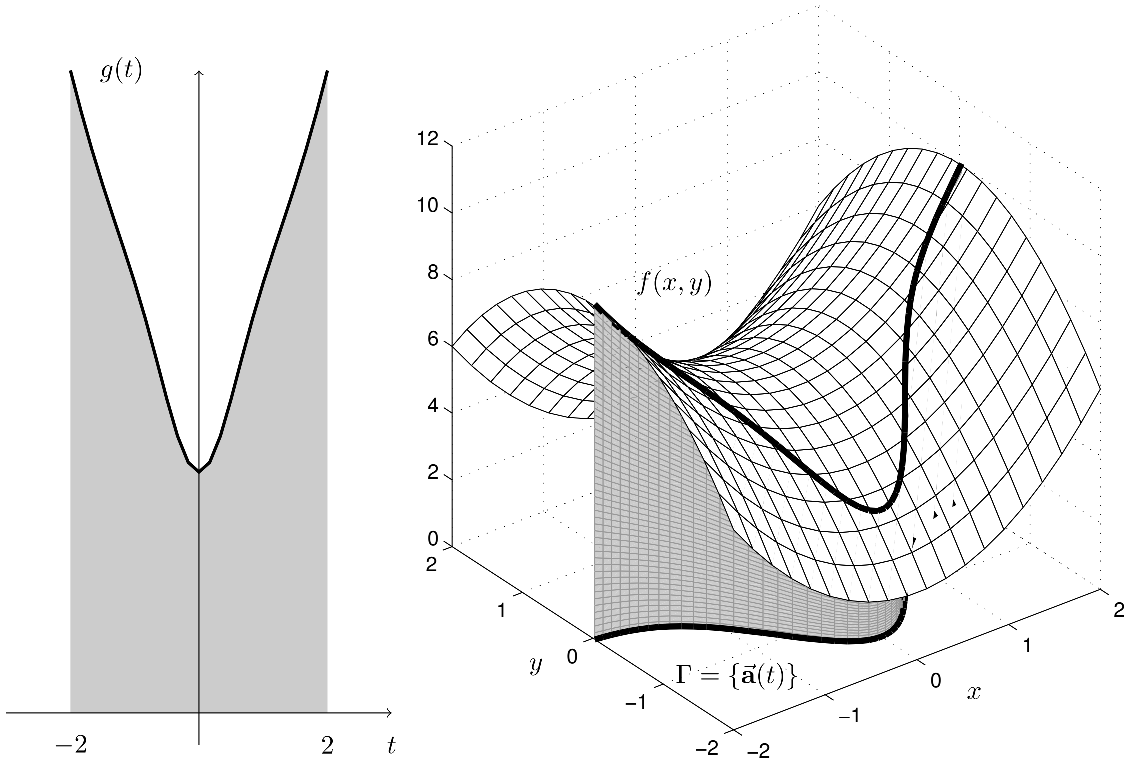


Figure 15: ***Left:*** *the integral* $\int_{-2}^{2} g(t)\,\mathrm{d}t$ *of the real function* $g(t) = 6 + t^2 - \frac{9}{4}(\mathrm{e}^{-x^2})^2$ *is the area of the shaded region between the graph of the function and the x-axis.*
Right: *the path* Γ *of the planar curve* $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} - \frac{3}{2}\mathrm{e}^{-t^2}\hat{\boldsymbol{\jmath}}$ *(lower thick plot) and the surface plot of the scalar field* $f = 6 + x^2 - y^2$ *(the mesh above). The line integral* $\int_\Gamma f\,\mathrm{d}s$ *of* f *over* Γ *is the area of the vertical surface lying between* Γ *itself and its projection* $\vec{\mathbf{a}}(t) + f(\vec{\mathbf{a}}(t))\hat{\boldsymbol{k}}$ *on the graph of* f*, i.e. the upper thick curve. (The shaded area in the left plot is equal to the projection on the xz-plane of the shaded area in the right plot, since* $g(t) = f(\vec{\mathbf{a}}(t))$ *and* $a_1(t) = t$*, but the integrals are different since the "stretching" given by the shape of* Γ *is not taken into account by* g*.)*

If the field is not always positive but changes sign in its domain, as in the scalar case, the integral is the difference between the area of the part of the surface above the xy-plane and the part below it.

In three dimensions (i.e. if all three components of $\vec{\mathbf{a}}(t)$ vary in t) it is not possible to have an easy geometric picture of the integral, since it would require a fourth dimension. In this case it is preferable to think at f as some pointwise mass (or charge) density along a wire and the line integral as its total mass (or charge).

How can we write this integral in a computable formula? We first write the formula and we justify it in Remark 2.1 below. The **line integral** of f along Γ is

$$\boxed{\int_\Gamma f\,\mathrm{d}s = \int_{t_I}^{t_F} f\big(\vec{\mathbf{a}}(t)\big)\left|\frac{d\vec{\mathbf{a}}}{dt}\right|\mathrm{d}t} = \int_{t_I}^{t_F} f\big(\vec{\mathbf{a}}(t)\big)\sqrt{\Big(\frac{da_1}{dt}\Big)^2 + \Big(\frac{da_2}{dt}\Big)^2 + \Big(\frac{da_3}{dt}\Big)^2}\,\mathrm{d}t, \qquad (41)$$

where $\mathrm{d}s$ denotes the infinitesimal element of the path Γ. The right-hand side of (41) is an ordinary one-dimensional integral over the interval $[t_I, t_F]$, involving the magnitude of the total derivative $\frac{d\vec{\mathbf{a}}}{dt}$ of the curve $\vec{\mathbf{a}}$. If $\vec{\mathbf{a}}$ is a loop (i.e. $\vec{\mathbf{a}}(t_I) = \vec{\mathbf{a}}(t_F)$), then the line integral along Γ is called **contour integral** and denoted by the symbol $\oint_\Gamma f\,\mathrm{d}s$.

⋆ **Remark 2.1** (Justification of the line integral formula (41))**.** To justify formula (41) we recall Riemann's sums for one-dimensional integrals (in a very simple case), which you might have already seen in other courses. Consider a "well-behaved" function $g : [0,1] \to \mathbb{R}$. Given a natural number $N \in \mathbb{N}$, we can split the domain $[0,1]$ in N subintervals $[0, \frac{1}{N}], [\frac{1}{N}, \frac{2}{N}], \dots, [\frac{N-1}{N}, \frac{N}{N}]$ with equal lengths (see Figure 16). In each small subinterval we approximate $g(t)$ with the constant function taking the value of g at the right extreme of the subinterval, i.e. $g(j/N)$. If the intervals are small enough, or equivalently if N is large enough, the integral of g on the subinterval is approximated by the integral of this constant function (which is simply $g(j/N)/N$, since the length of the interval is $j/N - (j-1)/N = 1/N$). Summing over the N subintervals, we obtain an approximation of the integral of g over $[0,1]$, (see Figure 16):

$$\int_0^1 g(t)\,\mathrm{d}t = \sum_{j=1}^N \int_{(j-1)/N}^{j/N} g(t)\,\mathrm{d}t \approx \sum_{j=1}^N g\Big(\frac{j}{N}\Big)\Big|\frac{j}{N} - \frac{j-1}{N}\Big| = \frac{1}{N}\sum_{j=1}^N g\Big(\frac{j}{N}\Big), \tag{42}$$

(here the symbol "≈" means "is approximately equal to"). If the integrand g is sufficiently "well-behaved" (for example it is continuous in the closed interval $[0,1]$), the approximation will actually converge to the desired integral, i.e.

$$\int_0^1 g(t)\,\mathrm{d}t = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N g\Big(\frac{j}{N}\Big).$$

This is the basic idea behind Riemann sums (although more general partitions of the interval and points of evaluation may be taken).

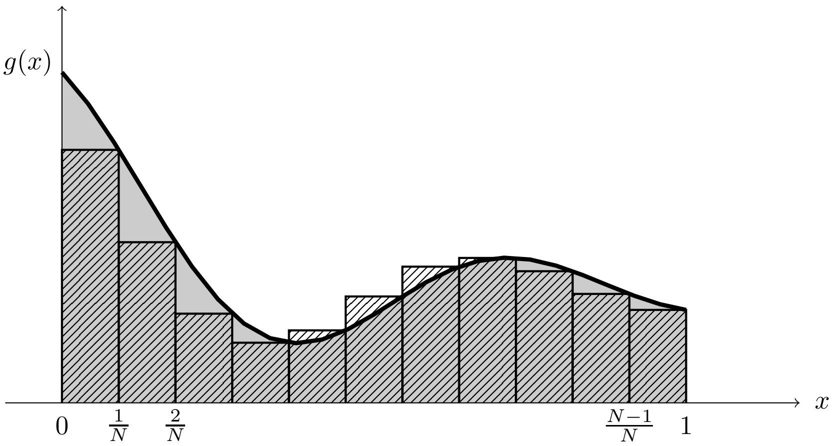


Figure 16: *The approximation of the integral of a real variable function $g : [0,1] \to \mathbb{R}$ by Riemann sums. The area of the grey part bounded by the plot of the integrand represents the integral of g, the area covered with diagonal lines is its Riemann sum approximation as in equation (42).*

How can we write Riemann sums for the line integral of a scalar field? Consider a path Γ parametrised by the curve $\vec{\mathbf{a}} : [0,1] \to \mathbb{R}^3$ and a scalar field f (both sufficiently smooth). Fix $N \in \mathbb{N}$; for all integers $1 \le j \le N$ call Γ_j the sub-path parametrised by the curve $\vec{\mathbf{a}}$ restricted to the subinterval $[(j-1)/N, j/N]$. We repeat the same procedure used for scalar functions: we use the additivity of the integral to split it in a sum over the sub-paths (step *(i)* in the equation below) and we approximate the integrals with those of the constant $f(\vec{\mathbf{a}}(j/n))$, *(ii)*. To compute the integrals of these constants we need to know the lengths of each sub-path Γ_j *(iii)* (compare with the assumptions made in (40)). We approximate again the length of Γ_j with the distance between its endpoints $\vec{\mathbf{a}}((j-1)/N)$ and $\vec{\mathbf{a}}(j/N)$, *(iv)*. We multiply and divide by $1/N$ *(v)* and we note that the ratio obtained is a difference quotient approximating the total derivative of $\vec{\mathbf{a}}(t)$ at $t = j/N$ *(vi)*. Using the Riemann sum (42) for real functions, we see that the formula obtained is an approximation of the integral in (41) and we conclude *(vii)–(viii)*. In formulas:

$$\begin{aligned}\int_\Gamma f\,\mathrm{d}s &\overset{\text{(i)}}{=} \sum_{j=1}^N \int_{\Gamma_j} f\,\mathrm{d}s\\ &\overset{\text{(ii)}}{\approx} \sum_{j=1}^N \int_{\Gamma_j} f\Big(\vec{\mathbf{a}}\Big(\frac{j}{N}\Big)\Big)\,\mathrm{d}s\\ &\overset{\text{(iii)}}{=} \sum_{j=1}^N f\Big(\vec{\mathbf{a}}\Big(\frac{j}{N}\Big)\Big)\,\mathrm{Length}(\Gamma_j)\end{aligned}$$

$$\stackrel{\text{(iv)}}{\approx} \sum_{j=1}^{N} f\Big(\vec{\mathbf{a}}\Big(\frac{j}{N}\Big)\Big)\, \Big|\vec{\mathbf{a}}\Big(\frac{j}{N}\Big) - \vec{\mathbf{a}}\Big(\frac{j-1}{N}\Big)\Big|$$

$$\stackrel{\text{(v)}}{=} \frac{1}{N} \sum_{j=1}^{N} f\Big(\vec{\mathbf{a}}\Big(\frac{j}{N}\Big)\Big)\, \frac{\Big|\vec{\mathbf{a}}\Big(\frac{j}{N}\Big) - \vec{\mathbf{a}}\Big(\frac{j-1}{N}\Big)\Big|}{\frac{j}{N} - \frac{j-1}{N}}$$

$$\stackrel{\text{(vi)}}{\approx} \frac{1}{N} \sum_{j=1}^{N} f\Big(\vec{\mathbf{a}}\Big(\frac{j}{N}\Big)\Big)\, \Big|\frac{d\vec{\mathbf{a}}}{dt}\Big(\frac{j}{N}\Big)\Big|$$

$$\stackrel{\text{(vii)}}{\approx} \sum_{j=1}^{N} \int_{(j-1)/N}^{j/N} f\big(\vec{\mathbf{a}}(t)\big)\Big|\frac{d\vec{\mathbf{a}}}{dt}(t)\Big|\,\mathrm{d}t$$

$$\stackrel{\text{(viii)}}{=} \int_0^1 f\big(\vec{\mathbf{a}}(t)\big)\Big|\frac{d\vec{\mathbf{a}}}{dt}(t)\Big|\,\mathrm{d}t.$$

We have derived (in a hand-waving way) formula (41) for the line integral of a scalar field, using only the Riemann sums for functions of real variable and the fundamental properties of the integrals stated in (40).

In this class we take the line integral formula (41) (and all the similar formulas for double, triple, surface and flux integrals in the rest of Section 2) as a definition; however, as we have seen, these can be (rigorously) derived from the general theory of Riemann integration.

Example 2.2 (Length of a path)**.** Integrating the constant field $f = 1$ (cf. (40)), we see that the length of a path Γ is

$$\boxed{\operatorname{length}(\Gamma) = \int_\Gamma \mathrm{d}s = \int_{t_I}^{t_F} \Big|\frac{d\vec{\mathbf{a}}}{dt}\Big|\,\mathrm{d}t.} \tag{43}$$

The line integral in (41) measures a quantity f distributed along Γ, thus it must be **independent of the special parametrisation** $\vec{\mathbf{a}}$. This is guaranteed by the factor $|\frac{d\vec{\mathbf{a}}}{dt}|$, which takes into account the "speed" of travel along Γ (if we think at the parameter t as time). Similarly, the travel direction along Γ (the curve orientation) does not affect the value of the line integral. In other words, f and Γ uniquely define the value of the integral in (41), independently of the choice of $\vec{\mathbf{a}}$ we use to compute it. This makes the notation $\int_\Gamma f\,\mathrm{d}s$ in (41) well-posed, without the need of specifying $\vec{\mathbf{a}}$. Example 2.3 demonstrates the invariance of the line integral with respect to the parametrisation.

Example 2.3 (Independence of parametrisation)**.** Consider the scalar field $f = y$ and the unit half circle C centred at the origin and located in the half plane $\{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ y \geq 0\}$, which can be defined by either of the two parametrisations

$$\vec{\mathbf{a}} : [0,\pi] \to \mathbb{R}^3, \quad \vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{b}} : [-1,1] \to \mathbb{R}^3, \quad \vec{\mathbf{b}}(\tau) = \tau\hat{\boldsymbol{\imath}} + \sqrt{1-\tau^2}\,\hat{\boldsymbol{\jmath}};$$

(see Figure 17). Verify that

$$\int_0^\pi f\big(\vec{\mathbf{a}}(t)\big)\Big|\frac{d\vec{\mathbf{a}}}{dt}\Big|\,\mathrm{d}t = \int_{-1}^1 f\big(\vec{\mathbf{b}}(\tau)\big)\Big|\frac{d\vec{\mathbf{b}}}{d\tau}\Big|\,\mathrm{d}\tau.$$

We start by computing the total derivatives of the two parametrisations

$$\frac{d\vec{\mathbf{a}}}{dt} = -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{b}}}{d\tau} = \hat{\boldsymbol{\imath}} + \frac{-\tau}{\sqrt{1-\tau^2}}\hat{\boldsymbol{\jmath}},$$

and their magnitudes

$$\Big|\frac{d\vec{\mathbf{a}}}{dt}\Big| = \sqrt{\sin^2 t + \cos^2 t} = 1, \qquad \Big|\frac{d\vec{\mathbf{b}}}{d\tau}\Big| = \sqrt{1 + \frac{\tau^2}{1-\tau^2}} = \frac{1}{\sqrt{1-\tau^2}}.$$

The values of the field f along the two curves are $f(\vec{\mathbf{a}}(t)) = \sin t$ and $f(\vec{\mathbf{b}}(\tau)) = \sqrt{1-\tau^2}$. We compute the two integrals and verify they give the same value:

$$\int_0^\pi f\big(\vec{\mathbf{a}}(t)\big)\Big|\frac{d\vec{\mathbf{a}}}{dt}\Big|\,\mathrm{d}t = \int_0^\pi \sin t\,1\,\mathrm{d}t = -\cos t\Big|_0^\pi = 1-(-1) = 2,$$

$$\int_{-1}^1 f\big(\vec{\mathbf{b}}(\tau)\big)\Big|\frac{d\vec{\mathbf{b}}}{d\tau}\Big|\,\mathrm{d}\tau = \int_{-1}^1 \sqrt{1-\tau^2}\frac{1}{\sqrt{1-\tau^2}}\,\mathrm{d}\tau = \int_{-1}^1 \mathrm{d}\tau = 2.$$

Note that, not only the two parametrisations travel along C with different speeds ($\vec{\mathbf{a}}$ has constant speed $|\frac{d\vec{\mathbf{a}}}{dt}| = 1$, while $\vec{\mathbf{b}}$ "accelerates" at the endpoints), but they also travel in opposite directions: $\vec{\mathbf{a}}$ from right to left and $\vec{\mathbf{b}}$ from left to right.

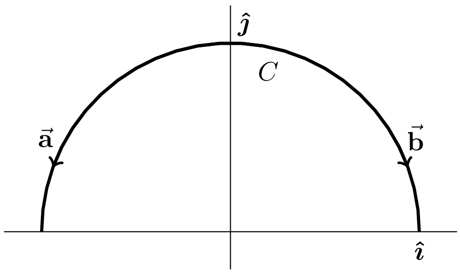


Figure 17: *The half circle parametrised by two curves $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ as in Exercise 2.3.*

Exercise 2.4. ▶ Draw the following curves (helix, cubic, Archimedean spiral; compare with Figure 8)

$$\begin{aligned} &\Gamma_A \quad \text{defined by} \quad \vec{\mathbf{a}}_A(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + t\hat{\boldsymbol{k}}, && 0 \le t \le 2\pi,\\ &\Gamma_B \quad \text{defined by} \quad \vec{\mathbf{a}}_B(t) = \frac{t^3}{3}\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{k}}, && 0 \le t \le 1,\\ &\Gamma_C \quad \text{defined by} \quad \vec{\mathbf{a}}_C(t) = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}}, && 0 \le t \le 10, \end{aligned}$$

and compute $\int_{\Gamma_A} f_A \,\mathrm{d}s$, $\int_{\Gamma_B} f_B \,\mathrm{d}s$ and $\int_{\Gamma_C} f_C \,\mathrm{d}s$ for the following fields:

$$f_A(\vec{\mathbf{r}}) = x + y + z, \qquad f_B(\vec{\mathbf{r}}) = \sqrt{9x^2 + z^2 + y^3}, \qquad f_C = \sqrt{\frac{x^2 + y^2}{1 + x^2 + y^2}}.$$

(Despite looking nasty, the third integral is quite easy to compute, find the trick!)

Example 2.5 (Length of a graph)**.** Formula (43) can be used to compute the length of the graph of a real function. Given a function $g : [t_I, t_F] \to \mathbb{R}$, its graph can be represented by the curve $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + g(t)\hat{\boldsymbol{\jmath}}$, whose total derivative is $\frac{d\vec{\mathbf{a}}}{dt}(t) = \hat{\boldsymbol{\imath}} + g'(t)\hat{\boldsymbol{\jmath}}$. Therefore, its length is

$$\text{length of the graph of } g = \int_{t_I}^{t_F} \left|\frac{d\vec{\mathbf{a}}}{dt}\right| \mathrm{d}t = \int_{t_I}^{t_F} |\hat{\boldsymbol{\imath}} + g'(t)\hat{\boldsymbol{\jmath}}| \,\mathrm{d}t = \int_{t_I}^{t_F} \sqrt{1 + \big(g'(t)\big)^2} \,\mathrm{d}t.$$

★ **Remark 2.6** (Line integrals of densities in physics)**.** If f represents the density of a wire with shape Γ, line integrals can be used to compute not only its total mass $\int_\Gamma f \,\mathrm{d}s$, but also its centre of mass (or barycentre) $(\int_\Gamma xf \,\mathrm{d}s)\hat{\boldsymbol{\imath}} + (\int_\Gamma yf \,\mathrm{d}s)\hat{\boldsymbol{\jmath}} + (\int_\Gamma zf \,\mathrm{d}s)\hat{\boldsymbol{k}}$ and its moment of inertia about a certain axis. You can find some examples in Section 15.3 of the textbook [1].

Exercise 2.7. ▶ Consider the following two planar curves corresponding to the logarithmic and the algebraic spirals shown in Figure 18:

$$\vec{\mathbf{a}}(t) = \mathrm{e}^{-t}\cos t\,\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}\sin t\,\hat{\boldsymbol{\jmath}}, \qquad t \in [0, \infty), \qquad \vec{\mathbf{b}}(\tau) = \frac{1}{\tau}\cos\tau\,\hat{\boldsymbol{\imath}} + \frac{1}{\tau}\sin\tau\,\hat{\boldsymbol{\jmath}}, \qquad \tau \in [1, \infty).$$

1. Compute the length of the logarithmic spiral traced by $\vec{\mathbf{a}}$.
2. Show that the algebraic spiral traced by $\vec{\mathbf{b}}$ has infinite length. (You might use that $\operatorname{arsinh}'\tau = 1/\sqrt{1+\tau^2}$, but an easier way is to bound from below the integral obtained in a smart way.)
3. Show that both paths spin infinitely many times around the origin.
4. Given a positive number ℓ can you construct a spiral $\vec{\mathbf{c}}$ that starts from $\hat{\boldsymbol{\imath}}$, swirls infinitely many times around the origin, eventually reaches $\vec{\mathbf{0}}$ and has length ℓ? What is the minimal ℓ possible?

You can find more exercises on path lengths and line integrals in Section 11.3 (exercises 13–20, p. 641) and Section 15.3 (exercises 1–9, p.861) of the textbook [1].

2.1.2 Line integrals of vector fields

Consider a curve $\vec{\mathbf{a}} : I \to \mathbb{R}^3$ as before, now together with a vector field $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ (we denote again by $\Gamma := \vec{\mathbf{a}}(I) \subset \mathbb{R}^3$ the path of the curve $\vec{\mathbf{a}}$). The **line integral** of $\vec{\mathbf{F}}$ along the path Γ is

$$\boxed{\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}} = \int_{t_I}^{t_F} \vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t) \,\mathrm{d}t.} \tag{44}$$

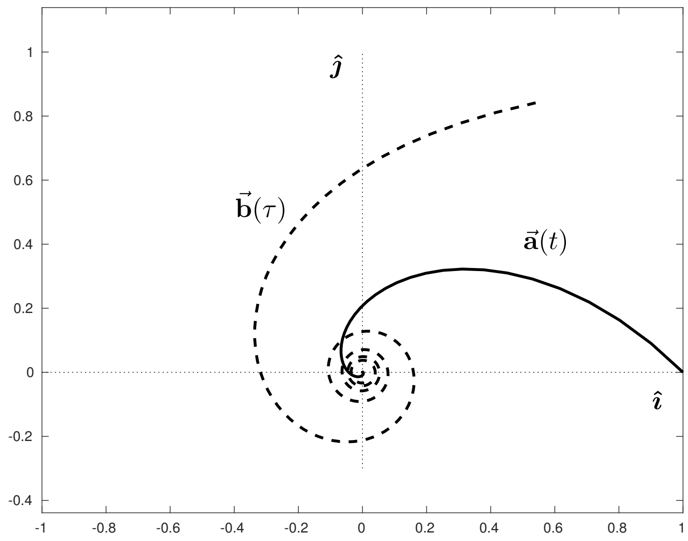


Figure 18: *The logarithmic and the algebraic spirals of Exercise 2.7. We see that both swirls infinitely many times around the origin but* $\vec{\mathbf{a}}$ *approaches the origin itself much more quickly than* $\vec{\mathbf{b}}$*. Indeed the path of* $\vec{\mathbf{a}}$ *has finite length, while the length of the path of* $\vec{\mathbf{b}}$ *is infinite.*

The right-hand side of (44) is an ordinary one-dimensional integral (note that $\vec{\mathbf{a}}$, $\frac{d\vec{\mathbf{a}}}{dt}$ and $\vec{\mathbf{F}}$ are vector-valued but the integral is scalar since it contains a scalar product). If $\vec{\mathbf{a}}$ is a loop, then the line integral of $\vec{\mathbf{F}}$ along Γ is called **contour integral** or **circulation** and denoted by the symbol $\oint_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$.

Similarly to scalar fields, different parametrisations of Γ give the same integral, if the travel direction is the same. **If the curve orientation is reversed, the sign of the integral changes** (recall that the sign does *not* change for line integrals of scalar fields!):

$$\text{if}\quad \vec{\mathbf{b}}(t) = \vec{\mathbf{a}}(1-t) \quad\text{then}\quad \vec{\mathbf{a}}([0,1]) = \vec{\mathbf{b}}([0,1]) = \Gamma \quad\text{and}\quad \int_\Gamma \vec{\mathbf{F}}(\vec{\mathbf{b}}) \cdot \mathrm{d}\vec{\mathbf{b}} = -\int_\Gamma \vec{\mathbf{F}}(\vec{\mathbf{a}}) \cdot \mathrm{d}\vec{\mathbf{a}}.$$

The sign change follows from $\frac{d\vec{\mathbf{b}}}{dt}(t) = -\frac{d\vec{\mathbf{a}}}{dt}(1-t)$ (when $\vec{\mathbf{b}}(t) = \vec{\mathbf{a}}(1-t)$). Example 2.10 demonstrates the invariance of the line integral and the sign change. Since, up to orientation, all the parametrisations of Γ give the same integral for a given field, we can denote the line integral itself by $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$, forgetting the dependence on the parametrisation $\vec{\mathbf{a}}$. In this notation, Γ is an **oriented path**, namely a path with a given direction of travelling; if we want to consider the integral taken in the reverse direction we can simply write $\int_{-\Gamma} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = -\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$. (In this notation, Γ and $-\Gamma$ are two different oriented paths sharing the same support.)

The integral in (44) represents **the integral along Γ of the component of $\vec{\mathbf{F}}$ tangential to Γ itself**. To see this, for a smooth curve $\vec{\mathbf{a}} : [t_I, t_F] \to \mathbb{R}^3$ (under the assumption $\frac{d\vec{\mathbf{a}}}{dt}(t) \neq \vec{\mathbf{0}}$ for all t), we define the **unit tangent vector $\hat{\boldsymbol{\tau}}$**:

$$\hat{\boldsymbol{\tau}} \;=\; \tau_1\hat{\boldsymbol{\imath}} + \tau_2\hat{\boldsymbol{\jmath}} + \tau_3\hat{\boldsymbol{k}} \;=\; \frac{1}{\left|\frac{d\vec{\mathbf{a}}}{dt}\right|}\frac{d\vec{\mathbf{a}}}{dt} \;=\; \frac{\frac{da_1}{dt}\hat{\boldsymbol{\imath}} + \frac{da_2}{dt}\hat{\boldsymbol{\jmath}} + \frac{da_3}{dt}\hat{\boldsymbol{k}}}{\sqrt{\left(\frac{da_1}{dt}\right)^2 + \left(\frac{da_2}{dt}\right)^2 + \left(\frac{da_3}{dt}\right)^2}}.$$

This is a vector field of unit length defined on Γ and tangent to it at every point. As mentioned, the line integral of the vector field $\vec{\mathbf{F}}$ is equal to the line integral of the scalar field $(\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}})$, the projection of $\vec{\mathbf{F}}$ on the tangent vector of Γ:

$$\boxed{\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}} \overset{(44)}{=} \int_{t_I}^{t_F} \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}t = \int_{t_I}^{t_F} \left(\vec{\mathbf{F}} \cdot \frac{\frac{d\vec{\mathbf{a}}}{dt}}{\left|\frac{d\vec{\mathbf{a}}}{dt}\right|}\right)\left|\frac{d\vec{\mathbf{a}}}{dt}\right|\mathrm{d}t = \int_{t_I}^{t_F} (\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}})\left|\frac{d\vec{\mathbf{a}}}{dt}\right|\mathrm{d}t \overset{(41)}{=} \boxed{\int_\Gamma (\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}})\,\mathrm{d}s.} \qquad (45)$$

This an important relation between the main definitions of Sections 2.1.1 and 2.1.2.

Remark 2.8. We introduce a notation that will be used in Section 3 and is often found in books. The line integrals of a scalar field f with respect to the elements $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ (over the oriented path $\Gamma = \vec{\mathbf{a}}([t_I, t_F])$) are defined as

$$\int_\Gamma f\,\mathrm{d}x \;:=\; \int_\Gamma f\,\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(45)}{=} \int_\Gamma f\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\tau}})\,\mathrm{d}s \overset{(44)}{=} \int_{t_I}^{t_F} f\big(\vec{\mathbf{a}}(t)\big)\frac{da_1}{dt}(t)\,\mathrm{d}t,$$

$$\int_\Gamma f\,\mathrm{d}y := \int_\Gamma f\,\hat{\boldsymbol{\jmath}}\cdot \mathrm{d}\vec{\mathbf{r}} \overset{(45)}{=} \int_\Gamma f\,(\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\tau}})\,\mathrm{d}s \overset{(44)}{=} \int_{t_I}^{t_F} f\big(\vec{\mathbf{a}}(t)\big)\frac{da_2}{dt}(t)\,\mathrm{d}t, \tag{46}$$
$$\int_\Gamma f\,\mathrm{d}z := \int_\Gamma f\,\hat{\boldsymbol{k}}\cdot \mathrm{d}\vec{\mathbf{r}} \overset{(45)}{=} \int_\Gamma f\,(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\tau}})\,\mathrm{d}s \overset{(44)}{=} \int_{t_I}^{t_F} f\big(\vec{\mathbf{a}}(t)\big)\frac{da_3}{dt}(t)\,\mathrm{d}t.$$

Therefore, we can expand the line integral of a vector field $\vec{\mathbf{F}} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}} + F_3\hat{\boldsymbol{k}}$ as the sum of the integrals of its components:

$$\int_\Gamma \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} = \int_\Gamma F_1\,\mathrm{d}x + \int_\Gamma F_2\,\mathrm{d}y + \int_\Gamma F_3\,\mathrm{d}z = \int_\Gamma \big(F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z\big). \tag{47}$$

One can also use the formal identity $\mathrm{d}x = \frac{dx}{dt}\,\mathrm{d}t$, identify $x = a_1(t)$ and obtain $\mathrm{d}x = \frac{da_1(t)}{dt}\,\mathrm{d}t$, which gives again the first equation in (46) (and similarly for the remaining two).

If the path Γ can be parametrised by the coordinate x, then it is possible to write the integral $\int_\Gamma f\,\mathrm{d}x$ more explicitly. Assume Γ to be parametrised by $\vec{\mathbf{a}}(x) = x\hat{\boldsymbol{\imath}} + y(x)\hat{\boldsymbol{\jmath}}$, for $x_L < x < x_R$ and for a continuous function $y : (x_L, x_R) \to \mathbb{R}$ (here it is important that the parametrisation satisfies $\frac{da_1}{dx} = 1$). Then, the line integral $\int_\Gamma f\,\mathrm{d}x$ can be written as the one-dimensional integral in x of the field evaluated along the graph of $y(x)$:

$$\int_\Gamma f\,\mathrm{d}x \overset{(46)}{=} \int_{x_L}^{x_R} f\big(\vec{\mathbf{a}}(x)\big)\hat{\boldsymbol{\imath}}\cdot\frac{d\vec{\mathbf{a}}}{dx}(x)\,\mathrm{d}x = \int_{x_L}^{x_R} f\big(x, y(x)\big)\frac{da_1}{dx}(x)\,\mathrm{d}x = \int_{x_L}^{x_R} f\big(x, y(x)\big)\,\mathrm{d}x. \tag{48}$$

In other words, the integral $\int_\Gamma f\,\mathrm{d}x$ is a standard integral in x *if* the path Γ is parametrised by a curve whose first component is $a_1(x) = x + \lambda$ for some $\lambda \in \mathbb{R}$. This fact also extends to curves in three dimensions. If the path is run in the opposite direction (right to left, or from higher to lower values of x) with a similar parametrisation, then the sign of the integral is reversed, as expected.

⋆ **Remark 2.9** (Work)**.** In physics, the line integral of a force field $\vec{\mathbf{F}}$ along a curve $\vec{\mathbf{a}}$, representing the trajectory of a body, is the **work** done by $\vec{\mathbf{F}}$.

Example 2.10 (Integrals of a vector field along curves with common endpoints)**.** Consider the following five curves, all connecting the points $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$:

$$\begin{aligned}
&\vec{\mathbf{a}}_A(t) = t\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}, && 0 \le t \le 1,\\
&\vec{\mathbf{a}}_B(t) = \sin t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, && 0 \le t \le \frac{\pi}{2},\\
&\vec{\mathbf{a}}_C(t) = \mathrm{e}^{-t}\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}}, && 0 \le t < \infty,\\
&\vec{\mathbf{a}}_D(t) = t^2\hat{\boldsymbol{\imath}} + t^4\hat{\boldsymbol{\jmath}}, && 0 \le t \le 1,\\
&\vec{\mathbf{a}}_E(t) = \begin{cases} t\hat{\boldsymbol{\imath}}, & 0 \le t \le 1,\\ \hat{\boldsymbol{\imath}} + (t-1)\hat{\boldsymbol{\jmath}}, & 1 < t \le 2.\end{cases}
\end{aligned}$$

The curves $\vec{\mathbf{a}}_A$ and $\vec{\mathbf{a}}_B$ run along the diagonal of the square $S = \{\vec{\mathbf{r}} \text{ s.t. } 0 \le x \le 1,\ 0 \le y \le 1\}$ at different speeds; $\vec{\mathbf{a}}_C$ travels along the same diagonal but in the opposite direction and in infinite time; $\vec{\mathbf{a}}_D$ and $\vec{\mathbf{a}}_E$ trace different paths (see Figure 19). Note that $\vec{\mathbf{a}}_E$ is not smooth. Compute $\int_{\Gamma_n} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{a}}_n$ for $\vec{\mathbf{F}} = 2y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}$ and $n = A, B, C, D, E$, where Γ_n is the trajectory described by $\vec{\mathbf{a}}_n$ (thus $\Gamma_A = \Gamma_B = \Gamma_C \ne \Gamma_D \ne \Gamma_E$).

We begin by computing the total derivatives of the curves

$$\frac{d\vec{\mathbf{a}}_A}{dt} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{a}}_B}{dt} = \cos t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{a}}_C}{dt} = -\mathrm{e}^{-t}\hat{\boldsymbol{\imath}} - \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{a}}_D}{dt} = 2t\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}},$$
$$\frac{d\vec{\mathbf{a}}_E}{dt} = \begin{cases} \hat{\boldsymbol{\imath}} & 0 \le t \le 1,\\ \hat{\boldsymbol{\jmath}} & 1 < t \le 2.\end{cases}$$

Now we insert them in formula (44):

$$\int_{\Gamma_A} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{a}}_A = \int_0^1 \big(2y(t)\hat{\boldsymbol{\imath}} - x(t)\hat{\boldsymbol{\jmath}}\big)\cdot(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 \big(2t\hat{\boldsymbol{\imath}} - t\hat{\boldsymbol{\jmath}}\big)\cdot(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 t\,\mathrm{d}t = \frac{t^2}{2}\bigg|_0^1 = \frac{1}{2},$$
$$\begin{aligned}
\int_{\Gamma_B} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{a}}_B &= \int_0^{\frac{\pi}{2}} \big(2\sin t\,\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}\big)\cdot(\cos t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}})\,\mathrm{d}t\\
&= \int_0^{\frac{\pi}{2}} \sin t\cos t\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t\,\mathrm{d}t = \frac{-\cos 2t}{4}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},
\end{aligned}$$
$$\int_{\Gamma_C} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{a}}_C = \int_0^\infty \big(2\mathrm{e}^{-t}\hat{\boldsymbol{\imath}} - \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}}\big)\cdot(-\mathrm{e}^{-t}\hat{\boldsymbol{\imath}} - \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^\infty (-\mathrm{e}^{-2t})\,\mathrm{d}t = \frac{\mathrm{e}^{-2t}}{2}\bigg|_0^\infty = \frac{0-1}{2} = -\frac{1}{2},$$

$$\int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_0^1 (2t^4\hat{\boldsymbol{\imath}} - t^2\hat{\boldsymbol{\jmath}}) \cdot (2t\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 (4t^5 - 4t^5)\,\mathrm{d}t = 0,$$
$$\int_{\Gamma_E} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = \int_0^1 (-t\hat{\boldsymbol{\jmath}}) \cdot \hat{\boldsymbol{\imath}}\,\mathrm{d}t + \int_1^2 \big(2(t-1)\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}\big) \cdot \hat{\boldsymbol{\jmath}}\,\mathrm{d}t = 0 + \int_1^2 (-1)\,\mathrm{d}t = -1,$$

As expected, we have

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_A = \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_B = -\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_C = \frac{1}{2} \neq \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = 0 \neq \int_{\Gamma_E} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = -1,$$

since the first and the second are different parametrisations of the same path, the third is a parametrisation of the same path with reverse direction, while the last two correspond to different paths.

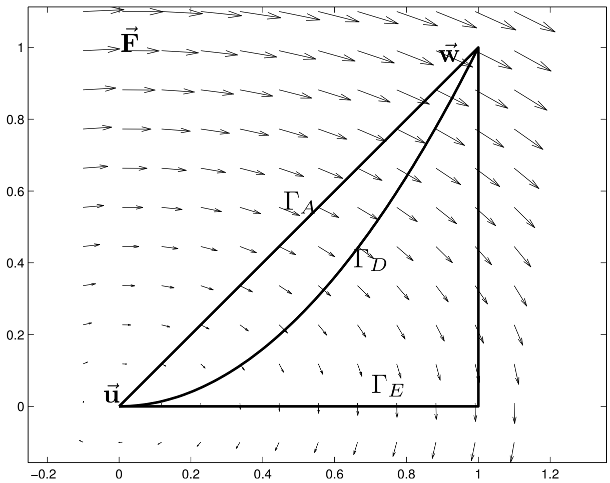


Figure 19: *The vector field* $\vec{\mathbf{F}} = 2y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}$ *and the curves described in Example 2.10. Note that* $\vec{\mathbf{F}}$ *is perpendicular to* Γ_D *in each point (thus* $\int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_{\Gamma_D} 0\,\mathrm{d}s = 0$*) and to* Γ_E *in its horizontal segment.*

Exercise 2.11. ▶ Compute the line integral of $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}}$ along the five curves described in Example 2.10. (Hint: recall the derivative of $\sin^3 t$.)

Exercise 2.12. ▶ Compute the line integrals of the position vector $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}$ along the two spirals defined in Exercise 2.7. Can you guess the signs of the integrals before computing them?

To practise more on line integrals of vector fields, see e.g. Problems 1–9 of [1, Section 15.4, p. 869].

★ **Remark 2.13** (Line integrals of vector fields as limits of Riemann sums)**.** An intuitive explanation of the relation between formula (44) and its interpretation as integral of the tangential component of $\vec{\mathbf{F}}$, can be obtained by considering a discrete approximation of the line integral with Riemann sums (similarly to what we have already done in Remark 2.1 for integrals of scalar fields). Consider the times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$, for $i = 0, \ldots, N$, and the corresponding points $\vec{\mathbf{a}}_i = \vec{\mathbf{a}}(t_i) \in \Gamma$. Define the increments $\delta\vec{\mathbf{a}}_i := \vec{\mathbf{a}}_i - \vec{\mathbf{a}}_{i-1}$ and $\delta t_i := t_i - t_{i-1}$. Then, the integral of the component of $\vec{\mathbf{F}}$ tangential to Γ in the interval (t_{i-1}, t_i) is approximated by $\vec{\mathbf{F}}(\vec{\mathbf{a}}_i) \cdot \delta\vec{\mathbf{a}}_i$ (recall the projection seen in Section 1.1.1), as depicted in Figure 20, and the integral on Γ is approximated by the sum of these terms:

$$\int_\Gamma \vec{\mathbf{F}}(\vec{\mathbf{a}}) \cdot \mathrm{d}\vec{\mathbf{r}} \;\approx\; \sum_{i=1}^N \vec{\mathbf{F}}(\vec{\mathbf{a}}_i) \cdot \delta\vec{\mathbf{a}}_i \;=\; \sum_{i=1}^N \vec{\mathbf{F}}(\vec{\mathbf{a}}_i) \cdot \frac{\delta\vec{\mathbf{a}}_i}{\delta t_i}\,\delta t_i.$$

In the limit $N \to \infty$ (if the times t_i are "well spaced" and $\vec{\mathbf{a}}$ is smooth), $\frac{\delta\vec{\mathbf{a}}_i}{\delta t_i}$ tends to the total derivative of $\vec{\mathbf{a}}$ and the sum of δt_i gives the integral in $\mathrm{d}t$, similarly to (42). Thus we recover formula (44).

2.1.3 Independence of path and line integrals for conservative fields

We consider the line integral of the gradient of a smooth scalar field φ along a curve $\vec{\mathbf{a}} : [t_I, t_F] \to \Gamma \subset \mathbb{R}^3$. Using the chain rule, since the function $t \mapsto \varphi(\vec{\mathbf{a}}(t))$ is a real function of real variable and the usual fundamental theorem of calculus applies, we obtain

$$\int_\Gamma (\vec{\nabla}\varphi) \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(44)}{=} \int_{t_I}^{t_F} \vec{\nabla}\varphi\big(\vec{\mathbf{a}}(t)\big) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t)\,\mathrm{d}t \overset{(37)}{=} \int_{t_I}^{t_F} \frac{d\big(\varphi(\vec{\mathbf{a}})\big)}{dt}(t)\,\mathrm{d}t = \varphi\big(\vec{\mathbf{a}}(t_F)\big) - \varphi\big(\vec{\mathbf{a}}(t_I)\big).$$

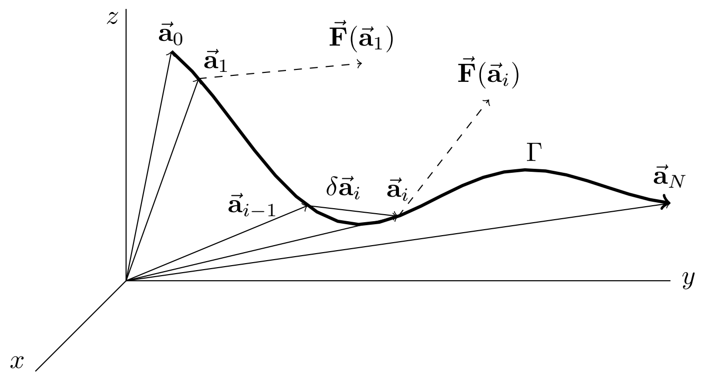


Figure 20: *A schematic representation of the approximation of the line integral of the vector field* $\vec{\mathbf{F}}$ *along* Γ *using discrete increments (see Remark 2.13).*

This is a simple but important result and we state it as a theorem. It is often called "fundamental theorem of vector calculus" (or "gradient theorem"), in analogy to the similar theorem known from one-dimensional calculus. This is the first relation between differential operators and integration we encounter; we will see several others in Section 3 (see also the comparisons in Table 3 and in Appendix A).

Theorem 2.14 (Fundamental theorem of vector calculus)**.** Consider a conservative vector field $\vec{\mathbf{F}}$ with scalar potential φ and a (piecewise smooth) curve $\vec{\mathbf{a}}: [t_I, t_F] \to \Gamma \subset \mathbb{R}^3$. Then

$$\boxed{\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma (\vec{\nabla}\varphi) \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi\big(\vec{\mathbf{a}}(t_F)\big) - \varphi\big(\vec{\mathbf{a}}(t_I)\big).} \tag{49}$$

The only information on the path Γ contained in the right-hand side of equation (49) are its endpoints $\vec{\mathbf{a}}(t_I)$ and $\vec{\mathbf{a}}(t_F)$: the actual path Γ is not needed to compute the line integral of $\vec{\mathbf{F}}$ on it (once we know the scalar potential φ). This means that **line integrals of conservative fields are independent of the particular integration path**, but depend only on the initial and the final points of integration. Thus, given a conservative field $\vec{\mathbf{F}}$ and two points $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ in its domain, we can use the notation

$$\int_{\vec{\mathbf{p}}}^{\vec{\mathbf{q}}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} := \int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}},$$

where Γ is *any* path connecting $\vec{\mathbf{p}}$ to $\vec{\mathbf{q}}$. (Note that $\int_{\vec{\mathbf{p}}}^{\vec{\mathbf{q}}}$ is not well-defined for non-conservative fields.)

Comparison with scalar calculus 2.15. The fundamental theorem of calculus states that $\int_a^b g(t)\,\mathrm{d}t = G(b) - G(a)$ for all continuous real functions $g : [a, b] \to \mathbb{R}$, where G is a primitive of g, namely $G' = g$. Similarly, the fundamental theorem of vector calculus gives $\int_{\vec{\mathbf{p}}}^{\vec{\mathbf{q}}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\vec{\mathbf{a}}(t_F)) - \varphi(\vec{\mathbf{a}}(t_I))$, where φ is a scalar potential of the vector field $\vec{\mathbf{F}}$. So we can consider scalar potentials as the line-integral equivalent of primitives. Recall also that both primitives and scalar potentials are defined up to constants.

Example 2.16. From Example 2.10 and formula (49), we immediately deduce that $\vec{\mathbf{F}} = 2y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}$ is not conservative, since integrals on different paths connecting the same endpoints give different values. The field $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}}$ of Exercise 2.11 might be conservative as its integral has the same value along three different curves; however, in order to prove it is conservative we should proceed as we did in Example 1.67.

Exercise 2.17. ▶ Use formula (49), together with the results of Example 1.67 and Exercise 1.68, to compute

$$\int_\Gamma \vec{\mathbf{r}} \cdot \mathrm{d}\vec{\mathbf{r}} \qquad \text{and} \qquad \int_\Gamma (z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{k}}) \cdot \mathrm{d}\vec{\mathbf{r}},$$

where Γ is any path connecting the point $\vec{\mathbf{p}} = \hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ to the point $\vec{\mathbf{q}} = -\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$.

In the next theorem we use path integrals to prove what was mentioned in Remark 1.70: while the implication "conservative ⇒ irrotational" is true for all vector fields, its converse "irrotational ⇒ conservative" holds for fields defined on suitable domains. We define a **star-shaped domain** (or star-like) to be a domain D such that for all $\vec{\mathbf{r}} \in D$ the straight segment connecting the origin to $\vec{\mathbf{r}}$ lies entirely inside D. All convex shapes, such as parallelepipeds and balls, containing the origin are star-shaped.

Theorem 2.18. Let D be a star-shaped domain, and $\vec{\mathbf{F}}$ be an irrotational vector field defined on D. Then, $\vec{\mathbf{F}}$ is conservative and a scalar potential is given by $\Phi(\vec{\mathbf{r}}) = \int_{\Gamma_{\vec{\mathbf{r}}}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$, where $\Gamma_{\vec{\mathbf{r}}}$ is the segment connecting the origin to $\vec{\mathbf{r}}$.

Proof. Given $\vec{\mathbf{r}} \in D$, $\vec{\mathbf{r}} \neq 0$, a parametrisation of the segment $\Gamma_{\vec{\mathbf{r}}}$ is $\vec{\mathbf{a}}(t) = t\vec{\mathbf{r}}$ for $0 \le t \le 1$ (recall Remark 1.24). We note that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ because $\vec{\mathbf{F}}$ is irrotational. Using this, together with product and chain rule, we can compute the partial derivative of Φ in x:

$$\begin{aligned}
\frac{\partial \Phi}{\partial x}(\vec{\mathbf{r}}) &= \frac{\partial}{\partial x}\int_{\Gamma_{\vec{\mathbf{r}}}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} && \text{(from the definition of } \Phi\text{)}\\
&= \frac{\partial}{\partial x}\int_0^1 \vec{\mathbf{F}}(t\vec{\mathbf{r}}) \cdot \vec{\mathbf{r}}\,\mathrm{d}t && \text{(from the parametrisation of } \Gamma_{\vec{\mathbf{r}}}\text{)}\\
&= \int_0^1 \frac{\partial}{\partial x}\big(xF_1(t\vec{\mathbf{r}}) + yF_2(t\vec{\mathbf{r}}) + zF_3(t\vec{\mathbf{r}})\big)\,\mathrm{d}t\\
&= \int_0^1 \left(F_1(t\vec{\mathbf{r}}) + x\frac{\partial\big(F_1(t\vec{\mathbf{r}})\big)}{\partial x} + y\frac{\partial\big(F_2(t\vec{\mathbf{r}})\big)}{\partial x} + z\frac{\partial\big(F_3(t\vec{\mathbf{r}})\big)}{\partial x}\right)\mathrm{d}t && \text{(from product rule)}\\
&= \int_0^1 \left(F_1(t\vec{\mathbf{r}}) + tx\frac{\partial F_1}{\partial x}(t\vec{\mathbf{r}}) + ty\frac{\partial F_2}{\partial x}(t\vec{\mathbf{r}}) + tz\frac{\partial F_3}{\partial x}(t\vec{\mathbf{r}})\right)\mathrm{d}t && \text{(from chain rule)}\\
&= \int_0^1 \left(F_1(t\vec{\mathbf{r}}) + tx\frac{\partial F_1}{\partial x}(t\vec{\mathbf{r}}) + ty\frac{\partial F_1}{\partial y}(t\vec{\mathbf{r}}) + tz\frac{\partial F_1}{\partial z}(t\vec{\mathbf{r}})\right)\mathrm{d}t && \text{(from } \vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}\text{)}\\
&= \int_0^1 \left(F_1(t\vec{\mathbf{r}}) + t\vec{\mathbf{r}} \cdot \vec{\nabla} F_1(t\vec{\mathbf{r}})\right)\mathrm{d}t\\
&= \int_0^1 \left(F_1(t\vec{\mathbf{r}}) + t\frac{d}{dt}F_1(t\vec{\mathbf{r}})\right)\mathrm{d}t && \text{(chain rule (37) and } \frac{d\vec{\mathbf{a}}}{dt} = \vec{\mathbf{r}}\text{)}\\
&= \int_0^1 \frac{d}{dt}\Big(t\,F_1(t\vec{\mathbf{r}})\Big)\,\mathrm{d}t && \text{(from the product rule for } \frac{d}{dt}\text{)}\\
&= 1F_1(\vec{\mathbf{r}}) - 0F_1(\vec{\mathbf{0}}) = F_1(\vec{\mathbf{r}}).
\end{aligned}$$

Similar identities can easily be derived for the derivatives in y and z, so we obtain $\vec{\nabla}\Phi = \vec{\mathbf{F}}$, i.e. Φ is a scalar potential of $\vec{\mathbf{F}}$ and this is conservative. □

In next Theorem 2.19, we use the fundamental theorem of vector calculus 2.14 to characterise conservative fields as those fields whose line integrals only depend on the endpoints. The proof of this theorem also provides a construction of the scalar potential.

Theorem 2.19. Consider a continuous vector field $\vec{\mathbf{F}}$. The following three conditions are equivalent:

(i) $\vec{\mathbf{F}}$ is conservative;

(ii) for all loops Γ
$$\oint_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0;$$

(iii) for all pairs of paths Γ_A and Γ_B with identical endpoints
$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}.$$

★ *Proof.* [*(i)⇒(ii)*] If $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ for some scalar potential φ. If the loop Γ is parametrised by $\vec{\mathbf{a}} : [t_I, t_F] \to \Gamma \subset \mathbb{R}^3$ with $\vec{\mathbf{a}}(t_I) = \vec{\mathbf{a}}(t_F)$, then by the fundamental theorem of vector calculus (49) we have $\oint_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\vec{\mathbf{a}}(t_F)) - \varphi(\vec{\mathbf{a}}(t_I)) = 0$.

[*(ii)⇒(iii)*] Assume condition *(ii)* is satisfied. The two paths Γ_A and Γ_B can be combined in a loop[13] $\Gamma = \Gamma_A - \Gamma_B$ (see Figure 21). Using the additivity property (40), the difference between the two path integrals is then

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} - \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{-\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \oint_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0,$$

[13] As we have done in Section 2.1.2, here we consider all paths Γ to be "oriented" paths, i.e. paths equipped with a preferred direction of travel (orientation). We denote by $-\Gamma$ the same path as Γ with the opposite orientation. If Γ_α and Γ_β are paths such that the final point of Γ_α coincides with the initial point of Γ_β, we denote by $\Gamma_\alpha + \Gamma_\beta$ their "concatenation", i.e., the path corresponding to their union (obtained travelling first along Γ_α and then along Γ_β). Thus $\Gamma_A - \Gamma_B$ is the concatenation obtained travelling first along Γ_A and then along Γ_B, the latter run in the opposite direction,

which is condition *(iii)*.

[*(iii)*⇒*(i)*] Fix a point $\vec{\mathbf{r}}_0$ in the domain of $\vec{\mathbf{F}}$. Then, for all points $\vec{\mathbf{r}}$, the scalar field $\Phi(\vec{\mathbf{r}}) := \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is well-defined (does not depend on the integration path). Consider the partial derivative

$$\frac{\partial \Phi}{\partial x}(\vec{\mathbf{r}}) = \lim_{h\to 0} \frac{1}{h}\big(\Phi(\vec{\mathbf{r}} + h\hat{\boldsymbol{\imath}}) - \Phi(\vec{\mathbf{r}})\big) = \lim_{h\to 0} \frac{1}{h}\bigg(\int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}+h\hat{\boldsymbol{\imath}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}\bigg).$$

Since in the integrals we can choose any path, in the first integral we consider the path passing through $\vec{\mathbf{r}}$ and moving in the x direction along the segment with endpoints $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}+h\hat{\boldsymbol{\imath}}$. This segment is parametrised by $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}} + t\hat{\boldsymbol{\imath}}$ for $0 \le t \le h$ (which has total derivative $\frac{d\vec{\mathbf{a}}}{dt}(t) = \hat{\boldsymbol{\imath}}$), thus

$$\begin{aligned}
\frac{\partial \Phi}{\partial x}(\vec{\mathbf{r}}) &= \lim_{h\to 0} \frac{1}{h}\bigg(\int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{\vec{\mathbf{r}}}^{\vec{\mathbf{r}}+h\hat{\boldsymbol{\imath}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}\bigg) \\
&= \lim_{h\to 0} \frac{1}{h} \int_{\vec{\mathbf{r}}}^{\vec{\mathbf{r}}+h\hat{\boldsymbol{\imath}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\
&\overset{(44)}{=} \lim_{h\to 0} \frac{1}{h} \int_0^h \vec{\mathbf{F}}(\vec{\mathbf{r}} + t\hat{\boldsymbol{\imath}}) \cdot \hat{\boldsymbol{\imath}} \, dt \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^h F_1\big((x+t)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}\big) \, dt \\
&= \lim_{h\to 0} F_1\big((x+\tau)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}\big) \qquad \text{for some } \tau \in [0,h] \\
&= F_1(\vec{\mathbf{r}})
\end{aligned}$$

(using, in the last two equalities, the integral version of the mean value theorem for real continuous functions $\frac{1}{b-a}\int_a^b f(t)\,dt = f(\tau)$ for a $\tau \in [a,b]$ and the continuity of F_1). Similarly, $\frac{\partial \Phi}{\partial y} = F_2$ and $\frac{\partial \Phi}{\partial z} = F_3$, thus $\vec{\mathbf{F}} = \vec{\nabla}\Phi$. The vector field $\vec{\mathbf{F}}$ is conservative and Φ is a scalar potential. □

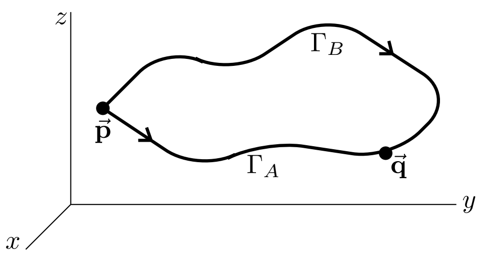


Figure 21: *Two paths Γ_A and Γ_B connecting the points $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ and defining a loop $\Gamma = \Gamma_A - \Gamma_B$, as in Theorem 2.19.*

We summarise what we learned here and in Section 1.5 about conservative fields in the following scheme:

$$\boxed{\begin{matrix} \vec{\mathbf{F}} = \vec{\nabla}\varphi \\ \text{(conservative)} \end{matrix} \iff \begin{matrix} \int_{\vec{\mathbf{p}}}^{\vec{\mathbf{q}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ \text{is path independent} \end{matrix} \iff \begin{matrix} \oint_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 \\ \forall \text{ loops } \Gamma \end{matrix} \implies \begin{matrix} \vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}} \\ \text{(irrotational)} \end{matrix}} \qquad (50)$$

If the domain of definition of $\vec{\mathbf{F}}$ is star-shaped, by Theorem 2.18 the converse of the last implication holds.

Exercise 2.20. ▶ Consider the two oriented paths from the point $\hat{\boldsymbol{\imath}}$ to the point $\hat{\boldsymbol{\jmath}}$ (see Figure 22):

$$\Gamma_a := \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ x+y=1,\ x,y \ge 0\big\}, \qquad \Gamma_b := \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ x^2+y^2=1,\ x,y \ge 0\big\},$$

and the vector field $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = (x^2+y^2)(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})$. Compute the line integrals $\int_{\Gamma_a} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, $\int_{\Gamma_b} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and use the values of these integrals to prove that $\vec{\mathbf{F}}$ is not conservative.

★ **Remark 2.21** (An irrotational, non-conservative field)**.** In Remark 1.71, we have seen that the field

$$\vec{\mathbf{F}} = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

is conservative in the half-space $D = \{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } x > 0\}$ and we have claimed (without justification) that no scalar potential can be defined in its larger domain of definition $E = \{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } x^2+y^2 > 0\}$. We can easily

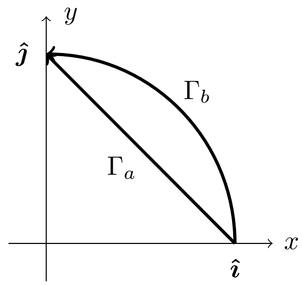


Figure 22: *The paths Γ_a and Γ_b in Exercise* (2.20).

compute the circulation of $\vec{\mathbf{F}}$ along the unit circle $C = \{\vec{\mathbf{r}} \in \mathbb{R}^3$, s.t. $x^2 + y^2 = 1, z = 0\}$ (parametrised by $\vec{\mathbf{a}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$ with $0 \le t \le 2\pi$):

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^{2\pi} (-\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}}) \cdot (-\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi.$$

Since this integral is not equal to zero, Theorem 2.19 proves that $\vec{\mathbf{F}}$ is not conservative in the domain E. (We have completed the proof of the claim made in Remark 1.71.)

★ **Remark 2.22.** From the results studied in this section we can learn the following idea. We already know that any continuous real functions $f : (t_I, t_F) \to \mathbb{R}$ can be written as the derivative of another function $F : (t_I, t_F) \to \mathbb{R}$ (the primitive of f, $F(t) = \int_{t_*}^{t} f(s)\,\mathrm{d}s$ for some t_*), even if sometimes computing F may be very difficult. Thus, the integral of $f = F'$ can be evaluated as the difference of two values of F. Similarly, a scalar field $g : D \to \mathbb{R}$, $D \subset \mathbb{R}^3$, can be written as a partial derivative (with respect to a chosen direction) of another scalar field. On the other hand, only *some* vector fields can be written as gradients of a scalar field (the potential) and their line integral computed as differences of two evaluations of the potential. These are precisely the conservative fields, characterised by the conditions in box (50).

2.2 Multiple integrals

We have learned how to compute integrals of scalar and vector fields along curves. We now focus on integrals on (flat and curved) surfaces and volumes.

2.2.1 Double integrals

For a two-dimensional region $R \subset \mathbb{R}^2$ and a scalar field $f : R \to \mathbb{R}$, we want to define and compute

$$\iint_R f(x, y)\,\mathrm{d}A, \tag{51}$$

namely the integral of f over R. The differential $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ represents the infinitesimal **area element**, which is the product of the infinitesimal length elements in the x and the y directions. Since the domain is two-dimensional, this is commonly called **double integral**.

The double integral $\iint_R f\,\mathrm{d}A$ over a region R represents **the (signed) volume of the portion of space delimited by the graph surface of the two-dimensional field f and the xy-plane**; see Figures 23 and 24. "Signed" means that the contribution given by the part of R in which f is negative is subtracted from the contribution from the part where f is positive.

In the same way as integrals of real functions are approximated by sums of areas of rectangles (recall Figure 16), double integrals are approximated by sums of volumes of parallelepipeds, see Figure 24.

For simplicity, we first consider only two-dimensional domains which can be written as

$$R = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ \text{s.t. } x_L < x < x_R,\ a(x) < y < b(x)\Big\}, \tag{52}$$

for two functions $a(x) < b(x)$ defined in the interval (x_L, x_R). The domains that can be expressed in this way are called **y-simple**. These are the domains such that their intersection with any vertical line is either a segment or the empty set. A prototypical example is shown in Figure 25. Examples of non-y-simple domains are the C-shaped domain $\{\vec{\mathbf{r}} \text{ s.t. } 2y^2 - 1 < x < y^2\}$, the "doughnut" $\{\vec{\mathbf{r}} \text{ s.t. } 1 < |\vec{\mathbf{r}}| < 2\}$ (draw them!) and any other domain containing a hole. These and other more complicated non-y-simple domains can be decomposed in the disjoint union of two or more y-simple or x-simple (similarly defined) domains.

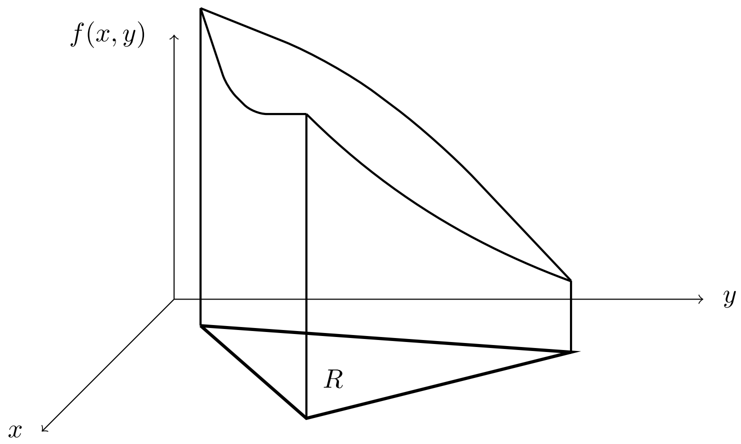


Figure 23: *The double integral* $\iint_R f \,\mathrm{d}A$ *over a domain* R *(a triangle in the picture) represents the (signed) volume delimited by the graph of the two-dimensional field* f *(the upper surface) and the* xy*-plane.*

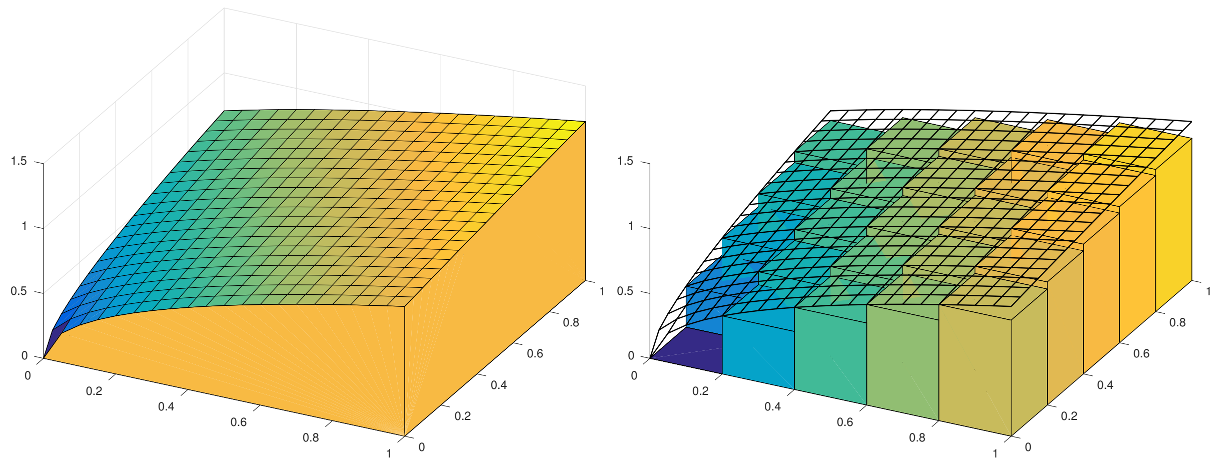


Figure 24: *The surface plot of the scalar field* $f = \sqrt{x + y/2}$ *over the square* $Q = (0,1) \times (0,1)$*. The double integral* $\iint_Q f \,\mathrm{d}A$ *measures the volume shown in the left figure. If* Q *is partitioned in smaller squares* $S_1, \ldots, S_n$*, the double integral can be approximated by the sum over* $j = 1, \ldots, n$ *of the volumes of the parallelepipeds with basis the small squares* S_j *and height the value of* f *at one point* $\vec{\mathbf{x}} \in S_j$*:* $\iint_Q f \,\mathrm{d}A \approx \sum_{j=1}^n f(\vec{\mathbf{x}}_j)\mathrm{Area}(S_j)$*. In this example the integral is approximated by* $n = 25$ *parallelepipeds. In the limit* $n \to \infty$ *(i.e. more smaller squares), the sum converges to the desired integral. This is how double integrals are actually rigorously defined. Recall the similar argument made in Remark 2.1 for the case of line integrals. (In this example,* $\iint_Q f \,\mathrm{d}A = \sqrt{2}(3^{5/2} - 1 - 2^{5/2})/15 \approx 0.842$*, and* $\sum_{j=1}^{25} f(\vec{\mathbf{x}}_j)\mathrm{Area}(S_j) \approx 0.734$*.)*

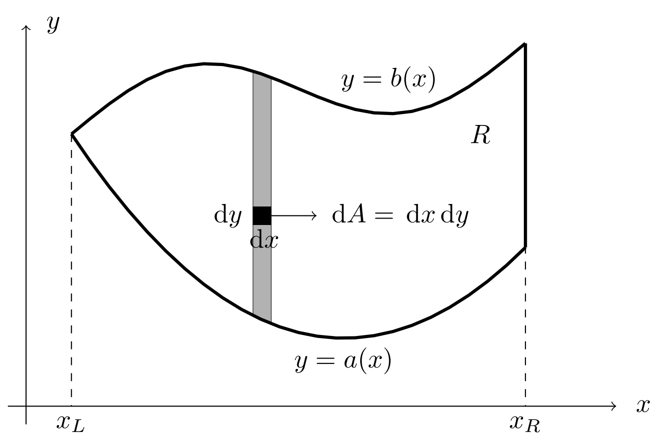


Figure 25: *A* y*-simple two-dimensional domain* R*, bounded by the functions* a *and* b *in the interval* (x_L, x_R)*, and the infinitesimal area element* $\mathrm{d}A$*.*

In order to compute the integral (51), we view it as an **iterated** or **nested integral**.[14] In other words, for each value of $x \in (x_L, x_R)$, we first consider the one-dimensional integral of f along the segment parallel to the y-axis contained in R, i.e.

$$I_f(x) = \int_{a(x)}^{b(x)} f(x,y)\,\mathrm{d}y.$$

Note that both the integrand and the integration endpoints depend on x. The integral I_f is a function of x, but is independent of y; we integrate it along the direction x:

$$\boxed{\iint_R f(x,y)\,\mathrm{d}A = \int_{x_L}^{x_R} I_f(x)\,\mathrm{d}x = \int_{x_L}^{x_R}\left(\int_{a(x)}^{b(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.} \tag{53}$$

Of course, the role of x and y are exchanged in x-simple domains.

We consider as concrete example the isosceles triangle

$$Q := \left\{\vec{\mathbf{r}} \in \mathbb{R}^2,\ \text{s.t. } 0 < y < 1-|x|\right\}; \tag{54}$$

see Figure 26. In this case, $x_L = -1$, $x_R = 1$, $a(x) = 0$, $b(x) = 1-|x|$. If we first integrate a field f along the y-axis and then along the x-axis, we obtain:

$$\iint_Q f(x,y)\,\mathrm{d}A = \int_{-1}^{1}\left(\int_0^{1-|x|} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_{-1}^{0}\left(\int_0^{1+x} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x + \int_0^1\left(\int_0^{1-x} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

We can also consider the same triangle as an x-simple domain and first integrate along the x-axis and then along the y-axis (right plot in Figure 26):

$$\iint_Q f(x,y)\,\mathrm{d}A = \int_0^1\left(\int_{y-1}^{1-y} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Of course, the two integrals will give the same value.

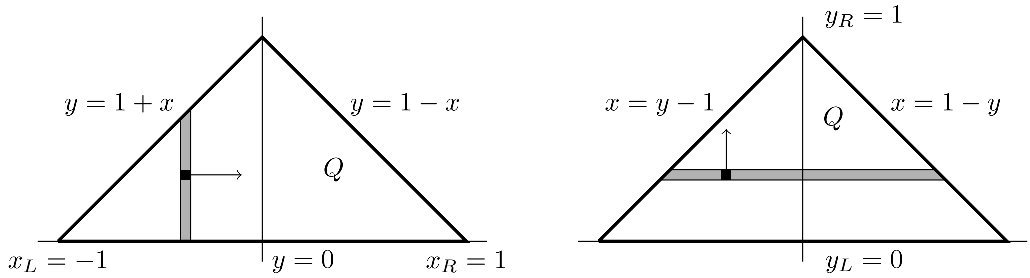


Figure 26: *The computation of the double integral* $\iint_Q f(x,y)\,\mathrm{d}A$ *on the triangle* Q*. The left plot represents* $\int_{-1}^1(\int_0^{1-|x|} f\,\mathrm{d}y)\,\mathrm{d}x$*, where the inner integral is taken along the shaded band (of infinitesimal width) and the outer integral corresponds to the movement of the band along the* x *axis. The right plot represent the integration where the variables are taken in the opposite order* $\int_0^1(\int_{y-1}^{1-y} f\,\mathrm{d}x)\,\mathrm{d}y$*.*

To have a more concrete example, we compute the integral of $f = x - y^2$ on the triangle Q in both the ways described above:

$$\begin{aligned}
\iint_Q (x-y^2)\,\mathrm{d}A &= \int_{-1}^0\left(\int_0^{1+x}(x-y^2)\,\mathrm{d}y\right)\mathrm{d}x + \int_0^1\left(\int_0^{1-x}(x-y^2)\,\mathrm{d}y\right)\mathrm{d}x\\
&= \int_{-1}^0\left(xy - \frac13 y^3\right)\Big|_{y=0}^{1+x}\mathrm{d}x + \int_0^1\left(xy-\frac13 y^3\right)\Big|_{y=0}^{1-x}\mathrm{d}x\\
&= \int_{-1}^0 -\frac13(x^3+1)\,\mathrm{d}x + \int_0^1\left(\frac13 x^3 - 2x^2 + 2x - \frac13\right)\mathrm{d}x\\
&= \left(-\frac1{12}x^4 - \frac13 x\right)\Big|_{-1}^0 + \left(\frac1{12}x^4 - \frac23 x^3 + x^2 - \frac13 x\right)\Big|_0^1 = \frac1{12} - \frac13 + \frac1{12} - \frac23 + 1 - \frac13 = -\frac16;\\
\iint_Q (x-y^2)\,\mathrm{d}A &= \int_0^1\left(\int_{y-1}^{1-y}(x-y^2)\,\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^1\left(\frac12 x^2 - xy^2\right)\Big|_{x=y-1}^{1-y}\mathrm{d}y = \int_0^1 2(y^3-y^2)\,\mathrm{d}y = \frac12 y^4 - \frac23 y^3\Big|_0^1 = -\frac16.
\end{aligned}$$

[14] You have already seen iterated integrals during the first year: see handout 5, from page 23 on, of the Calculus module.

Exercise 2.23. ▶ Compute the following double integrals (drawing a sketch of the domain might be helpful):

$$\iint_R e^{3y} \sin x \,dx\,dy \qquad \text{where } R = (0,\pi)\times(1,2) = \big\{x\hat{\boldsymbol\imath} + y\hat{\boldsymbol\jmath} \text{ s.t. } 0<x<\pi,\ 1<y<2\big\},$$

$$\iint_Q y\,dx\,dy \qquad \text{where } Q \text{ is the triangle with vertices } \vec{\mathbf{0}},\ \hat{\boldsymbol\jmath},\ 5\hat{\boldsymbol\imath}+\hat{\boldsymbol\jmath},$$

$$\iint_S \cos x\,dx\,dy \qquad \text{where } S = \Big\{x\hat{\boldsymbol\imath} + y\hat{\boldsymbol\jmath} \text{ s.t. } 0<x<\frac{\pi}{2},\ 0<y<\sin x\Big\}.$$

You can find plenty of exercises on double integrals in Section 14.2 (p. 802, exercises 1–28) of [1].

2.2.2 Change of variables

Comparison with scalar calculus 2.24. We learned in first-year calculus how to compute one-dimensional integrals by **substitution** of the integration variable, for example

$$\int_0^1 e^x\sqrt{1+e^x}\,dx \overset{\substack{\xi(x)=1+e^x\\ d\xi=e^x\,dx}}{=} \int_{\xi(0)}^{\xi(1)} (\xi-1)\sqrt{\xi}\frac{dx}{d\xi}\,d\xi = \int_2^{1+e}\sqrt{\xi}\,d\xi = \frac{2}{3}\xi^{\frac32}\Big|_2^{1+e} = \frac{2}{3}(1+e)^{\frac32} - \frac{4\sqrt2}{3} \approx 2.894.$$

How do we extend this technique to double integrals? In this one-dimensional example, the interval of integration $x\in(0,1)$ is mapped[15] into a new interval $(2,1+e)$ (whose variable is denoted by ξ) by the change of variable $T: x\mapsto \xi = 1+e^x$, which is designed to make "simpler" (more precisely: easier to integrate) the function to be integrated. The factor $dx/d\xi$ takes care of the "stretching" of the variable. In two dimensions, we can use a change of variables not only to make the integrand simpler, but also to obtain a domain of integration with a simpler shape (e.g. a rectangle instead of a curvilinear domain).

Consider a region $R\subset\mathbb{R}^2$ on which a two-dimensional, injective vector field $\vec{\mathbf{T}}: R\to\mathbb{R}^2$ is defined. We can interpret[16] this field as a "change of variables", i.e. a deformation of the domain R into a new domain $\vec{\mathbf{T}}(R)$. We will informally call $\vec{\mathbf{T}}$ a "transformation" or "mapping" of the region R. Departing from the usual notation, we denote by $\xi(x,y)$ and $\eta(x,y)$ ("xi" and "eta") the components of $\vec{\mathbf{T}}$. We understand ξ and η as the Cartesian coordinates describing the transformed domain: R is a subset of the plane described by x and y, $\vec{\mathbf{T}}(R)$ is a subset of the plane described by ξ and η. We denote by $x(\xi,\eta)$ and $y(\xi,\eta)$ the components of the inverse transformation $\vec{\mathbf{T}}^{-1}$ from the $\xi\eta$-plane to the xy-plane[17]:

$$\vec{\mathbf{T}}: R\to\vec{\mathbf{T}}(R) \qquad\qquad \vec{\mathbf{T}}^{-1}: \vec{\mathbf{T}}(R)\to R$$
$$x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath} \mapsto \xi(x,y)\hat{\boldsymbol\xi}+\eta(x,y)\hat{\boldsymbol\eta} \qquad\qquad \xi\hat{\boldsymbol\xi}+\eta\hat{\boldsymbol\eta} \mapsto x(\xi,\eta)\hat{\boldsymbol\imath}+y(\xi,\eta)\hat{\boldsymbol\jmath},$$

or, with a different notation,

$$(x,y)\mapsto\big(\xi(x,y),\eta(x,y)\big) \qquad\qquad (\xi,\eta)\mapsto\big(x(\xi,\eta),y(\xi,\eta)\big).$$

For example, we consider the following (affine) change of variables[18] $\vec{\mathbf{T}}$:

$$\begin{aligned} &\xi(x,y) = \frac{-x-y+1}{2}, && \eta(x,y) = \frac{x-y+1}{2},\\ \text{whose inverse } \vec{\mathbf{T}}^{-1} \text{ corresponds to } \quad & x(\xi,\eta) = -\xi+\eta, && y(\xi,\eta) = -\xi-\eta+1. \end{aligned} \tag{55}$$

(The transformation $\vec{\mathbf{T}}$ associated to ξ and η is a combination of a translation of length 1 in the negative y direction, an anti-clockwise rotation of angle $3\pi/4$, and a shrinking of factor $\sqrt2$ in every direction.) The triangle Q in (54) in the xy-plane is mapped by $\vec{\mathbf{T}}$ into the triangle with vertices $\vec{\mathbf{0}}$, $\hat{\boldsymbol\xi}$ and $\hat{\boldsymbol\eta}$ in the $\xi\eta$-plane, as shown in Figure 27. Other examples of changes of coordinates are displayed in Figure 28.

[15] In case you are not familiar with the mathematical use of the verb "to map", it is worth recalling it. Given a function T defined in the set A and taking values in the set B (which we can simply write as $T: A\to B$), we say that T "maps" an element $a\in A$ into $b\in B$ if $T(a)=b$, and "maps" the subset $C\subset A$ into $T(C)\subset B$. Thus, a function is often called a "map" or a "mapping"; we will use this words to denote the functions related to changes of variables or operators. (If you think about it, you see that a map in the non-mathematical meaning is nothing else than a function that associates to every point in a piece of paper a point on the surface of the Earth. We will also define "charts" to be some special maps from a flat surface to a curvilinear one.)

[16] Another simple and useful interpretation of a change of variables is the following. Each point $\vec{\mathbf{r}} = x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath}\in R\subset\mathbb{R}^2$ is identified by two numbers, the coordinates x and y. A change of variables is simply a rule to associate to each point in R other two numbers, the coordinates ξ and η.

[17] Here and in the following, we use the vectors $\hat{\boldsymbol\xi}$ and $\hat{\boldsymbol\eta}$. They can be understood either as the unit vectors of the canonical basis of the $\xi\eta$-plane (exactly in the same role of $\hat{\boldsymbol\imath}$ and $\hat{\boldsymbol\jmath}$ in the xy-plane) or as vector fields with unit length, defined in R, pointing in the direction of increase of ξ and η, respectively. The first interpretation is more helpful at this stage, while the second will be useful in Section 2.3.

[18] This change of variables is "affine", meaning that its components ξ and η are polynomials of degree one in x and y. Affine change of variables translate, rotate, dilate and stretch the coordinates but do not introduce curvature: straight lines are mapped into straight lines.

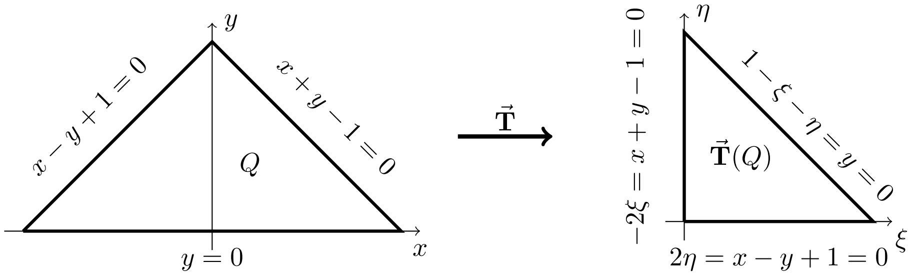


Figure 27: *The triangle Q in the xy-plane and in the $\xi\eta$-plane. Along the edges are shown the equations of the corresponding lines.*

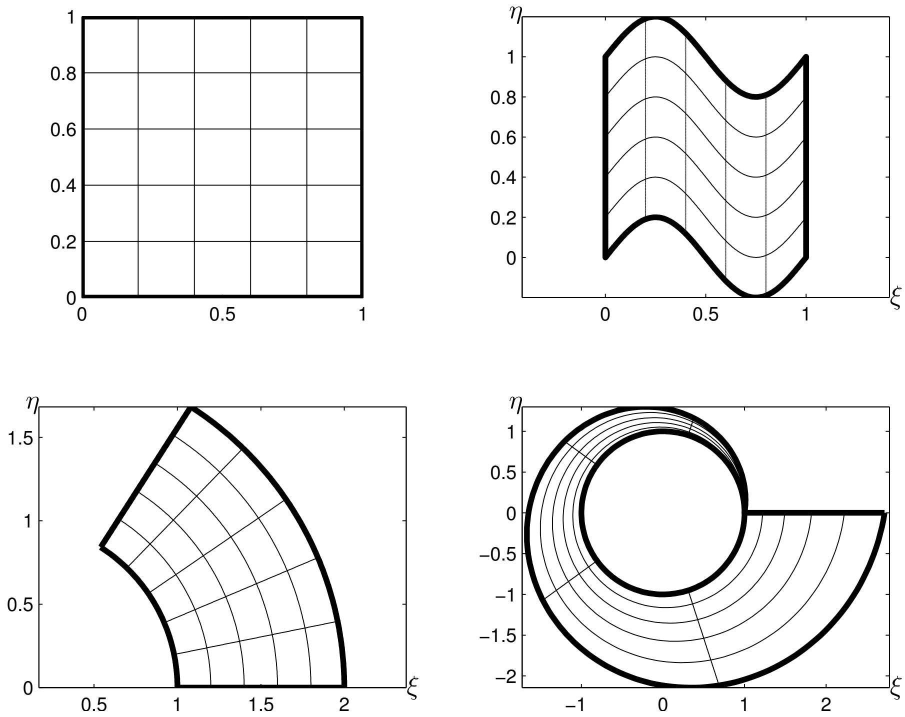


Figure 28: *The unit square $0 < x, y < 1$ (upper left) under three different mappings: $\xi = x, \eta = y + \frac{1}{5}\sin(2\pi x)$ (upper right); $\xi = (1+x)\cos y, \eta = (1+x)\sin y$ (lower left); $\xi = \mathrm{e}^{xy}\cos(2\pi y), \eta = \mathrm{e}^{xy}\sin(2\pi y)$ (lower right).*

A transformation $\vec{\mathbf{T}} : (x, y) \mapsto (\xi, \eta)$ warps and stretches the plane, if we want to compute an integral in the transformed variables we need to take this into account. The infinitesimal area element $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ is modified by the transformation in the $\xi\eta$-plane.

We recall that the Jacobian matrices of $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}^{-1}$ are 2×2 matrix fields containing their first order partial derivatives (see Section 1.3.3). We denote their **determinants** by

$$\boxed{\frac{\partial(\xi,\eta)}{\partial(x,y)} := \det(J\vec{\mathbf{T}}) = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x},} \qquad \text{and} \qquad \boxed{\frac{\partial(x,y)}{\partial(\xi,\eta)} := \det\big(J(\vec{\mathbf{T}}^{-1})\big) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}.} \tag{56}$$

These are called **Jacobian determinants**[19]; their absolute values are exactly the factors needed to compute double integrals under the change of coordinates $\vec{\mathbf{T}}$:

$$\boxed{\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\vec{\mathbf{T}}(R)} f\big(x(\xi,\eta), y(\xi,\eta)\big) \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta,} \tag{57}$$

for any scalar field $f : R \to \mathbb{R}$. This is the fundamental formula for the change of variables in a double integral. In other words, we can say that the infinitesimal surface elements in the xy-plane and in the

[19]In some books, when used as noun, the word "Jacobian" stands for Jacobian determinant, as opposed to Jacobian matrix. Again, this can be a source of ambiguity.

$\xi\eta$-plane are related to each other by the formulas:

$$\boxed{\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta, \qquad \mathrm{d}\xi\,\mathrm{d}\eta = \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y.}$$

We also have

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(\xi,\eta)}}. \tag{58}$$

Formula (58) is often useful when the partial derivatives of only one of the transformations $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}^{-1}$ are easy to compute. In other words, if you need $\frac{\partial(x,y)}{\partial(\xi,\eta)}$ for computing an integral via a change of variable as in (57), but $J\vec{\mathbf{T}}$ is easier to obtain than $J(\vec{\mathbf{T}})^{-1}$, you can compute $\frac{\partial(\xi,\eta)}{\partial(x,y)}$ and then apply (58).

Comparison with scalar calculus 2.25. Recall that in the one-dimensional integration by substitution in 2.24 we use the factor $\frac{d\xi}{dx}$. The corresponding factor for double integrals is the Jacobian determinant $\frac{d(\xi,\eta)}{d(x,y)}$ (56).

⋆ **Remark 2.26.** Note that here we are implicitly assuming that the Jacobian matrices of $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}^{-1}$ are never singular, i.e. their determinants do not vanish in any point of the domain, otherwise equation (58) would make no sense. Under this assumption, it is possible to prove that $J(\vec{\mathbf{T}}^{-1}) = (J\vec{\mathbf{T}})^{-1}$, namely the Jacobian matrix of the inverse transform is equal to the inverse matrix of the Jacobian of $\vec{\mathbf{T}}$ itself; this is part of the Inverse Function Theorem you might study in a future class.

We return to the affine transformations (55). Their Jacobian matrices are (recall the definition in (16))

$$J\vec{\mathbf{T}} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad J(\vec{\mathbf{T}}^{-1}) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Note that this case is quite special: since the transformations are affine (namely polynomials of degree one), their Jacobian are constant in the whole plane $\mathbb{R}^2$. From this, we compute the Jacobian determinants

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big) - \frac{1}{2}\Big(-\frac{1}{2}\Big) = \frac{1}{2}, \qquad \frac{\partial(x,y)}{\partial(\xi,\eta)} = (-1)(-1) - 1(-1) = 2. \tag{59}$$

Note that, as expected, $\frac{\partial(x,y)}{\partial(\xi,\eta)} = 2$ is the ratio between the areas of Q and $\vec{\mathbf{T}}(Q)$ (since in this simple case the Jacobian of $\vec{\mathbf{T}}$ and its determinant are constant in Q). Now we can easily integrate on the triangle Q in (54) the scalar field $f = (1-x-y)^5$, for example, using the change of variables (55) which maps Q into $\vec{\mathbf{T}}(Q)$ as in Figure 27:

$$\begin{aligned}
\iint_Q f(x,y)\,\mathrm{d}x\,\mathrm{d}y &\overset{(55),(57)}{=} \iint_{\vec{\mathbf{T}}(Q)} f(-\xi+\eta, -\xi-\eta+1)\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta \\
&\overset{(59)}{=} \iint_{\vec{\mathbf{T}}(Q)} (1+\xi-\eta+\xi+\eta-1)^5\, 2\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \iint_{\vec{\mathbf{T}}(Q)} (2\xi)^5\, 2\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\overset{(51)}{=} 64\int_0^1 \xi^5 \Big(\int_0^{1-\xi} 1\,\mathrm{d}\eta\Big)\mathrm{d}\xi \\
&= 64\int_0^1 (\xi^5-\xi^6)\,\mathrm{d}\xi \\
&= 64\Big(\frac{1}{6}\xi^6 - \frac{1}{7}\xi^7\Big)\Big|_0^1 = \frac{64}{42} \approx 1.524.
\end{aligned}$$

Remark 2.27. Note that in all the formulas related to change of variables the domain of integration (appearing at subscript of the integral sign) must be a subset of the plane spanned by the integration variables (appearing in the differential term $\mathrm{d}\ldots\mathrm{d}\ldots$). In the example above, the integral $\iint_Q$ is always associated to the differential $\mathrm{d}x\,\mathrm{d}y$, meaning that Q is a set in the xy-plane; the integral $\iint_{\vec{\mathbf{T}}(Q)}$ always appears with the differential $\mathrm{d}\xi\,\mathrm{d}\eta$, as $\vec{\mathbf{T}}(Q)$ is a set in the $\xi\eta$-plane.

Example 2.28 (Area of regions)**.** As we required in (40), the **area** of a domain $R \subset \mathbb{R}^2$ can be computed by integrating on it the constant one: $\mathrm{Area}(R) = \iint_R 1\,\mathrm{d}x\,\mathrm{d}y$. If R is y-simple, one can use the iterated integral (53). Otherwise, if R is complicated but can be mapped into a simpler shape, the change of variable formula (57) can be used to compute its area.

We now want to use formula (57) to compute the areas of the curvilinear domains plotted in Figure 28. Each of them is obtained from the unit square $S = \{0 < x < 1, 0 < y < 1\}$ (upper left plot) from the transformations listed in the figure caption. We denote the domains by R_{UR}, R_{LL}, R_{LR} (as upper right, lower left and lower right plot) to distinguish one another, and use a similar notation for the change of variable transformations and coordinates (e.g. $\vec{\mathbf{T}}_{UR} : S \to R_{UR}$). In the first case (upper right plot), the transformation, its Jacobian matrix and determinant are

$$\xi_{UR} = x, \qquad \eta_{UR} = y + \frac{1}{5}\sin(2\pi x), \qquad J\vec{\mathbf{T}}_{UR} = \begin{pmatrix} 1 & 0 \\ \frac{2}{5}\pi\cos(2\pi x) & 1 \end{pmatrix}, \qquad \frac{\partial(\xi_{UR}, \eta_{UR})}{\partial(x,y)} = 1.$$

From this we obtain

$$\text{Area}(R_{UR}) = \iint_{R_{UR}} \mathrm{d}\xi_{UR}\,\mathrm{d}\eta_{UR} = \iint_{\vec{\mathbf{T}}_{UR}^{-1}(R_{UR})} \frac{\partial(\xi_{UR}, \eta_{UR})}{\partial(x,y)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 1\,\mathrm{d}x\,\mathrm{d}y = 1,$$

(which we could have guessed from the picture!). For the second picture we have

$$\xi_{LL} = (1+x)\cos y, \qquad \eta_{LL} = (1+x)\sin y,$$

$$J\vec{\mathbf{T}}_{LL} = \begin{pmatrix} \cos y & -(1+x)\sin y \\ \sin y & (1+x)\cos y \end{pmatrix}, \qquad \frac{\partial(\xi_{LL}, \eta_{LL})}{\partial(x,y)} = 1 + x,$$

which leads to

$$\text{Area}(R_{LL}) = \iint_{R_{LL}} \mathrm{d}\xi_{LL}\,\mathrm{d}\eta_{LL} = \iint_S \frac{\partial(\xi_{LL}, \eta_{LL})}{\partial(x,y)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 (1+x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 (1+x)\,\mathrm{d}x = \frac{3}{2}.$$

Exercise 2.29. ▶ Compute the area of the domain R_{LR} in the lower-right plot of Figure 28, obtained from the unit square $S = \{0 < x < 1, 0 < y < 1\}$ via the change of variables

$$\xi_{LR} = \mathrm{e}^{xy}\cos(2\pi y), \qquad \eta_{LR} = \mathrm{e}^{xy}\sin(2\pi y).$$

(If you do it in the right way, i.e. with no mistakes, despite looking nasty the computation is actually easy).

Exercise 2.30. ▶ *(i)* Draw the quadrilateral $Q \subset \mathbb{R}^2$ and compute its area:

$$Q = \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, \text{ such that } x = (4-\eta)\xi, y = (\pi - \xi)\eta, \text{ for } 0 < \xi < 1,\ 0 < \eta < 1\big\}.$$

(ii) More generally, given two constants $a, b \geq 2$ compute the area of

$$Q_{ab} = \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, \text{ such that } x = (a-\eta)\xi, y = (b - \xi)\eta, \text{ for } 0 < \xi < 1,\ 0 < \eta < 1\big\}.$$

Exercise 2.31. ▶ Use the change of variables

$$\begin{cases} \xi = \xi(x,y) = \frac{x}{4} + y - 1, \\ \eta = \eta(x,y) = y - \frac{x}{4}, \end{cases} \tag{60}$$

to compute the double integral $\iint_P f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$, where $f(\xi,\eta) = \mathrm{e}^{-\xi-\eta}$, and where the two-dimensional domain P in the $\xi\eta$-plane is the square with vertices $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, -\hat{\boldsymbol{\xi}}, -\hat{\boldsymbol{\eta}}$.

★ **Remark 2.32** (Intuitive justification of the change of variable formula)**.** Where does the change of variable formula (57) come from? Why are the infinitesimal surface elements related to the Jacobian determinant?

Consider the square S with vertices

$$\vec{\mathbf{p}}_{SW} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{p}}_{SE} = (x+h)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{p}}_{NW} = x\hat{\boldsymbol{\imath}} + (y+h)\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{p}}_{NE} = (x+h)\hat{\boldsymbol{\imath}} + (y+h)\hat{\boldsymbol{\jmath}},$$

where $h > 0$ is "very small". Of course, it has area equal to $|\vec{\mathbf{p}}_{SE} - \vec{\mathbf{p}}_{SW}||\vec{\mathbf{p}}_{NW} - \vec{\mathbf{p}}_{SW}| = h^2$. Under a transformation $(x,y) \mapsto (\xi,\eta)$, the first three vertices are mapped into

$$\begin{aligned}
\vec{\mathbf{q}}_{SW} &= \xi(x,y)\hat{\boldsymbol{\xi}} + \eta(x,y)\hat{\boldsymbol{\eta}}, \\
\vec{\mathbf{q}}_{SE} &= \xi(x+h,y)\hat{\boldsymbol{\xi}} + \eta(x+h,y)\hat{\boldsymbol{\eta}} \approx \Big(\xi(x,y) + h\frac{\partial\xi}{\partial x}(x,y)\Big)\hat{\boldsymbol{\xi}} + \Big(\eta(x,y) + h\frac{\partial\eta}{\partial x}(x,y)\Big)\hat{\boldsymbol{\eta}}, \\
\vec{\mathbf{q}}_{NW} &= \xi(x,y+h)\hat{\boldsymbol{\xi}} + \eta(x,y+h)\hat{\boldsymbol{\eta}} \approx \Big(\xi(x,y) + h\frac{\partial\xi}{\partial y}(x,y)\Big)\hat{\boldsymbol{\xi}} + \Big(\eta(x,y) + h\frac{\partial\eta}{\partial y}(x,y)\Big)\hat{\boldsymbol{\eta}},
\end{aligned}$$

where we have approximated the values of the fields ξ and η with their Taylor's expansions centred a $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (which is a good approximation if h is small).

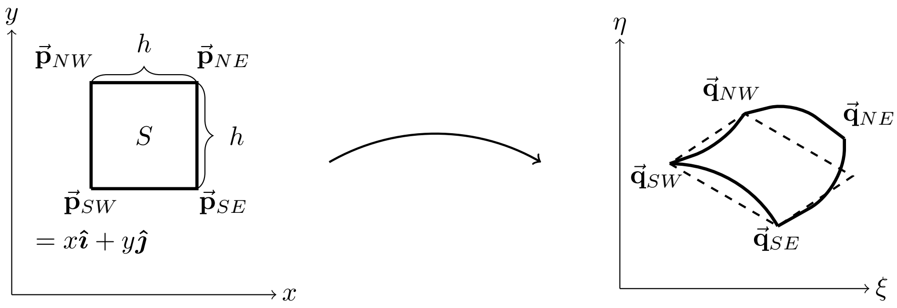


Figure 29: *As described in Remark 2.32, the square S of area h^2 in the xy-plane is mapped into a curvilinear shape in the $\xi\eta$-plane. The Jacobian determinant (multiplied by h^2) measures the area of the dashed parallelogram in the $\xi\eta$-plane, which approximates the area of the image of S.*

We then approximate the image of S in the $\xi\eta$-plane with the parallelogram with edges $[\vec{\mathbf{q}}_{SE}, \vec{\mathbf{q}}_{SW}]$ and $[\vec{\mathbf{q}}_{NW}, \vec{\mathbf{q}}_{SW}]$, see Figure 29. Using the geometric characterisation of the vector product as in Section 1.1.2, we see that this parallelogram has area equal to the magnitude of the vector product of two edges:

$$\begin{aligned}\mathrm{Area}\big(\vec{\mathbf{T}}(S)\big) \approx \big|(\vec{\mathbf{q}}_{SE}-\vec{\mathbf{q}}_{SW})\times(\vec{\mathbf{q}}_{NW}-\vec{\mathbf{q}}_{SW})\big| &= \left|\Big(h\frac{\partial\xi}{\partial x}\hat{\boldsymbol\xi}+h\frac{\partial\eta}{\partial x}\hat{\boldsymbol\eta}\Big)\times\Big(h\frac{\partial\xi}{\partial y}\hat{\boldsymbol\xi}+h\frac{\partial\eta}{\partial y}\hat{\boldsymbol\eta}\Big)\right|\\ &= h^2\left|\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}\right| \\ &\overset{(56)}{=} h^2\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|.\end{aligned} \tag{61}$$

Therefore, the Jacobian determinant is the ratio between the area of the parallelogram approximating the image of S, and the area of the square S itself. In the limit $h\to0$, the approximation error vanishes and the Jacobian determinant is thus equal to the ratio between the "infinitesimal areas" in the xy-plane and in the $\xi\eta$-plane. When we perform the change of variables $(x,y)\mapsto(\xi,\eta)$ inside an integral, in order to preserve the value of the integral itself, we need to replace the contribution given by every "infinitesimally small" square S with the contribution given by its transform, which is the same as multiplying with $\frac{\partial(\xi,\eta)}{\partial(x,y)}$.

To pass from an infinitesimal element to a finite domain, we use two-dimensional Riemann sums. Consider a domain R partitioned in N disjoint small squares $\{S_j\}_{j=1,\dots,N}$ with area h^2; for example R can be the unit square in the upper left plot of Figure 28, where $N=25$. The domain $\vec{\mathbf{T}}(R)$ obtained by a change of variables $(x,y)\mapsto(\xi,\eta)$ is composed of N curvilinear subdomains $\{\vec{\mathbf{T}}(S_j)\}_{j=1,\dots,N}$, see e.g. one of the other three plots in Figure 28. Denoting by $f_j := f(\vec{\mathbf{p}}_j)$ the value of the integrand f (a continuous scalar field) in a point $\vec{\mathbf{p}}_j\in S_j$, for all $1\le j\le N$, we can decompose the integral at the right-hand side of (57) over all the subdomains and approximate each of them with the integral of the constant f_j. Then, approximating the area of each subdomain with (61), we obtain an approximation of the change of variables formula (57):

$$\begin{aligned}\iint_{\vec{\mathbf{T}}(R)} f\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta &= \sum_{j=1}^N\iint_{\vec{\mathbf{T}}(S_j)} f\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta\\ &\approx \sum_{j=1}^N f_j\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}(\vec{\mathbf{p}}_j)\right|\mathrm{Area}\big(\vec{\mathbf{T}}(S_j)\big)\\ &\overset{(61)}{\approx}\sum_{j=1}^N f_j\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}(\vec{\mathbf{p}}_j)\right| h^2\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}(\vec{\mathbf{p}}_j)\right|\\ &\overset{(58)}{=}\sum_{j=1}^N f_j h^2 \quad\approx\quad \iint_R f\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

If the partition of R in subdomains is sufficiently regular, this approximate identity can be made rigorous and is an equality in the limit $N\to\infty$.

Example 2.33 (Change of variables for y-simple domains)**.** The change of variables formula (57) can be immediately applied for computing integrals in y-simple domains, as an alternative to (53). Indeed, the transformation with components

$$\xi(x,y)=x, \qquad \eta(x,y)=\frac{y-a(x)}{b(x)-a(x)}$$

maps the domain R of (52) into the rectangle

$$(x_L, x_R) \times (0,1) = \left\{\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}} \text{ s. t. } x_L < \xi < x_R,\ 0 < \eta < 1\right\}.$$

The inverse transformation is

$$x(\xi,\eta) = \xi, \qquad y(\xi,\eta) = \big(b(\xi) - a(\xi)\big)\eta + a(\xi),$$

whose Jacobian determinant reads

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} 1 & 0 \\ \eta b'(\xi) + (1-\eta)a'(\xi) & b(\xi) - a(\xi) \end{vmatrix} = b(\xi) - a(\xi).$$

For instance, we compute the integral of $f(x,y) = \frac{1}{y}$ in the "smile" domain (see left plot in Figure 30):

$$R = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2, \text{ s.t. } -1 < x < 1, \quad 2x^2 - 2 < y < x^2 - 1\Big\};$$

(note that $a(x) = 2x^2 - 2$, $b(x) = x^2 - 1$, $x_L = -1$ and $x_R = 1$, in the notation of (52)). Using the change of variables formula (57), we have

$$\begin{aligned}
\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{(-1,1)\times(0,1)} f\Big(x(\xi,\eta), y(\xi,\eta)\Big)\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta \\
&= \iint_{(-1,1)\times(0,1)} f\Big(\xi, \big(b(\xi) - a(\xi)\big)\eta + a(\xi)\Big)\big|b(\xi) - a(\xi)\big|\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \iint_{(-1,1)\times(0,1)} f\Big(\xi, (1-\xi^2)\eta + 2\xi^2 - 2\Big)(1-\xi^2)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \iint_{(-1,1)\times(0,1)} \frac{1}{(1-\xi^2)(\eta-2)}(1-\xi^2)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \int_{-1}^{1} \mathrm{d}\xi \int_0^1 \frac{-1}{2-\eta}\,\mathrm{d}\eta = 2\log(2-\eta)\Big|_0^1 = -2\log 2 \approx -1.386.
\end{aligned}$$

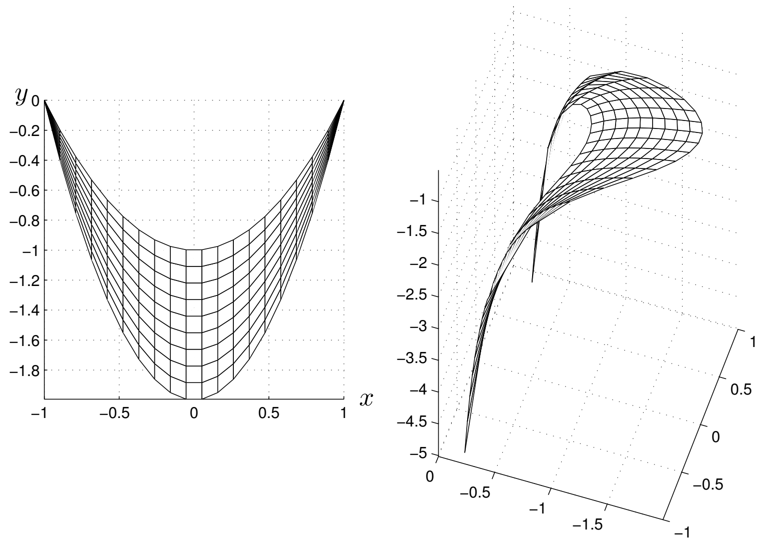


Figure 30: *The "smile domain" described in Example 2.33 (left plot) and the field $f = \frac{1}{y}$ (right plot). (Note that the values attained by the field approach $-\infty$ at the two "tips" of the domain, however the value of the integral $\int_D f\,\mathrm{d}x\,\mathrm{d}y$ is bounded since the tips are thin.)*

Example 2.34 (Integrals in domains bounded by level sets)**.** In order to apply the change of variable formula to the computation of areas of domains delimited by four curvilinear paths, it is sometimes useful to express the paths as level sets of some field. For instance, we want to compute the area of the domain R bounded by the parabolas

$$y = x^2, \qquad y = 2x^2, \qquad x = y^2, \qquad x = 3y^2$$

(see sketch in Figure 31). We note that these parabolas can be written as level curves for the two scalar fields

$$\xi(x,y) = \frac{x^2}{y} \qquad \text{and} \qquad \eta(x,y) = \frac{y^2}{x}.$$

In other words, ξ and η constitute a change of variables that transforms R in the rectangle $\frac{1}{2} < \xi < 1$ and $\frac{1}{3} < \eta < 1$ in the $\xi\eta$-plane. The corresponding Jacobian determinants are

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} 2\frac{x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & 2\frac{y}{x} \end{vmatrix} = 3, \qquad \Rightarrow \qquad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{3},$$

from which we compute the area of R

$$\iint_R \mathrm{d}x\,\mathrm{d}y = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{3}}^{1} \frac{1}{3}\,\mathrm{d}\eta\,\mathrm{d}\xi = \frac{1}{9}.$$

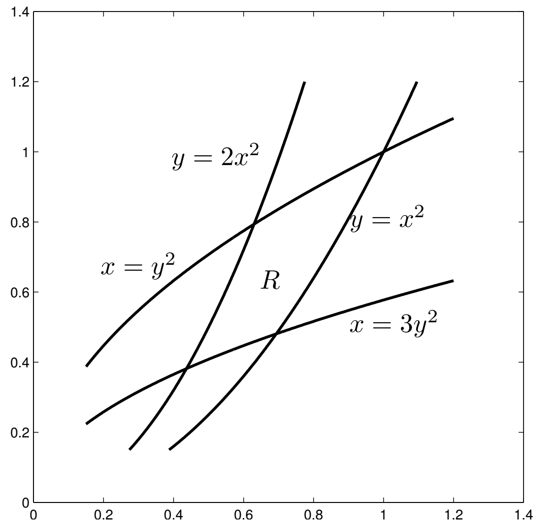


Figure 31: *The curvilinear quadrilateral R bounded by four parabolas as described in Example 2.34.*

Remark 2.35 (How to find a change of variables?)**.** So far we have seen several examples where a change of variables $\vec{\mathbf{T}}$ between two domains R and $\vec{\mathbf{T}}(R)$ was given. Often the explicit expression of $\vec{\mathbf{T}}$ is not known, and different exercises might be posed. Given two regions R and Q, how can we construct a suitable map $\vec{\mathbf{T}} : R \to Q$? Or, given a single domain R, how can we find a change of variable $\vec{\mathbf{T}}$ such that the image $\vec{\mathbf{T}}(R)$ is particularly simple to be used as domain of integration (typically a rectangle)?

The possible changes of variables are never unique, so it is not possible to give a general answer to these questions. (Note that this situation is very similar to that encountered for the parametrisation of a path, see Remark 1.24.) The only requirements on $\vec{\mathbf{T}}$ are that it must be bijective from R to Q and that the Jacobian matrices $J\vec{\mathbf{T}}$ and $J(\vec{\mathbf{T}}^{-1})$ are invertible in all points. Under these conditions, we usually require the change of variables to be "as simple as possible", in particular to have Jacobian determinants that (when multiplied with the given integrand, if this is given) are easy to integrate. Here we consider only some examples; to understand them you need to draw pictures of the domains involved.

1. A typical technique to construct changes of variables consists in matching the edges and the vertices of the two desired domains. For example, consider the triangles Q and $\vec{\mathbf{T}}(Q)$ in Figure 27. The line $\{y = 1 - x\}$ (or equivalently $\{x + y - 1 = 0\}$) is part of the boundary of Q and must be mapped to the line $\{\xi = 0\}$. A way to ensure this is to choose a change of coordinates such that $\xi = (x + y - 1)A$, with A to be chosen (in this case A will be a constant as the transformation is affine, more in general it might be a function $A(x, y)$). Similarly, since the line $\{y = 1 + x\}$ is mapped to the line $\{\eta = 0\}$, we have the condition $\eta = (x - y + 1)B$. To fix A and B we look at the vertices of the triangles: $\hat{\boldsymbol{\imath}}$ must be mapped to $\hat{\boldsymbol{\eta}}$, thus plugging its coordinates $x = 1$ and $y = 0$ into $\eta = (x - y + 1)B$ must give $\eta = 1$, which is possible if $B = 1/2$. Similarly, the vertex $-\hat{\boldsymbol{\imath}}$ must be mapped to $\hat{\boldsymbol{\xi}}$, giving $A = -1/2$. Substituting the values of A and B found, we obtain the change of variables (55).

2. We consider another similar example. Let $P = \{0 < x < 1, 0 < y < 2 - x\} \subset \mathbb{R}^2$ be the trapezoid with vertices $\vec{\mathbf{0}}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, 2\hat{\boldsymbol{\jmath}}$ (draw it). We look for a change of variables mapping P to the unit square $(0, 1)^2$. The vertical lines $\{x = 0\}$ and $\{x = 1\}$ are mapped into $\{\xi = 0\}$ and $\{\xi = 1\}$, so we can simply choose $\xi = x$. The horizontal line $\{y = 0\}$ is also preserved (i.e. mapped into $\{\eta = 0\}$) so we choose $\eta = yA$, where now A will depend on x. The edge with equation $\{y = 2 - x\}$ is mapped into $\{\eta = 1\}$, so we

have $1=\eta=yA=(2-x)A$ so $A(x)=1/(2-x)$ and the transformation, with its inverse, reads[20]

$$\begin{cases}\xi=x,\\ \eta=y/(2-x),\end{cases}\qquad\qquad \begin{cases}x=\xi,\\ y=(2-\xi)\eta.\end{cases}$$

3. If R has four edges, denoted $\Gamma_S,\Gamma_E,\Gamma_N$ and Γ_W, we can look for two two-dimensional continuous fields $\xi,\eta:R\to\mathbb{R}^2$ such that each edge corresponds to a level set:

$$\Gamma_S=\{\eta=c_S\},\qquad \Gamma_E=\{\xi=c_E\},\qquad \Gamma_N=\{\eta=c_N\},\qquad \Gamma_W=\{\xi=c_W\},$$

   where $c_S<c_N$ and $c_W<c_E$ are real numbers. Then, if the map $\xi(x,y)\hat{\boldsymbol{\xi}}+\eta(x,y)\hat{\boldsymbol{\eta}}$ is injective on R, it is a change of variables with image the rectangle $(c_W,c_E)\times(c_S,c_N)$. A prototypical case is described in Example 2.34.

4. If a domain is x- or y-simple (see (52)), it can be mapped to a rectangle with the procedure described in Example 2.33

5. If the domain has some circular symmetry, polar coordinates can be used, as described in Section 2.3.1.

Another instructive example is in Exercise 2.36.

Exercise 2.36 (Assignment 3, MA2VC 2013). ▶ Consider the unit square $S=(0,1)^2=\{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}},\ 0<x<1,\ 0<y<1\}$ and the triangle $T=\{\xi\hat{\boldsymbol{\xi}}+\eta\hat{\boldsymbol{\eta}},\ 0<\xi<\eta<1\}$ with vertices $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\eta}}$.
(i) Find a simple (bijective) change of variables $(x,y)\mapsto(\xi,\eta)$ that maps S into T.
Hint: consider a polynomial transformation that deforms the x variable only.
(ii) Use the change of variables you found to compute $\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$.

2.2.3 Triple integrals

All what we have said about double integrals immediately extends to triple integrals, namely integrals on three-dimensional domains. It is not possible to have an easy geometrical interpretation of triple integrals as that in Figure 25 for double integrals. The volume integral of a positive field f on a domain D may be understood as total mass of a body of shape D and density f.

As before, triple integrals can be computed as iterated integrals. For instance, consider a "z-simple" domain $D\subset\mathbb{R}^3$, defined as

$$D=\Big\{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}}\ \text{ s. t. }\ x_L<x<x_R,\quad a(x)<y<b(x),\quad \alpha(x,y)<z<\beta(x,y)\Big\}$$

for two real numbers x_L and x_R, two real functions a and b, and two scalar fields (in two variables) α and β. Then, the integral in D of a scalar field f can be written as

$$\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_{x_L}^{x_R}\left(\int_{a(x)}^{b(x)}\left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x. \qquad (62)$$

The infinitesimal volume element is defined as $\mathrm{d}V=\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

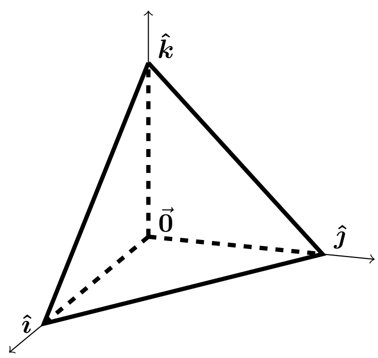


Figure 32: *The tetrahedron with vertices* $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ *and* $\hat{\boldsymbol{k}}$.

For example, the tetrahedron B with vertices $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ depicted in Figure 32 can be written as

$$B=\Big\{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}}\ \text{ s. t. }\ 0<x<1,\quad 0<y<1-x,\quad 0<z<1-x-y\Big\}.$$

[20] You can visualise this change of coordinates in Matlab/Octave using the function `VCplotter.m` (available on the course web page), with the command: `VCplotter(6, @(x,y) x, @(x,y) y*(2-x),0,1,0,1);`

In this example, $x_L = 0$, $x_R = 1$, $a(x) = 0$, $b(x) = 1 - x$, $\alpha(x, y) = 0$ and $\beta(x, y) = 1 - x - y$.

The change of variable formula (57) extends to three dimensions in a straightforward fashion. If the transformation $\vec{\mathbf{T}} : D \to \vec{\mathbf{T}}(D)$ has components

$$\xi(x, y, z), \qquad \eta(x, y, z), \qquad \sigma(x, y, z),$$

and its Jacobian determinant is

$$\frac{\partial(\xi, \eta, \sigma)}{\partial(x, y, z)} = \det(J\vec{\mathbf{T}}) = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \sigma}{\partial x} & \frac{\partial \sigma}{\partial y} & \frac{\partial \sigma}{\partial z} \end{vmatrix},$$

then the change of variable formula reads

$$\boxed{\iiint_D f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \int_{\vec{\mathbf{T}}(D)} f\big(x(\xi, \eta, \sigma), y(\xi, \eta, \sigma), z(\xi, \eta, \sigma)\big) \left| \frac{\partial(x, y, z)}{\partial(\xi, \eta, \sigma)} \right| \mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\sigma,} \tag{63}$$

for any scalar field f defined in D.

Returning to the previous example, the tetrahedron B is the image of the unit cube $0 < x, y, z < 1$ under the transformation $\vec{\mathbf{T}}$ with components

$$\xi = (1 - z)(1 - y)x, \qquad \eta = (1 - z)y, \qquad \sigma = z. \tag{64}$$

The corresponding Jacobian is

$$\frac{\partial(\xi, \eta, \sigma)}{\partial(x, y, z)} = \det(J\vec{\mathbf{T}}) = \begin{vmatrix} (1 - z)(1 - y) & -(1 - z)x & -(1 - y)x \\ 0 & (1 - z) & -y \\ 0 & 0 & 1 \end{vmatrix} = (1 - y)(1 - z)^2.$$

In order to demonstrate the formulas of this section, we compute the volume of the tetrahedron B integrating the constant 1 first with the iterated integral (62) and then with the change of variables (63):

$$\begin{aligned}
\mathrm{Vol}(B) &= \iiint_B 1 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&\overset{(62)}{=} \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^1 \left(\int_0^{1-x} (1 - x - y) \,\mathrm{d}y \right) \mathrm{d}x = \int_0^1 \frac{1}{2}(1 - x)^2 \,\mathrm{d}x = \frac{1}{2}\Big(x - x^2 + \frac{1}{3}x^3 \Big)\Big|_0^1 = \frac{1}{6}, \\
\mathrm{Vol}(B) &= \iiint_B 1 \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\sigma \\
&\overset{(63)}{=} \iiint_{\vec{\mathbf{T}}^{-1}(B)} \frac{\partial(\xi, \eta, \sigma)}{\partial(x, y, z)} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= \int_0^1 \int_0^1 \int_0^1 (1 - y)(1 - z)^2 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \left(\int_0^1 \mathrm{d}x \right) \left(\int_0^1 (1 - y) \,\mathrm{d}y \right) \left(\int_0^1 (1 - z)^2 \,\mathrm{d}z \right) = 1 \, \frac{1}{2} \, \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

2.2.4 Surface integrals

There is another kind of integrals we have not encountered yet: we have seen integrals along curves, on flat surfaces, and on volumes, what is still missing are integrals on curved surfaces. This section introduces surface integrals of scalar fields; Section 2.2.5 considers surface integrals of vector fields.

Comparison with scalar calculus 2.37. Surfaces might be thought as the two-dimensional generalisation of paths: as paths are "curvilinear segments", surfaces are "curvilinear planar regions". Intuitively, paths and surfaces are one- and two-dimensional subsets of $\mathbb{R}^3$, respectively. As paths can be defined by two equations in x, y and z, surfaces can be written with a single equation, e.g.

$$\begin{aligned}
\big\{ \vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + y^2 = 1,\ z = 0 \big\} &\quad \text{is a path (the unit circle)}, \\
\big\{ \vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1 \big\} &\quad \text{is a surface (the unit sphere)}.
\end{aligned}$$

However, we have seen in Section 2.1 that to compute integrals along a path Γ we need a parametrisation $\vec{\mathbf{a}}$, i.e. a bijective function mapping an interval to the path itself. Similarly, computing surface integrals will require a parametrisation, which in this case will be a map (denoted $\vec{\mathbf{X}}$) from a planar region to the surface itself.

We will consider surfaces satisfying the following definition.

Definition 2.38. We say that a set $S \subset \mathbb{R}^3$ is a **parametric surface** if there exist a two-dimensional open region $R \subset \mathbb{R}^2$ and a continuous, injective vector field $\vec{\mathbf{X}} : R \to \mathbb{R}^3$ such that S is the image of R under $\vec{\mathbf{X}}$, i.e. $S = \{\vec{\mathbf{r}} \in \mathbb{R}^3$ such that $\vec{\mathbf{r}} = \vec{\mathbf{X}}(\vec{\mathbf{p}})$, for some $\vec{\mathbf{p}} \in R\}$. The field $\vec{\mathbf{F}}$ is called **parametrisation**, or **chart**, of S.

We always tacitly assume that the field $\vec{\mathbf{X}}$ is smooth. For a given (non-empty) surface S there exist infinitely many different parametrisations $\vec{\mathbf{X}}$. The assumption that $\vec{\mathbf{X}}$ be injective (i.e. if $\vec{\mathbf{p}}, \vec{\mathbf{q}} \in R$ are different $\vec{\mathbf{p}} \neq \vec{\mathbf{q}}$, then they have different images $\vec{\mathbf{X}}(\vec{\mathbf{p}}) \neq \vec{\mathbf{X}}(\vec{\mathbf{q}})$) is needed to avoid self-intersections of S.

Remark 2.39. The three components X_1, X_2 and X_3 of the field $\vec{\mathbf{X}}$ are the components of the points $\vec{\mathbf{r}} \in S$, thus to avoid confusion we will usually denote the two input variables (living in $R \subset \mathbb{R}^2$) of $\vec{\mathbf{X}}$ with the letters u and w instead of x and y. So we will write $\vec{\mathbf{X}}(u, w) = \vec{\mathbf{X}}(u\hat{\boldsymbol{u}} + w\hat{\boldsymbol{w}}) = X_1(u, w)\hat{\boldsymbol{\imath}} + X_2(u, w)\hat{\boldsymbol{\jmath}} + X_3(u, w)\hat{\boldsymbol{k}}$. The scalars u and w play a similar role to t for curves, but being two we can not interpret them as time.

Example 2.40. The graph of the scalar field in two variables $f = x^2 - y^2$ in Figure 5 is a surface and is defined by the equation $S = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ z = x^2 - y^2\}$. Thus a chart for it is $\vec{\mathbf{X}}(u, w) = u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} + (u^2 - w^2)\hat{\boldsymbol{k}}$. If we want to consider the whole graph of f as an unbounded surface we take the region R as the whole of $\mathbb{R}^2$. If we want to define S to be only the bounded portion shown in Figure 5, we choose the region R to be the square $(-2, 2) \times (-2, 2) = \{|u|, |w| < 2\}$. If we want S to look like a Pringles crisp we choose a circular region $R = \{u^2 + w^2 < 1\}$.

Example 2.41. The unit sphere $\{x^2 + y^2 + z^2 = 1\}$ is clearly a surface, but can not be written in the simple framework of Definition 2.38. However the "punctured unit sphere" $\{x^2 + y^2 + z^2 = 1, z \neq 1\}$ i.e. the unit sphere with the north pole point removed is a parametric surface. The chart $\vec{\mathbf{X}} : \mathbb{R}^2 \to S$ is the famous "stereographic projection" (see Figure 33):

$$\vec{\mathbf{X}}(u, w) = \frac{u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} + \left(\frac{u^2}{4} + \frac{w^2}{4} - 1\right)\hat{\boldsymbol{k}}}{1 + \frac{u^2}{4} + \frac{w^2}{4}}. \tag{65}$$

Exercise 2.42. ▶ Show that the stereographic projection (65) maps $R = \mathbb{R}^2$ into the punctured unit sphere. Compute the inverse map $\vec{\mathbf{X}}^{-1} : S \to \mathbb{R}^2$.

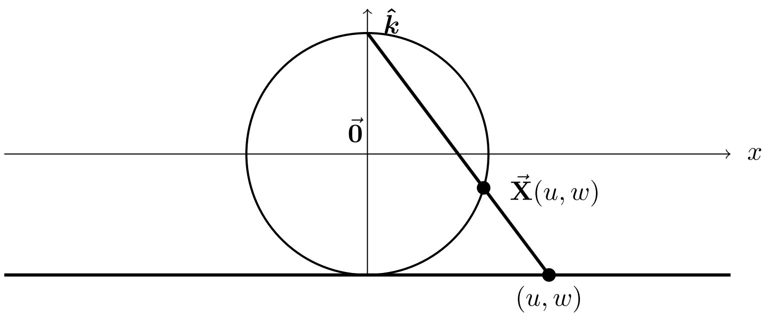


Figure 33: *A representation of the stereographic projection $\vec{\mathbf{X}} : \mathbb{R}^2 \to S$ in (65), S being the punctured unit sphere. The figure shows the vertical section $\{y = 0\}$. We identify the uw-plane with the horizontal plane $\{z = -1\}$ (the lower horizontal line in the figure). A point $\vec{\mathbf{p}} = u\hat{\boldsymbol{u}} + w\hat{\boldsymbol{w}} = u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$ in this plane is mapped by $\vec{\mathbf{X}}$ into the intersection between the unit sphere and the segment connecting $\vec{\mathbf{p}}$ itself to the north pole $\hat{\boldsymbol{k}}$. The point $u = 0, w = 0$ is mapped to the south pole $-\hat{\boldsymbol{k}}$, the points in the circle $\{u^2 + w^2 = 2\}$ are mapped to the equator $\{x^2 + y^2 = 1, z = 0\}$, and the point "at infinity" is mapped to the north pole $\vec{\mathbf{k}}$.*

Exercise 2.43. ▶ Fix two numbers $0 < b < a$. Try to draw the surface defined by the chart

$$\vec{\mathbf{X}}(u, w) = (a + b\cos w)\cos u\hat{\boldsymbol{\imath}} + (a + b\cos w)\sin u\hat{\boldsymbol{\jmath}} + b\sin w\hat{\boldsymbol{k}} \qquad -\pi < u < \pi, \qquad -\pi < w < \pi.$$

From Example 2.41 and Exercise 2.43 we see that some simple surfaces, such as the sphere, cannot be written as parametric surfaces according to Definition 2.38. A more general definition could be given to include these surfaces but it is beyond the scope of this course. However, Example 2.40 suggests that this definition includes a very important class of surfaces, those defined as graphs of fields in two variables. We describe them in the next very important remark.

Remark 2.44. Given a region $R \subset \mathbb{R}^2$ and a planar scalar field $g : R \to \mathbb{R}$, the **graph** of g is the surface

$$S_g = \left\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in R,\ z = g(x,y)\right\} \subset \mathbb{R}^3. \tag{66}$$

A chart for S_g is given by the vector field $\vec{\mathbf{X}}_g(u,w) = u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} + g(u,w)\hat{\boldsymbol{k}}$, which is defined on R. In this case, we can identify u with x and w with y. The main property of a graph surface is that for each point $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ in the planar region R there is *exactly one* point in S that lies in the vertical line through $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, namely the point $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + h(x,y)\hat{\boldsymbol{k}}$.

A graph surface can be written in brief as $S_g = \{z = g(x,y)\}$, meaning that S_g is the set of the points $\vec{\mathbf{r}}$ whose coordinates x, y and z are solutions of the equation in braces. Some examples of graph surfaces are the paraboloid of revolution $\{z = x^2 + y^2\}$ (e.g. a satellite dish is a portion of it), the hyperbolic paraboloid $\{z = x^2 - y^2\}$ of Example 2.40, the upper half sphere $\{z = \sqrt{1 - x^2 - y^2},\ \ x^2 + y^2 < 1\}$. The entire sphere is not a graph, since to each point $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ in the plane, with $x^2 + y^2 < 1$, correspond two different points on the sphere; on the other hand the sphere can be written as union of two graphs, the north and south hemispheres.

Given a surface S defined by the chart $\vec{\mathbf{X}}$ and the region R, and given a point $\vec{\mathbf{r}}_0 = \vec{\mathbf{X}}(u_0, w_0) \in S$, the partial derivatives of the vector field $\vec{\mathbf{F}}$, i.e. $\frac{\partial \vec{\mathbf{F}}}{\partial u}(u_0, w_0)$ and $\frac{\partial \vec{\mathbf{F}}}{\partial w}(u_0, w_0)$ represent two **tangent vectors** to S. Assuming that the tangent vectors $\frac{\partial \vec{\mathbf{F}}}{\partial u}$ and $\frac{\partial \vec{\mathbf{F}}}{\partial w}$ are linearly independent, it follows that their vector product $\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}$ is perpendicular to S.

Exercise 2.45. ▶ Consider the surface graph S_g for the field $g(x,y) = 16xy(1-x)(1-y)$ defined in $R = (0,1)^2 = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ 0 < x, y < 1\}$. Draw S_g and compute its tangent vector fields.

★ **Remark 2.46** (Tangent plane to a surface)**.** Given a parametric surface S and a point $\vec{\mathbf{r}}_0 = \vec{\mathbf{X}}(u_0, w_0) \in S$, the plane

$$TS_{\vec{\mathbf{r}}_0} = \left\{\vec{\mathbf{X}}(u_0, w_0) + u\frac{\partial \vec{\mathbf{X}}}{\partial u}(u_0, w_0) + w\frac{\partial \vec{\mathbf{X}}}{\partial w}(u_0, w_0),\ \text{for all } u, w \in \mathbb{R}\right\}$$

is called the **tangent plane** to S in $\vec{\mathbf{r}}_0$. When we say that a vector is perpendicular to a surface S in a point $\vec{\mathbf{r}}_0$, what we actually mean is that it is perpendicular to $TS_{\vec{\mathbf{r}}_0}$. If $\vec{\mathbf{r}}_0$ is placed on a vertex or an edge of S, then $\vec{\mathbf{X}}$ is not differentiable at $u_0\hat{\boldsymbol{u}} + w_0\hat{\boldsymbol{w}}$ and no tangent plane nor perpendicular vectors can be defined.

For a graph surface S_g as in (66), using the chart $\vec{\mathbf{X}}_g(u,w) = u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} + g(u,w)\hat{\boldsymbol{k}}$, we see that the tangent vectors are $\frac{\partial \vec{\mathbf{X}}_g}{\partial u} = \hat{\boldsymbol{\imath}} + \frac{\partial g}{\partial u}\hat{\boldsymbol{k}}$ and $\frac{\partial \vec{\mathbf{X}}_g}{\partial w} = \hat{\boldsymbol{\jmath}} + \frac{\partial g}{\partial w}\hat{\boldsymbol{k}}$. Thus, renaming $x = u + u_0$, $y = w + w_0$, $x_0 = u_0$, $y_0 = w_0$, the tangent plane can be written as

$$\left\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \Big(g(x_0, y_0) + \vec{\nabla} g(x_0, y_0) \cdot \big((x - x_0)\hat{\boldsymbol{\imath}} + (y - y_0)\hat{\boldsymbol{\jmath}}\big)\Big)\hat{\boldsymbol{k}},\ \text{for all } x, y \in \mathbb{R}\right\}.$$

This coincides with the formula already obtained in Remark 1.40.

We now have all the tools needed to compute integrals over surfaces. Given a scalar field f defined on a parametric surface S, given by the chart $\vec{\mathbf{X}}$ and the region R, the integral of f over S is

$$\boxed{\iint_S f \,\mathrm{d}S \ = \ \iint_R f(\vec{\mathbf{X}})\left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| \mathrm{d}A \ = \ \iint_R f\big(\vec{\mathbf{X}}(u,w)\big)\left|\frac{\partial \vec{\mathbf{X}}}{\partial u}(u,w) \times \frac{\partial \vec{\mathbf{X}}}{\partial w}(u,w)\right| \mathrm{d}u \,\mathrm{d}w.} \tag{67}$$

where the integral at the right-hand side is a double integral on the (flat) region R as those studied in Section 2.2.1. Note that we used the symbol $\iint$, since the domain of integration is two-dimensional.

The symbol $\mathrm{d}S$ denotes the infinitesimal area element on S; it is the "curvilinear analogue" of $\mathrm{d}A$ or the "two-dimensional analogue" of $\mathrm{d}s$. Formula (67) states that the ratio between the area element on S and that on the parametric region R equals the magnitude of the vector product between the partial derivatives of the chart.

The surface integral (67) is independent of the chart: if $\vec{\mathbf{X}} : R \to S$ and $\vec{\mathbf{Y}} : Q \to S$ are different charts of the same surface S, then it is possible to prove[21] that $\iint_R f(\vec{\mathbf{X}})|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}|\,\mathrm{d}A = \iint_Q f(\vec{\mathbf{Y}})|\frac{\partial \vec{\mathbf{Y}}}{\partial u} \times \frac{\partial \vec{\mathbf{Y}}}{\partial w}|\,\mathrm{d}A$, thus the notation $\iint_S f\,\mathrm{d}S$ is well-defined (recall Example 2.3 for path integrals).

We can easily specialise formula (68) to the case of a graph surface. If S_g is the graph of a field g as in (66), the chart $\vec{\mathbf{X}}(u,w) = u\hat{\boldsymbol{\imath}} + w\hat{\boldsymbol{\jmath}} + g(u,w)\hat{\boldsymbol{k}}$ has derivatives

$$\frac{\partial \vec{\mathbf{X}}}{\partial u} = \hat{\boldsymbol{\imath}} + \frac{\partial g}{\partial u}\hat{\boldsymbol{k}}, \qquad \frac{\partial \vec{\mathbf{X}}}{\partial w} = \hat{\boldsymbol{\jmath}} + \frac{\partial g}{\partial w}\hat{\boldsymbol{k}}, \quad \Rightarrow \quad \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} = -\frac{\partial g}{\partial u}\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial w}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}.$$

[21] This follows from the fact that $\vec{\mathbf{Y}}^{-1} \circ \vec{\mathbf{X}} : R \to Q$ is a change of variables.

Plugging this into the surface integral formula (67) and identifying $u = x$ and $w = y$, we have that the integral on S_g of a scalar field f can be computed as

$$\iint_{S_g} f \, \mathrm{d}S = \iint_R f \sqrt{1 + |\vec{\nabla} g|^2} \, \mathrm{d}A = \iint_R f\big(x, y, g(x, y)\big) \sqrt{1 + \Big(\frac{\partial g}{\partial x}(x, y)\Big)^2 + \Big(\frac{\partial g}{\partial y}(x, y)\Big)^2} \, \mathrm{d}A, \qquad (68)$$

The area of a parametric surface (or of a graph) are computed by integrating the constant scalar field $f = 1$ (recall the basic properties of integrals in (40) and the path length formula (43)):

$$\mathrm{Area}(S) = \iint_R \left| \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} \right| \mathrm{d}u \, \mathrm{d}w, \qquad \mathrm{Area}(S_g) = \iint_R \sqrt{1 + |\vec{\nabla} g|^2} \, \mathrm{d}x \, \mathrm{d}y. \qquad (69)$$

Remark 2.47. The measure factor $\sqrt{1 + |\vec{\nabla} g|^2}$ in (68) should recall you the similar coefficient $\sqrt{1 + (g'(t))^2}$ we found in the computation of integrals along the graph of a real function in Example 2.5.

⋆ **Remark 2.48** (Relation between change of variables and surface integrals)**.** A two-dimensional change of variables $\vec{\mathbf{T}} : R \to \vec{\mathbf{T}}(R)$ can be considered the chart of a "flat parametric surface" by identifying $\vec{\mathbf{X}} = \vec{\mathbf{T}}$, $u = \xi$, $w = \xi$. Since $X_3 = 0$, we have $\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} = (\frac{\partial T_1}{\partial \xi} \frac{\partial T_2}{\partial \eta} - \frac{\partial T_2}{\partial \xi} \frac{\partial T_1}{\partial \eta}) \hat{\boldsymbol{k}} = \frac{\partial(x,y)}{\partial(\xi,\eta)}$, so the change of variables formula (57) is a special case of the surface integral formula (67). Thus a justification of the latter follows along the lines of Remark 2.32. (In particular formula (61) shows how the vector product enters the area element.)

Exercise 2.49. ▶ Compute the area of the parametric surface defined by the chart $\vec{\mathbf{X}}(u, w) = (u + w)\hat{\boldsymbol{\imath}} + (u - w)\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ on the region $R = (0, 1)^2 = 0 < u, w < 1$.

Exercise 2.50. ▶ Compute the area of the surface

$$S = \Big\{ \vec{\mathbf{r}} \in \mathbb{R}^3, \ z = \frac{2}{3}\big(x^{\frac{3}{2}} + y^{\frac{3}{2}}\big), \ 0 < x < 1, \ 0 < y < 1 \Big\}.$$

Exercise 2.51. ▶ *(i)* Compute the area of the parametric surface defined in Exercise 2.43. You can choose $a = 2$ and $b = 1$ for simplicity.

(ii) Compute the integral on S of the field $f(\vec{\mathbf{r}}) = (x^2 + y^2)^{-1/2}$.

We will examine other examples of surface integrals in the next sections.

2.2.5 Unit normal fields, orientations and fluxes (surface integrals of vector fields)

Remark 2.52 (Physical motivation for the definition of flux)**.** Imagine a pipe containing a fluid which moves with stationary (i.e. independent of time) velocity $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ at each point $\vec{\mathbf{r}}$. Denote by S a section of the pipe, which we model as a surface. Given $\vec{\mathbf{F}}$ and S, how can we compute the flux through the pipe, i.e. the volume of fluid passing through the section S in a unit of time? This must be an integral, summing the contributions from all portions of the section. Moreover, it must be independent of the tangential component of $\vec{\mathbf{F}}$ on S, as particles moving tangentially do not cross the surface. We also need to be able to distinguish the amount of flux crossing S in one direction from that crossing it in the opposite direction, thus to distinguish the two "sides" of S. This situation motivates the following definitions.

(We consider for a moment general surfaces, even those not included in Definition 2.38.) For every point $\vec{\mathbf{r}}$ on a smooth surface S, it is possible to define exactly two unit vectors $\hat{\boldsymbol{n}}_A$ and $\hat{\boldsymbol{n}}_B$ that are orthogonal to S in $\vec{\mathbf{r}}$. These two vectors are opposite to each other, i.e. $\hat{\boldsymbol{n}}_A = -\hat{\boldsymbol{n}}_B$, and they are called **unit normal vectors**. If for every point $\vec{\mathbf{r}} \in S$ we fix a unit normal vector $\hat{\boldsymbol{n}}(\vec{\mathbf{r}})$ in a continuous fashion (i.e. $\hat{\boldsymbol{n}}$ is a continuous unit normal vector field defined on S), then the pair $(S, \hat{\boldsymbol{n}})$ is called **oriented surface**. Note that the pairs $(S, \hat{\boldsymbol{n}})$ and $(S, -\hat{\boldsymbol{n}})$ are two *different* oriented surfaces, even though they occupy the same portion of the space.[22]

Not all surfaces admit a continuous unit normal vector field, the most famous example of non-orientable surface is the Möbius strip shown in Figure 34. On the other hand, the surfaces in the following three important families admit an orientation.

[22]Oriented surfaces share many properties with the oriented paths seen in Section 2.1. Also in that case, a path Γ supports two different orientations, corresponding to travel directions, leading to two different oriented paths. In both cases, the orientation is important to integrate vector fields, while is not relevant for the integration of scalar fields; can you guess why?

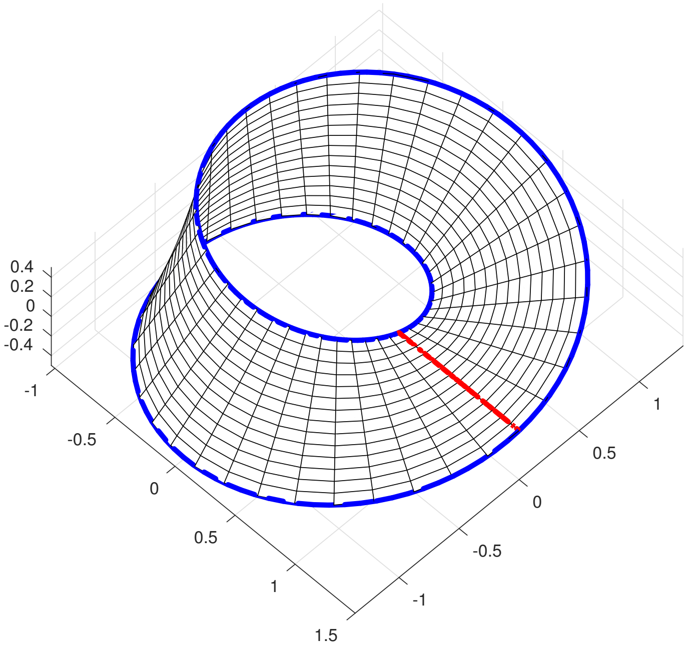


Figure 34: *The Möbius strip. This surface is not orientable: if we continuously move a unit normal vector along the surface, after one turn it will point in the direction opposite to the direction it started from. You can easily build a Möbius strip with a strip of paper.*
The chart $\vec{\mathbf{X}} = (1 + w\cos(u/2))\cos u\,\hat{\boldsymbol{\imath}} + (1 + w\cos(u/2))\sin u\,\hat{\boldsymbol{\jmath}} + w\sin(u/2)\hat{\boldsymbol{k}}$ *for* $0 < u < 2\pi$ *and* $-0.5 < w < 0.5$ *gives the Möbius strip without the the segment* $\{0.5 < x < 1.5,\ y = z = 0\}$ *(in red in the figure), corresponding to* $u = 0$*. Indeed, if we cut the surface along this segment, we cannot complete a turn along the surface itself and the surface obtained is orientable.*

(i) If S is a parametric surface as in Definition 2.38 with chart $\vec{\mathbf{X}} : R \to S$, then it is orientable with unit normal vector

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \frac{\dfrac{\partial\vec{\mathbf{X}}}{\partial u} \times \dfrac{\partial\vec{\mathbf{X}}}{\partial w}}{\left|\dfrac{\partial\vec{\mathbf{X}}}{\partial u} \times \dfrac{\partial\vec{\mathbf{X}}}{\partial w}\right|}. \tag{70}$$

A special case is when S_g is the **graph** of a two-dimensional scalar field g as in (66), then (70) reads

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \frac{-\dfrac{\partial g}{\partial x}\hat{\boldsymbol{\imath}} - \dfrac{\partial g}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}. \tag{71}$$

Since the z-component of $\hat{\boldsymbol{n}}$ in this formula is positive, this is the unit normal that points upwards.

(ii) Another family of surfaces admitting a unit normal vector fields are the **boundaries** of three-dimensional domains. The boundary of a domain $D \subset \mathbb{R}^3$ is commonly denoted by ∂D and is a surface (if D is "smooth enough"). In this case, we usually fix the orientation on ∂D by choosing the **outward-pointing unit normal** vector field.

(iii) If the surface S is the **level surface** of a smooth scalar field f satisfying $\vec{\nabla} f \neq \vec{\mathbf{0}}$ near S (see Section 1.2.1), then $\hat{\boldsymbol{n}} = \vec{\nabla} f/|\vec{\nabla} f|$ is an admissible unit normal field and $(S, \hat{\boldsymbol{n}})$ is an oriented surface[23]. This is a consequence of Part 4 of Proposition 1.33. We have seen a special case of this situation in Example 1.38. These surfaces are defined by the equation $S = \{f(\vec{\mathbf{r}}) = \lambda\}$.

Note that if S is the graph of g as in item (i), then it is also the level set of the field $f(x, y, z) = z - g(x, y)$, whose gradient is $\vec{\nabla} f = -\frac{\partial g}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$. So, the unit normal vector fields in (71) is a special case of both (70) and of item (iii) of the list above.

[23] Can you imagine what happens if $\vec{\nabla} f = \vec{\mathbf{0}}$?

★ **Remark 2.53** (Normal unit vectors on piecewise smooth surfaces)**.** The situation is more complicated when the surface S is not smooth but only "piecewise smooth", for instance is the boundary of a polyhedron. In this case it is not possible to define a *continuous* unit normal vector field. For instance, on the boundary $\partial C = \{\max\{|x|,|y|,|z|\} = 1\}$ of the unit cube $C = \{\max\{|x|,|y|,|z|\} \leq 1\}$, the outward-pointing unit normal vectors on two faces meeting at an edge are orthogonal to each other, so when crossing the edge they suddenly "jump", i.e. they are not continuous. However, it is possible to give a precise definition of orientation also in this case, formalising the idea that $\hat{\boldsymbol{n}}$ stays on "the same side" of S (see [1, page 881]). In all practical cases, with a bit of geometric intuition it should be clear how to define a normal field in such a way that it stays on the same side of the surface (and so it determines the surface orientation).

Given an oriented surface $(S, \hat{\boldsymbol{n}})$ and a vector field $\vec{\mathbf{F}}$ defined on S we define the **flux** of $\vec{\mathbf{F}}$ through $(S, \hat{\boldsymbol{n}})$ as the value of the integral over S of the normal component of $\vec{\mathbf{F}}$:

$$\boxed{\text{flux of } \vec{\mathbf{F}} \text{ through } (S,\hat{\boldsymbol{n}}) := \iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} := \iint_S \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S.} \tag{72}$$

Note that the integrand $\vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}$ is nothing else than a scalar field defined on S, so the flux can be computed as a surface integral studied in Section 2.2.4.

Combining the integral formula (67) and the expression (70) of the unit normal vector, we immediately obtain the formula for the flux of a vector field through a parametric surface S with chart $\vec{\mathbf{X}} : R \to S$:

$$\boxed{\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_R \vec{\mathbf{F}} \cdot \Big(\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} \Big) \, \mathrm{d}A} \tag{73}$$
$$= \iint_R \vec{\mathbf{F}}\big(\vec{\mathbf{X}}(u,w)\big) \cdot \Big(\frac{\partial \vec{\mathbf{X}}}{\partial u}(u,w) \times \frac{\partial \vec{\mathbf{X}}}{\partial w}(u,w) \Big) \, \mathrm{d}u \, \mathrm{d}w.$$

Similarly, on the graph surface (66) of g, the area element $\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}$ in (68) and the denominator in the unit normal (71) cancel each other and the integral in the flux (72) simplifies to[24]

$$\boxed{\iint_{S_g} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_R \Big(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \Big) \, \mathrm{d}A} \tag{74}$$
$$= \iint_R \Big(-F_1\big(x,y,g(x,y)\big) \frac{\partial g}{\partial x}(x,y) - F_2\big(x,y,g(x,y)\big) \frac{\partial g}{\partial y}(x,y) + F_3\big(x,y,g(x,y)\big) \Big) \, \mathrm{d}x \, \mathrm{d}y.$$

If the surface is a domain boundary ∂D, the surface integral and the flux are often denoted by the symbols

$$\oiint_{\partial D} f \, \mathrm{d}S \qquad \text{and} \qquad \oiint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}.$$

Remark 2.54. The flux $\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$ measures "how much" of the vector field $\vec{\mathbf{F}}$ crosses the surface S in the direction given by $\hat{\boldsymbol{n}}$. If $\vec{\mathbf{F}}$ is tangent to the surface in every point then the flux is zero.

Exercise 2.55. ▶ Compute the flux of the vector field $\vec{\mathbf{F}} = \vec{\mathbf{r}} \times \hat{\boldsymbol{\imath}} = z\hat{\boldsymbol{\jmath}} - y\hat{\boldsymbol{k}}$ through the hyperbolic paraboloid $S = \{z = x^2 - y^2,\ 0 < x, y < 1\}$ (see Figure 5).

Exercise 2.56. ▶ Consider the graph surface S_g of Exercise 2.45 (i.e. $g(x,y) = 16x(1-x)y(1-y)$ and $R = (0,1)^2$). Let $\vec{\mathbf{F}} = \hat{\boldsymbol{\imath}}$ and $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}}$ be two vector fields. Equip S_g with the unit normal vector of (71) (you do not need to compute $\hat{\boldsymbol{n}}$!). Compute the fluxes of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ through S_g.

Can you guess the value of one of this fluxes and the sign of the other before computing them?

Exercise 2.57. ▶ Compute the flux of the position vector $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}$ through the parametric surface of Exercise 2.43. You can assume again $a = 2$ and $b = 1$. Hint: the computations already done in Exercise 2.51 might be helpful.

A surface S may have a boundary ∂S, which is the path of a curve. For instance, the boundary of the upper half sphere $S = \{|\vec{\mathbf{r}}| = 1, z \geq 0\}$ is the unit circle $\partial S = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, x^2 + y^2 = 1\}$ (the equator), while the complete sphere $\{|\vec{\mathbf{r}}| = 1\}$ has no boundary. The boundary of the open cylindrical surface $C = \{x^2 + y^2 = 1,\ |z| < 1\}$ is composed of the two circles $\partial C = \{x^2 + y^2 = 1,\ z = \pm 1\}$.

[24] Note that when we write, for example, $\iint_R F_1 \frac{\partial g}{\partial x} \, \mathrm{d}A$ what we actually mean is the integral over the region R of $F_1\big(x,y,g(x,y)\big) \frac{\partial g}{\partial x}(x,y)$: the two dimensional field g (giving the shape of the surface S) is defined on $R \subset \mathbb{R}^2$, while the three-dimensional field F_1 is defined only on the surface $S \subset \mathbb{R}^3$, whose points can be written as $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + g(x,y)\hat{\boldsymbol{k}}$.

We have defined orientations for both paths and surfaces, we now relate the orientation of a surface with the that of its boundary. An oriented surface $(S, \hat{\boldsymbol{n}})$ "**induces an orientation**" on its boundary, i.e. there is a standard way of choosing one of the two possible path orientations of its boundary. **Given S and $\hat{\boldsymbol{n}}$, the induced path orientation on ∂S is the travel direction such that walking along ∂S, on the side of S defined by $\hat{\boldsymbol{n}}$, we leave S itself on our left.** This definition may be very confusing, try to visualise it with a concrete example, e.g. using a sheet of paper as model for a surface; see also Figure 35.

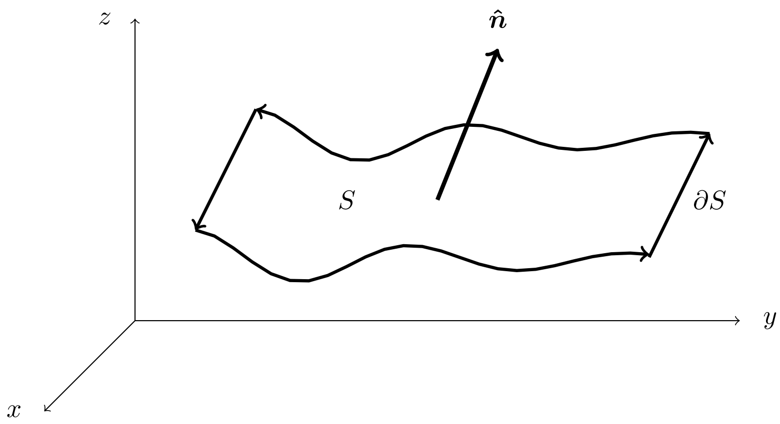


Figure 35: *The relation between the orientation of an oriented surface, given by the continuous unit normal field $\hat{\boldsymbol{n}}$, and the orientation of the boundary ∂S (the direction of travel along the path). Imagine to walk along the path ∂S, according to the arrows, and staying on the same side of the vector $\hat{\boldsymbol{n}}$ (i.e. above the surface). Then S is always at our left. So the path-orientation of ∂S is that one induced by the surface-orientation of S.*

⋆ **Remark 2.58.** In Section 2 we have defined several different kinds of integrals. Single, double and triple integrals, denoted by the differentials $\mathrm{d}t$, $\mathrm{d}A$, $\mathrm{d}V$, are standard or iterated integrals computed on "**flat**" (1-, 2-, or 3-dimensional) domains. Line and surface integrals denoted by the differentials $\mathrm{d}s$, $\mathrm{d}\vec{\mathbf{r}}$, $\mathrm{d}S$, $\mathrm{d}\vec{\mathbf{S}}$ require a **parametrisation** describing the shape of the domain, which is a curve or a surface and might be not flat.

⋆ **Remark 2.59** (The boundary of a boundary)**.** Note that if the surface S is the boundary of a volume $D \subset \mathbb{R}^3$, i.e. $S = \partial D$, then S has empty boundary $\partial S = \emptyset$. You may find this fact stated as $\partial^2 = \emptyset$, where the symbol "∂" stands for the action "taking the boundary". Try to think at some geometric examples.

Warning: do not mix up the different objects, the boundary ∂D of a volume D is a surface (the skin of an apple), the boundary ∂S of a surface S is a path (the edge of a sheet, the border of a country).

⋆ **Remark 2.60** (Boundaries and derivatives)**.** Why are boundaries and derivatives denoted by the same symbol "∂"? This notation comes from the profound theory of differential forms, however, we can see that these two apparently unrelated objects share some properties. A fundamental property of derivatives is the product rule: $(fg)' = f'g + fg'$ for scalar functions, or $\vec{\nabla}(fg) = (\vec{\nabla} f)g + f\vec{\nabla} g$ for scalar fields, (28). Now consider the Cartesian product of the two segments $[a, b] \subset \mathbb{R}$ and $[c, d] \subset \mathbb{R}$, namely the rectangle $R = [a, b] \times [c, d] = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ a \le x \le b,\ c \le y \le d\}$. Its boundary is the union of four edges $[a, b] \times \{c\}$, $[a, b] \times \{d\}$, $\{a\} \times [c, d]$, and $\{b\} \times [c, d]$. The boundaries of the segments are $\partial[a, b] = \{a, b\}$ and $\partial[c, d] = \{c, d\}$. Therefore we can write

$$\begin{aligned}
\partial\big([a, b] \times [c, d]\big) &= \big([a, b] \times \{c\}\big) \cup \big([a, b] \times \{d\}\big) \cup \big(\{a\} \times [c, d]\big) \cup \big(\{b\} \times [c, d]\big)\\
&= \big([a, b] \times \{c, d\}\big) \cup \big(\{a, b\} \times [c, d]\big)\\
&= \big([a, b] \times \partial[c, d]\big) \cup \big(\partial[a, b] \times [c, d]\big)
\end{aligned}$$

which closely resembles the product rule for derivatives. Here the Cartesian product ($\times$) plays the role of multiplication, the set union ($\cup$) that of addition and the action "take the boundary" plays the role of differentiation. A similar formula holds true for much more general domains; try to see what is the boundary of the Cartesian product of a circle and a segments, two circles, or a planar domain and a segment. If you are curious about this analogy take a look at the post http://mathoverflow.net/questions/46252.

2.3 Special coordinate systems

Among the possible changes of coordinates described in Section 2.2.2, some are particularly important. Polar coordinates are used for two-dimensional problems with some circular symmetry around a centre, while cylindrical and spherical coordinates are used in three dimensions. Several other special (more complicated) coordinate systems exist, we do not consider them here. You should remember polar coordinates from page 31 of handout 5 of last year calculus class. In Table 2 at the end of this section we summarise the most important facts to keep in mind for the three system of coordinates studied here[25].

These systems of coordinates are meant to simplify computations in problems with some geometrical symmetry. For example, cylindrical coordinates are related to axial symmetry and spherical coordinates to central symmetry. So the former system will be useful to compute triple integrals on cylinders and shapes like barrels or funnels; the latter for shapes like spheres or portions of them. On the other hand, if you try to use these systems on different geometries, like tetrahedra or parallelepipeds, you will end up with horrible computations, as they are not meant to be used in these cases.

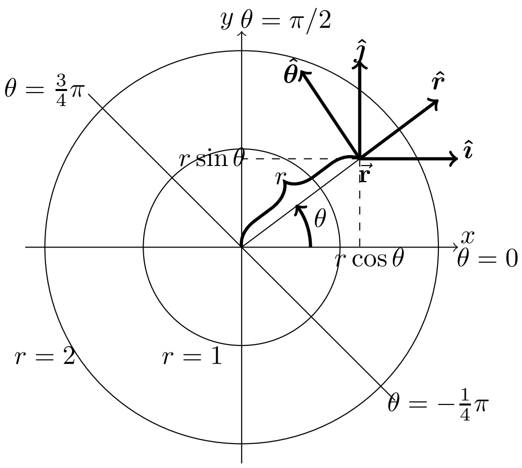
In table 4 in Section G at the end of these notes, you can find a list of exercises from the textbook where you can combine special coordinate systems with the techniques learned in the previous sections, such as line integrals, multiple integrals, potentials.

2.3.1 Polar coordinates

The polar coordinates r and θ are defined by the relations[26]

$$\boxed{\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \tan^{-1}\dfrac{y}{x}. \end{cases}} \tag{75}$$

The xy-plane $\mathbb{R}^2$ is mapped by this transformation into the strip $r \geq 0, -\pi < \theta \leq \pi$. Given a point $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, the value of r is the distance from the origin, while the value of θ is the (signed) angular distance between the direction $\hat{\boldsymbol{\imath}}$ of the x axis and the direction of the point itself. If $r = 0$, i.e. for the origin of the Cartesian axes, the angular coordinate θ is not defined.



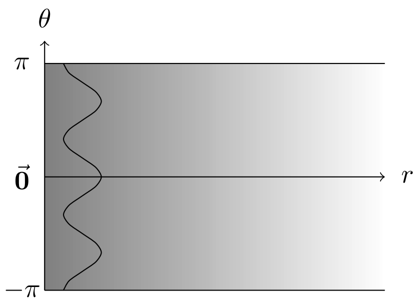


Figure 36: ***Left:*** *the polar coordinate system.*
Right: *the change of variables $(x, y) \mapsto (r, \theta)$ maps the punctured plane $\mathbb{R}^2 \setminus \{\vec{\mathbf{0}}\}$ to the semi-infinite strip $(0, \infty) \times (-\pi, \pi]$ in the $r\theta$-plane.*

We compute the Jacobian of the direct and the inverse transformations:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

[25]The page http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates contains a useful summary of these and many other identities and formulas (in 3D only). However, the notation used there is completely different from that we have chosen in this notes: to compare those formulas with ours, r and ρ must be exchanged with each other and similarly θ and ϕ! (See also footnote 29.) Our notation is chosen to be consistent with that of [1, Section 10.6].

[26]Here we are being quite sloppy. The "inverse" of the tangent function is usually called arctan and takes values in $(-\frac{\pi}{2}\,\frac{\pi}{2})$. Here $\tan^{-1}\frac{y}{x}$ is meant to be equal to $\arctan\frac{y}{x}$ only when $x > 0$ and to take values in $(-\pi, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ otherwise. For the exact definition, see http://en.wikipedia.org/wiki/Atan2. However, $\theta(x, y)$ can be easily computed "visually" as the angle determined by the point $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$. E.g. if $x > 0, y > 0$, then $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ is in the first quadrant so θ belongs to the interval $(0, \frac{\pi}{2})$; if $\widetilde{x} = -x < 0, \widetilde{y} = -y < 0$ then $\widetilde{x}\hat{\boldsymbol{\imath}} + \widetilde{y}\hat{\boldsymbol{\jmath}}$ belongs to the third quadrant and $\widetilde{\theta}$ must be in the interval $(-\pi, -\frac{\pi}{2})$, even if $\frac{\widetilde{y}}{\widetilde{x}} = \frac{y}{x}$ and $\arctan\frac{\widetilde{y}}{\widetilde{x}} = \arctan\frac{y}{x}$.

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{pmatrix}. \tag{76}$$

(The second matrix can also be calculated directly using $\frac{\partial(\tan^{-1} t)}{\partial t} = \frac{1}{1+t^2}$.) The Jacobian determinants are immediately computed as

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r(\cos^2\theta + \sin^2\theta) = r, \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}. \tag{77}$$

Therefore, the infinitesimal area element in polar coordinates is

$$\boxed{\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta.}$$

Example 2.61 (Area of the disc)**.** As a simple application of the polar coordinate system, we compute the area of the disc of radius $a > 0$ centred at the origin, i.e. $Q = \{x\hat{\imath} + y\hat{\jmath} \text{ s.t. } x^2 + y^2 < a\}$:

$$\text{Area}(Q) = \iint_Q \mathrm{d}x\,\mathrm{d}y = \int_0^a \int_{-\pi}^{\pi} r\,\mathrm{d}\theta\,\mathrm{d}r = \int_0^a 2\pi r\,\mathrm{d}r = 2\pi \frac{r^2}{2}\bigg|_0^a = \pi a^2,$$

which agrees with the formula we know from elementary geometry.

In the "punctured plane" $\mathbb{R}^2 \setminus \{\vec{\mathbf{0}}\}$, we define two vector fields $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ of unit magnitude:

$$\begin{aligned} \hat{\boldsymbol{r}}(x,y) &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}, \qquad &\Rightarrow \qquad \hat{\imath}(r,\theta) &= \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}, \\ \hat{\boldsymbol{\theta}}(x,y) &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\frac{y}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{x}{\sqrt{x^2+y^2}}\hat{\jmath}, & \hat{\jmath}(r,\theta) &= \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}. \end{aligned} \tag{78}$$

In every point $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$ the two unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are orthogonal one another and point in the direction of increase of the radial coordinate r and of the angular one θ, respectively; see Figure 36. In other words, $\hat{\boldsymbol{r}}$ points away from the origin and $\hat{\boldsymbol{\theta}}$ points in the anti-clockwise direction. Using (78) every planar vector field defined in $\mathbb{R}^2 \setminus \{\vec{\mathbf{0}}\}$ can be expanded in $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$:

$$\vec{\mathbf{F}} = F_1\hat{\imath} + F_2\hat{\jmath} = (\cos\theta F_1 + \sin\theta F_2)\hat{\boldsymbol{r}} + (-\sin\theta F_1 + \cos\theta F_2)\hat{\boldsymbol{\theta}} = (\vec{\mathbf{F}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} + (\vec{\mathbf{F}}\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}.$$

Example 2.62 (Curves in polar coordinates and area of domains delimited by polar graphs)**.** If the path of a curve $\vec{\mathbf{a}}$ can be expressed in the form $r = g(\theta)$ for some positive function g, it is sometimes called a "polar graph". In this kind of curves, the magnitude is function of the angle and the path can be seen as the graph of the function g in the $r\theta$-plane. In this case we can choose $t = \theta$ and write $\vec{\mathbf{a}}(t) = g(t)(\cos t\hat{\imath} + \sin t\hat{\jmath})$ for $-\pi < t \leq \pi$. For example, a circle of radius a can be written as $r = a$, i.e. the function g has constant value a, thus we find the usual parametrisation $\vec{\mathbf{a}}(t) = a(\cos t\hat{\imath} + \sin t\hat{\jmath})$.[27]

The domain R delimited by the curve $\vec{\mathbf{a}}$ is "r-simple" in the $r\theta$-plane (compare with the definition of y-simple domain in (52)):

$$\begin{aligned} R &= \Big\{\vec{\mathbf{r}} = x\hat{\imath} + y\hat{\jmath} \in \mathbb{R}^2, \text{ s.t. } \sqrt{x^2+y^2} < g\big(\theta(x,y)\big)\Big\} \\ &= \Big\{\vec{\mathbf{r}} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} \in \mathbb{R}^2, \text{ s.t. } -\pi < \theta \leq \pi,\ 0 \leq r < g(\theta)\Big\}. \end{aligned}$$

R is called "star-shaped region"[28] and its area can be computed as

$$\text{Area}(R) = \iint_R \mathrm{d}x\,\mathrm{d}y = \int_{-\pi}^{\pi}\left(\int_0^{g(\theta)} r\,\mathrm{d}r\right)\mathrm{d}\theta = \int_{-\pi}^{\pi} \frac{1}{2}r^2\bigg|_0^{g(\theta)}\mathrm{d}\theta = \frac{1}{2}\int_{-\pi}^{\pi} g^2(\theta)\,\mathrm{d}\theta. \tag{79}$$

For example, we compute the area of the domain delimited by the "propeller" curve $r = (2 + \cos 3\theta)$ depicted in left plot of Figure 37: using $\cos^2 t = \frac{1}{2}(t + \sin t\cos t)'$,

$$\text{Area}(R) = \frac{1}{2}\int_{-\pi}^{\pi}(2+\cos 3\theta)^2\,\mathrm{d}\theta = \left(2\theta + \frac{\theta}{4} + \frac{\sin 3\theta\cos 3\theta}{12} + \frac{2}{3}\sin 3\theta\right)\bigg|_{-\pi}^{\pi} = \frac{9}{2}\pi.$$

[27] The "clover" curve $\vec{\mathbf{c}}(t) = (\cos 3t)(\cos t)\hat{\imath} + (\cos 3t)(\sin t)\hat{\jmath}$ in Figure 8 might be written in polar form as $r = \cos 3\theta$. Note that here we are allowing negative values for the radial component! What does it mean? Compare with the curve plot.

[28] We have already defined star-shaped domains in Section 2.1.3, can you see the relation between the two definitions?

(We can deduce from this computation that the area delimited by $r = (2 + \cos n\theta)$ is independent of $n \in \mathbb{N}$, i.e. all the "flower" domains in the centre plot of Figure 37 have the same area.) The path of the curve $r = (2 + \cos 3\theta)$ in the $r\theta$-plane is the wave in the right plot of Figure 36.

We can also compute the area of more complicated shapes, for instance the domain between the two spirals $r = \theta$ and $r = 2\theta$ (with $0 < \theta < 2\pi$) in the right plot of Figure 37:

$$\int_0^{2\pi}\left(\int_\theta^{2\theta} r\,\mathrm{d}r\right)\mathrm{d}\theta = \int_0^{2\pi}\frac{1}{2}\big(4\theta^2 - \theta^2\big)\,\mathrm{d}\theta = \frac{(2\pi)^3}{2} = 4\pi^3 \approx 124.$$

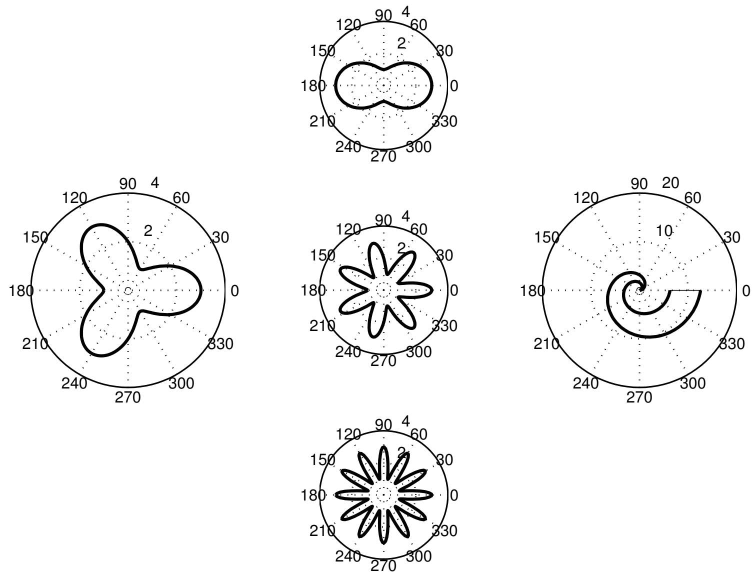


Figure 37: *The polar graph $r = (2 + \cos n\theta)$ for $n = 3$ (left), $n = 2$ (upper centre), $n = 7$ (centre centre), $n = 12$ (lower centre). As seen in Example 2.62, they all have the same area. In the right plot, the two spirals $r = \theta$ and $r = 2\theta$ (with $0 < \theta < 2\pi$). These curves can be plotted with Matlab's command* `polar`.

★ **Remark 2.63.** Any scalar field in two variables $f(x, y)$ can be expressed in polar coordinates as $F(r, \theta) = f(r\cos\theta, r\sin\theta)$. Here F is a function of two variables (as f) which represents the field f but has a different functional expression (we have already used this fact several times without spelling it out). The gradient and the Laplacian of f can be computed in polar coordinates using the chain rule:

$$\begin{aligned}
\vec{\nabla} f(x,y) &= \frac{\partial f(x,y)}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f(x,y)}{\partial y}\hat{\boldsymbol{\jmath}}\\
&= \frac{\partial F\big(r(x,y),\theta(x,y)\big)}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial F\big(r(x,y),\theta(x,y)\big)}{\partial y}\hat{\boldsymbol{\jmath}}\\
&= \left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\hat{\boldsymbol{\jmath}} \qquad \text{(chain rule (38))}\\
&= \frac{\partial F}{\partial r}\left(\frac{\partial r}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial r}{\partial y}\hat{\boldsymbol{\jmath}}\right) + \frac{\partial F}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial \theta}{\partial y}\hat{\boldsymbol{\jmath}}\right)\\
&\overset{(76)}{=} \frac{\partial F}{\partial r}\left(\frac{x}{r}\hat{\boldsymbol{\imath}} + \frac{y}{r}\hat{\boldsymbol{\jmath}}\right) + \frac{\partial F}{\partial \theta}\left(\frac{-y}{r^2}\hat{\boldsymbol{\imath}} + \frac{x}{r^2}\hat{\boldsymbol{\jmath}}\right) \overset{(78)}{=} \boxed{\frac{\partial F}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\boldsymbol{\theta}} = \vec{\nabla} F(r,\theta).}
\end{aligned} \tag{80}$$

Thus, whenever we have a scalar field F expressed in polar coordinates, we can compute its gradient using formula (80) (do not forget the $\frac{1}{r}$ factor!) without the need of transforming the field in Cartesian coordinates.

2.3.2 Cylindrical coordinates

There are two possible extensions of polar coordinates to three dimensions. The first one is used to treat problems with some axial, or cylindrical, symmetry. By convention, the axis of symmetry is fixed to be the axis z. The cylindrical coordinate r, θ and z corresponds to the polar coordinates r, θ in the xy-plane and the height z above this plane:

$$\boxed{\begin{cases} x = r\cos\theta,\\ y = r\sin\theta,\\ z = z, \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2},\\ \theta = \tan^{-1}\dfrac{y}{x},\\ z = z, \end{cases}} \qquad \begin{matrix} r \geq 0,\\ -\pi < \theta \leq \pi,\\ z \in \mathbb{R}. \end{matrix} \tag{81}$$

Proceeding as in (77), it is immediate to verify that the Jacobian determinants and the infinitesimal volume element are

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r, \qquad \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}, \qquad \boxed{\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.} \tag{82}$$

This can be used to compute the volume of the **solids of revolution**

$$D_g = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } z_{\mathrm{bot}} < z < z_{\mathrm{top}},\ r < g(z)\big\}, \tag{83}$$

where $g : (z_{\mathrm{bot}}, z_{\mathrm{top}}) \to \mathbb{R}$ is a non-negative function, as

$$\mathrm{Vol}(D_g) = \iiint_{D_g} \mathrm{d}V = \int_{z_{\mathrm{bot}}}^{z_{\mathrm{top}}} \left(\int_{-\pi}^{\pi} \left(\int_0^{g(z)} r\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}z = \int_{z_{\mathrm{bot}}}^{z_{\mathrm{top}}} \left(\int_{-\pi}^{\pi} \frac{g^2(z)}{2}\,\mathrm{d}\theta\right)\mathrm{d}z = \boxed{\pi \int_{z_{\mathrm{bot}}}^{z_{\mathrm{top}}} g^2(z)\,\mathrm{d}z.} \tag{84}$$

Example 2.64 (Volume of solids of revolution)**.** Compute the volume of the "pint" domain E in Figure 38: $E = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } 0 < z < 1,\ r < \frac{1}{3} + \frac{1}{2}z^2 - \frac{1}{3}z^3\big\}$.

$$\mathrm{Vol}(E) \overset{(84)}{=} \pi \int_0^1 \left(\frac{1}{3} + \frac{1}{2}z^2 - \frac{1}{3}z^3\right)^2 \mathrm{d}z = \pi \int_0^1 \left(\frac{1}{9} + \frac{1}{4}z^4 + \frac{1}{9}z^6 + \frac{1}{3}z^2 - \frac{2}{9}z^3 - \frac{1}{3}z^5\right)\mathrm{d}z$$
$$= \pi\left(\frac{1}{9} + \frac{1}{20} + \frac{1}{63} + \frac{1}{9} - \frac{1}{18} - \frac{1}{18}\right) = \frac{223}{1260} \approx 0.177.$$

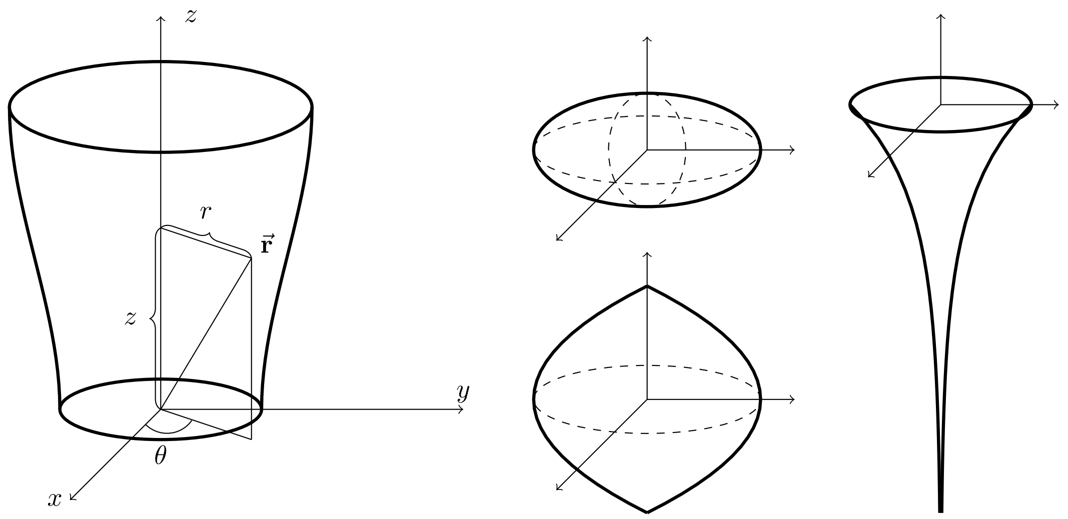


Figure 38: *The domains described in Examples 2.64, 2.65 and Exercise 2.66: the pint E (left), the ellipsoid $x^2 + y^2 + 4z^2 < 1$ (upper centre), the rugby ball B (lower centre) and the funnel F (right).*

Exercise 2.65 (Volume of the ellipsoid)**.** ▶ Compute the volume of the "oblate ellipsoid of revolution" bounded by the surface $x^2 + y^2 + \frac{z^2}{c^2} = 1$, for a real number $c > 0$.

Exercise 2.66. ▶ Calculate the volume of the rugby ball B and the (infinite) funnel F in Figure 38:

$$B = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } -1 < z < 1,\ r < 1 - z^2\big\}, \qquad F = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } -\infty < z < 0,\ r < \mathrm{e}^z\big\}.$$

Exercise 2.67. ▶ Cylindrical coordinates can also be used to deal with domains which are not solids of revolution. Can you draw the shape of $D = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } -\frac{\pi}{2} < z < \frac{\pi}{2},\ r < (\cos z)(2 + \sin 3\theta)\big\}$? Can you compute its volume? You need to derive a slightly more general formula than (84) (compare with formula (79) for the area of polar graphs).

Exercise 2.68 (Triple integrals on a cone)**.** ▶ Consider the cone $C = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } 0 < z < 1,\ r < z\big\}$. Verify the following computations:

$$\iiint_C z\,\mathrm{d}V = \frac{\pi}{4}, \qquad \iiint_C x\,\mathrm{d}V = 0, \qquad \iiint_C (x^2 + y^2 + z^2)\,\mathrm{d}V = \frac{3\pi}{10}.$$

(The domain is easily defined in cylindrical coordinates, while the last two integrands are defined in Cartesian coordinates, figure out how to deal with this fact.) Prove that, for any function $g : (0,1) \to \mathbb{R}$,

$$\iiint_C g(z)\,\mathrm{d}V = \pi \int_0^1 g(z) z^2\,\mathrm{d}z.$$

★ **Remark 2.69.** As we did in (78) for the polar coordinates, also for the cylindrical system we can compute the vector fields with unit length lying in the direction of increase of the three coordinates:

$$\begin{aligned} \hat{\boldsymbol{r}}(x,y,z) &= \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}, & \hat{\boldsymbol{\imath}} &= \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}, \\ \hat{\boldsymbol{\theta}}(x,y,z) &= -\frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\imath}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{\jmath}}, & \hat{\boldsymbol{\jmath}} &= \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}, \\ \hat{\boldsymbol{z}}(x,y,z) &= \hat{\boldsymbol{k}}, & \hat{\boldsymbol{k}} &= \hat{\boldsymbol{z}}. \end{aligned} \tag{85}$$

★ **Remark 2.70** (Vector differential operators in cylindrical coordinates)**.** As we did in equation (80) for polar coordinates, we can compute the gradient of a scalar field f expressed in cylindrical coordinates. Since we are now considering a three-dimensional space, we also compute divergence and curl of a vector field $\vec{\mathbf{G}} = G_r\hat{\boldsymbol{r}} + G_\theta\hat{\boldsymbol{\theta}} + G_z\hat{\boldsymbol{z}}$. We only write the final results (they can be derived from (81), (85) and the chain rule, but the computation are quite complicated):

$$\begin{aligned} \vec{\nabla} f(r,\theta,z) &= \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}, \\ \vec{\nabla}\cdot\vec{\mathbf{G}}(r,\theta,z) &= \frac{1}{r}\frac{\partial(rG_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z} = \frac{\partial G_r}{\partial r} + \frac{1}{r}G_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \\ \vec{\nabla}\times\vec{\mathbf{G}}(r,\theta,z) &= \Big(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\Big)\hat{\boldsymbol{r}} + \Big(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\Big)\hat{\boldsymbol{\theta}} + \Big(\frac{\partial F_\theta}{\partial r} + \frac{1}{r}F_\theta - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\Big)\hat{\boldsymbol{z}}. \end{aligned} \tag{86}$$

Exercise 2.71. ▶ Consider the vector field $\vec{\mathbf{F}} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$. Show that in cylindrical coordinates it reads $\vec{\mathbf{F}} = r\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{z}}$. Use (86) to prove that $\vec{\mathbf{F}}$ is solenoidal. This might be the field that gave the name to all solenoidal fields: the streamlines of $\vec{\mathbf{F}}$ are the helices (see Figure 8), which have the shape of a "solenoid" used in electromagnetism (although Wikipedia suggests a different origin of the name).

Example 2.72 (Surface integrals in cylindrical coordinates)**.** Polar and cylindrical coordinates can be used to compute surface integrals. Consider the conic surface of unit radius and unit height (written in Cartesian and cylindrical coordinates):

$$S = \Big\{\vec{\mathbf{r}}\in\mathbb{R}^3,\ z = \sqrt{x^2+y^2},\ x^2+y^2<1\Big\} = \Big\{\vec{\mathbf{r}}\in\mathbb{R}^3,\ z=r,\ r<1\Big\}.$$

(Note that S is part of boundary of the cone C in Exercise 2.68.) The surface S is the graph of the two-dimensional field $g(r,\theta) = r$ defined on the unit disc $Q = \{\vec{\mathbf{r}}\in\mathbb{R}^3,\ r<1\}$. Thus, its area can be computed using the graph surface integral formula (68), choosing $f=1$. We also exploit the fact that the gradient of $g = \sqrt{x^2+y^2} = r$ satisfies $\vec{\nabla} g = \frac{\partial r}{\partial r}\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}$, when written in polar coordinates, see (80), so it has length one. We obtain:

$$\text{Area}(S) \overset{(68)}{=} \iint_Q 1\sqrt{1+|\vec{\nabla} g|^2}\,\mathrm{d}A = \int_{-\pi}^{\pi}\int_0^1 \sqrt{1+1}\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\sqrt{2}\int_0^1 r\,\mathrm{d}r = \pi\sqrt{2}\approx 4.443,$$

where in the second equality we used $|\vec{\nabla} g| = |\hat{\boldsymbol{r}}| = 1$.

2.3.3 Spherical coordinates

In order to deal with three-dimensional situations involving a centre of symmetry, we introduce the spherical coordinate system. The three coordinates, denoted by ρ, ϕ and θ, are called **radius**, **colatitude** and **longitude** (or **azimuth**), respectively[29]. (The latitude commonly used in geography corresponds to $\frac{\pi}{2}-\phi$, hence the name colatitude.) For a point $\vec{\mathbf{r}}$, the radius $\rho\ge 0$ is its distance from the origin (the magnitude of $\vec{\mathbf{r}}$); the colatitude $0\le\phi\le\pi$ is the angle between the direction of $\vec{\mathbf{r}}$ and the unit vector $\hat{\boldsymbol{k}}$ on the axis z; the longitude $-\pi<\theta\le\pi$ is the polar angle of the projection of $\vec{\mathbf{r}}$ on the xy-plane. In formulas:[30]

$$\boxed{\begin{cases} \rho = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}, \\ \phi = \arccos\frac{z}{\rho} = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}, \\ \theta = \tan^{-1}\frac{y}{x}, \end{cases} \qquad \begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi, \end{cases}} \qquad \begin{matrix} \rho\ge 0, \\ 0\le\phi\le\pi, \\ -\pi<\theta\le\pi. \end{matrix} \tag{87}$$

[29] We assign to colatitude and longitude the same symbols used in the textbook (see [1, Page 598]). However, in many other references, the symbols θ and ϕ are swapped, e.g. in Calvin Smith's "vector calculus primer" you might have used. According to `http://en.wikipedia.org/wiki/Spherical_coordinates`, the first notation is more common in mathematics and the second in physics. Different books, pages of Wikipedia and websites use different conventions: this might be a continuous source of mistakes, watch out!

Also the naming of the radial variable can be an issue: we use ρ for spherical coordinates and r for cylindrical as in [1, Section 10.6], but, for instance, most pages of Wikipedia swap the two letters.

[30] You do not need to remember by hearth the formulas in (87). On the other hand, if you know their geometrical meaning it is easy to derive them using some basic trigonometry. You need to remember the formula of the volume element (88).

They are depicted in Figure 39. Spherical and cylindrical coordinate systems are related one another by the following formulas (note that the variable θ plays exactly the same role in the two cases):

$$\begin{cases}\rho=\sqrt{r^2+z^2},\\ \phi=\arctan\frac{r}{z},\\ \theta=\theta,\end{cases}\qquad\qquad \begin{cases}r=\rho\sin\phi,\\ \theta=\theta,\\ z=\rho\cos\phi.\end{cases}$$

The corresponding Jacobian matrix and the two determinants are:

$$\begin{pmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix},$$

$$\Rightarrow\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\rho^2\sin\phi,\qquad \frac{\partial(\rho,\phi,\theta)}{\partial(x,y,z)}=\frac{1}{\rho^2\sin\phi},\qquad \boxed{\mathrm{d}V=\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.}\tag{88}$$

Exercise 2.73. Verify the derivation of the Jacobian determinant in (88).

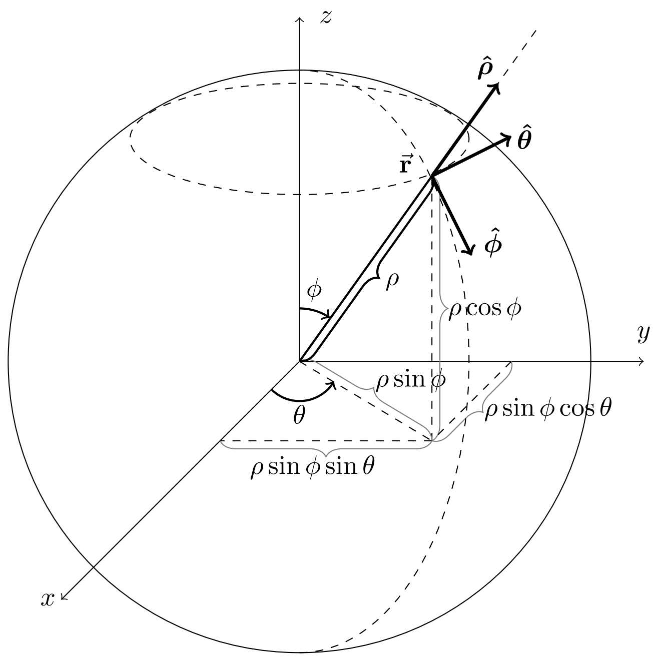


Figure 39: *A representation of the spherical coordinates ρ, ϕ and θ for the position vector $\vec{\mathbf r}\in\mathbb{R}^3$. Note the angles ϕ and θ, it is important to understand well their definition. The big sphere is the set of points with equal radius ρ. The dashed horizontal circle is the set of points with equal radius ρ and equal colatitude ϕ. The dashed vertical half-circle is the set of points with equal radius ρ and equal longitude θ. The set of points with equal colatitude ϕ and equal longitude θ is the half line starting at the origin and passing through $\vec{\mathbf r}$. The unit vectors $\hat{\boldsymbol\rho}$, $\hat{\boldsymbol\phi}$ and $\hat{\boldsymbol\theta}$ point in the directions of increase of the corresponding coordinates (see formulas (89)).*

★ **Remark 2.74.** The unit vector fields in the direction of the spherical coordinates, pictured in Figure 39, are

$$\begin{aligned}\hat{\boldsymbol\rho}(x,y,z)&=\frac{1}{\sqrt{(x^2+y^2+z^2)}}\big(x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath}+z\hat{\boldsymbol k}\big),\\ \hat{\boldsymbol\phi}(x,y,z)&=\frac{1}{\sqrt{(x^2+y^2+z^2)}\sqrt{x^2+y^2}}\big(xz\hat{\boldsymbol\imath}+yz\hat{\boldsymbol\jmath}-(x^2+y^2)\hat{\boldsymbol k}\big),\\ \hat{\boldsymbol\theta}(x,y,z)&=\frac{1}{\sqrt{x^2+y^2}}\big(-y\hat{\boldsymbol\imath}+x\hat{\boldsymbol\jmath}\big).\end{aligned}\tag{89}$$

★ **Remark 2.75** (The gradient in spherical coordinates)**.** The gradient of a scalar field f expressed in spherical coordinates reads:

$$\vec\nabla f(\rho,\phi,\theta)=\frac{\partial f}{\partial\rho}\hat{\boldsymbol\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol\phi}+\frac{1}{\rho\sin\theta}\frac{\partial f}{\partial\theta}\hat{\boldsymbol\theta}.$$

Since their expression is quite complicated, for the formulas of divergence and the curl of a vector field $\vec{\mathbf{F}}$ in spherical coordinates we refer to Section 16.7 of [1] (compare with those in cylindrical coordinates shown in Remark 2.70).

Example 2.76 (Domain volume in spherical coordinates)**.** Consider the star-shaped domain

$$D = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ \text{s.t. } \rho < r(\phi,\theta)\big\},$$

where $r : [0,\pi] \times (-\pi,\pi] \to \mathbb{R}$ is a positive two-dimensional field. Its volume is

$$\mathrm{Vol}(D) = \iiint_D \mathrm{d}V = \int_{-\pi}^{\pi}\bigg(\int_0^{\pi} \sin\phi\Big(\int_0^{r(\phi,\theta)} \rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\phi\bigg)\,\mathrm{d}\theta = \frac{1}{3}\int_{-\pi}^{\pi}\bigg(\int_0^{\pi} r^3(\phi,\theta)\sin\phi\,\mathrm{d}\phi\bigg)\,\mathrm{d}\theta.$$

For $r(\phi,\theta) = 1$, we recover the volume of the unit sphere $\frac{4}{3}\pi$ (verify the computation as exercise).

For example, we compute the volume of the pumpkin-shaped domain B of Figure 40 (right)

$$B = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ \text{s.t. } \rho < (\sin\phi)(5+\cos 7\theta)\big\}.$$

From the formula above, we have

$$\begin{aligned}
\mathrm{Vol}(B) &= \frac{1}{3}\int_{-\pi}^{\pi}\bigg(\int_0^{\pi} \sin^3\phi\,(5+\cos 7\theta)^3\sin\phi\,\mathrm{d}\phi\bigg)\,\mathrm{d}\theta\\
&= \frac{1}{3}\bigg(\int_0^{\pi}\sin^4\phi\,\mathrm{d}\phi\bigg)\bigg(\int_{-\pi}^{\pi}(5+\cos 7\theta)^3\,\mathrm{d}\theta\bigg)\\
&= \frac{1}{3}\bigg(\int_0^{\pi}\Big(\frac{1-\cos 2\phi}{2}\Big)^2\mathrm{d}\phi\bigg)\bigg(\int_{-\pi}^{\pi}(125+25\cos 7\theta+5\cos^2 7\theta+\cos^3 7\theta)\,\mathrm{d}\theta\bigg)\\
&= \frac{1}{3}\bigg(\int_0^{\pi}\Big(\frac{1}{4}-\frac{\cos 2\phi}{2}+\frac{\cos^2 2\phi}{4}\Big)\,\mathrm{d}\phi\bigg)\\
&\qquad\cdot\bigg(\int_{-\pi}^{\pi}\Big(125+25\cos 7\theta+5\frac{1+\cos 14\theta}{2}+\cos 7\theta(1-\sin^2 7\theta)\Big)\,\mathrm{d}\theta\bigg)\\
&= \frac{1}{3}\bigg(\int_0^{\pi}\Big(\frac{1}{4}-\frac{\cos 2\phi}{2}+\frac{\cos 4\phi+1}{8}\Big)\,\mathrm{d}\phi\bigg)\bigg(\frac{255}{2}\theta+\frac{26}{7}\sin 7\theta+\frac{5}{28}\sin 14\theta-\frac{1}{21}\sin^3 7\theta\bigg)\bigg|_{-\pi}^{\pi}\\
&= \frac{1}{3}\Big(\frac{1}{4}\phi-\frac{\sin 2\phi}{4}+\frac{\sin 4\phi}{32}+\frac{1}{8}\phi\Big)\bigg|_0^{\pi}(255\pi) = \frac{1}{3}\Big(\frac{3}{8}\pi\Big)(255\pi) = \frac{255}{8}\pi^2 \approx 315,
\end{aligned}$$

where we used twice the double-angle formula $\cos 2t = 1-2\sin^2 t = 2\cos^2 t - 1$ to expand the square of a sine or a cosine, while the cubic power is integrated using $\cos^3 t = \cos t - \cos t\sin^2 t = (\sin t - \frac{1}{3}\sin^3 t)'$.

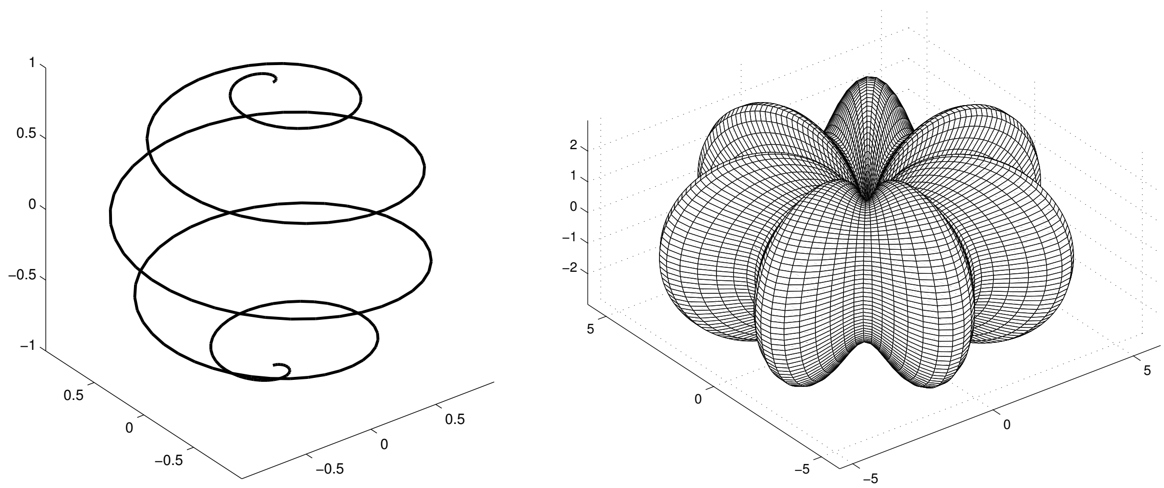


Figure 40: ***Left:*** *the curve* $\{\theta = 10\phi, \rho = 1\}$ *lying on the unit sphere.*
Right: *the pumpkin domain whose volume is computed in Example 2.76.*

Exercise 2.77. ▶ Compute the integral of the field $f = \rho^4(\sin^2\phi + \cos 12\theta)$ on the unit ball B (i.e. the ball centred at the origin with radius 1).

Example 2.78 (Surface integrals in spherical coordinates)**.** Not surprisingly, spherical coordinates can be useful to compute surface integrals. The field

$$\vec{\mathbf{X}}(\phi,\theta) = a\sin\phi\cos\theta\hat{\boldsymbol{\imath}} + a\sin\phi\sin\theta\hat{\boldsymbol{\jmath}} + a\cos\phi\hat{\boldsymbol{k}} \qquad \text{defined on the region } R=(0,\pi)\times(-\pi,\pi)$$

is a chart (in the sense of Definition 2.38) for the sphere of radius $a>0$ centred at the origin. To be more precise, the image of $\vec{\mathbf{X}}$ is the unit sphere without the meridian in the half plane $\{y=0, x\le 0\}$ (recall that a chart must be defined in a open region R). Here ϕ and θ play the roles of u and w. Note that this chart is different from the stereographic projection of Example 2.41. Deriving the expression of $\vec{\mathbf{X}}$, we obtain

$$\frac{\partial\vec{\mathbf{X}}}{\partial\phi} = a\cos\phi\cos\theta\hat{\boldsymbol{\imath}} + a\cos\phi\sin\theta\hat{\boldsymbol{\jmath}} - a\sin\phi\hat{\boldsymbol{k}}, \qquad \frac{\partial\vec{\mathbf{X}}}{\partial\theta} = -a\sin\phi\sin\theta\hat{\boldsymbol{\imath}} + a\sin\phi\cos\theta\hat{\boldsymbol{\jmath}},$$

$$\frac{\partial\vec{\mathbf{X}}}{\partial\phi}\times\frac{\partial\vec{\mathbf{X}}}{\partial\theta} = a^2(\sin^2\phi\cos\theta\hat{\boldsymbol{\imath}} - \sin^2\phi\sin\theta\hat{\boldsymbol{\jmath}} + \sin\phi\cos\phi\hat{\boldsymbol{k}}),$$

$$\left|\frac{\partial\vec{\mathbf{X}}}{\partial\phi}\times\frac{\partial\vec{\mathbf{X}}}{\partial\theta}\right|^2 = a^4(\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi) = a^4\sin^2\phi.$$

From this, together with the surface integral formula (67), we have that the curvilinear area element is

$$\mathrm{d}S = a^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$

(recall that $0\le\phi\le\pi$, so $\sin\phi\ge 0$). For example, we compute the area of the sphere S of radius a (the absence of a meridian does not affect the integral, as the meridian has zero area):

$$\mathrm{Area}(S) = \iint_S \mathrm{d}S = \int_{-\pi}^{\pi}\left(\int_0^{\pi} a^2\sin\phi\,\mathrm{d}\phi\right)\mathrm{d}\theta = 4\pi a^2.$$

We compute the flux (72) of the vector field $\vec{\mathbf{F}} = z\hat{\boldsymbol{k}} = \rho\cos\phi\hat{\boldsymbol{k}}$ through S, equipped with the outward pointing unit normal vector $\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \hat{\boldsymbol{\rho}} = \frac{\vec{\mathbf{r}}}{\rho} = \frac{\vec{\mathbf{r}}}{a}$:

$$\begin{aligned}\iint_S \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} &= \iint_S z\,\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\rho}}\,\mathrm{d}S \qquad\qquad \Big(\text{using } \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\rho}} \overset{(89)}{=} \frac{z}{\rho} \overset{(87)}{=} \cos\phi\Big)\\ &= \int_{-\pi}^{\pi}\left(\int_0^{\pi}(a\cos\phi)(\cos\phi)a^2\sin\phi\,\mathrm{d}\phi\right)\mathrm{d}\theta\\ &= 2\pi a^3\int_0^{\pi}\cos^2\phi\,\sin\phi\,\mathrm{d}\phi \;=\; 2\pi a^3\Big(-\frac{1}{3}\cos^3\phi\Big)\Big|_{\phi=0}^{\pi} \;=\; \frac{4}{3}\pi^2 a^3.\end{aligned}$$

(Draw the field $\vec{\mathbf{F}}$ and the sphere S, can you deduce the sign of $\iint_S \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}}$ from the plot?)

Exercise 2.79. ▶ The borders of the state of Colorado are defined by the parallels with latitude $37°$ north and $41°$ north and by the meridians with longitude $102°03'$ west and $109°03'$ west. Assuming that the Earth is a sphere with radius 6371 km, compute the area of the state.

	Coordinate system	Coordinates	Measure	Domain
$\mathbb{R}^2$	Cartesian	x, y	$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$	$x, y\in\mathbb{R}$
	Polar	$x = r\cos\theta$ $y = r\sin\theta$	$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$r\ge 0$ $\theta\in(-\pi,\pi]$
$\mathbb{R}^3$	Cartesian	x, y, z	$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$	$x, y, z\in\mathbb{R}$
	Cylindrical	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$r\ge 0$ $\theta\in(-\pi,\pi]$ $z\in\mathbb{R}$
	Spherical	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta$	$\rho\ge 0$ $\phi\in[0,\pi]$ $\theta\in(-\pi,\pi]$

Table 2: Summary of the coordinate systems described in Section 2.3

3 Green's, divergence and Stokes' theorems

The first two sections of these notes were mainly devoted to the study of "derivatives" (better: differential operators) and integrals of scalar and vector fields. This third section will focus on some important relations between differential operators and integrals.

We recall that the **fundamental theorem of calculus** gives the basic relation between differentiation and integration of real functions. It states that, for any continuous function $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative, and for any two real numbers $a < b$, the formula

$$\int_a^b \frac{df}{dt}(t)\,\mathrm{d}t = f(b) - f(a) \tag{90}$$

holds. This is often understood as "integration reverses differentiation". The **fundamental theorem of vector calculus** (49) of Section 2.1.3 extends this result to the integration of gradients along curves:

$$\int_{\vec{\mathbf{p}}}^{\vec{\mathbf{q}}} \vec{\nabla}\varphi(\vec{\mathbf{r}}) \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\vec{\mathbf{q}}) - \varphi(\vec{\mathbf{p}}). \tag{91}$$

Here the integral at the left-hand side is the line integral along any path going from $\vec{\mathbf{p}}$ to $\vec{\mathbf{q}}$. How can we extend this to multiple integrals and partial derivatives? We will see several different extensions. The integral at the left-hand side of equation (90) will become a double or a triple integral and the derivative will become a vector differential operator involving partial derivatives. How is the evaluations of f at the domain endpoints (i.e. $f(b) - f(a)$) generalised to higher dimensions? The set of the two values $\{a, b\} \subset \mathbb{R}$ can be thought as the boundary of the interval (a, b), and similarly the points $\{\vec{\mathbf{p}}, \vec{\mathbf{q}}\} \subset \mathbb{R}^3$ are the boundary of a path Γ. In the same way, when integrating a differential operator applied to a field, over a two- or three-dimensional domain D, we will obtain a certain integral of the same field over the **boundary** ∂D of D.[31]

In the fundamental theorem of calculus (either the scalar one (90) or the vector version (91)), the values at the endpoints are summed according to a precise choice of the signs: the value at the "initial point" ($f(a)$ or $\varphi(\vec{\mathbf{q}})$) is subtracted from the value at the "final point" ($f(b)$ or $\varphi(\vec{\mathbf{p}})$). This suggests that the boundary integrals will involve **oriented paths** (for the boundaries of two-dimensional domains) and **oriented surfaces** (for the boundaries of three-dimensional domains); see Section 2.2.5. The **unit tangent field** $\hat{\boldsymbol{\tau}}$ and the **unit normal field** $\hat{\boldsymbol{n}}$ will play an important role in assigning a "sign" to the integrals of vector fields on the boundaries of two- and three-dimensional domains, respectively.

The most important results of this sections are collected in three main theorems. **Green's theorem** 3.4 allows to compute the double integral of a component of the curl of a vector field as a path integral. The **divergence theorem** 3.14 (see also Theorem 3.10) states that the volume integral of the divergence of a vector field equals the flux of the same fields through the domain boundary. This key theorem holds in any dimensions, is probably the most used in applications, and is the most direct generalisation of the fundamental theorem of calculus to multiple integrals. Finally, **Stokes' theorem** 3.28 generalises Green's theorem to oriented surfaces, equalling the flux of the curl of a field to the boundary circulation of the same field[32]. All their proof are quite similar to each other and rely on the use of the fundamental theorem of calculus. We will also prove several other important identities. Table 3 at the end of the section collects the main formulas obtained.

To fix the notation, we will use the letter R to denote two-dimensional domains (or regions) and the letter D to denote three-dimensional domains. We will always assume that they are **piecewise smooth**, namely their boundaries are unions of smooth parts (∂R is union of smooth paths and ∂D is union of smooth surfaces). Green's, divergence and Stokes theorems concern planar regions R, three-dimensional domains D and surfaces S, respectively.

Green's, divergence and Stokes' theorems are treated, together with several examples and exercises, in Sections 16.3, 16.4 and 16.5 of the textbook [1].

★ **Remark 3.1** (Smoothness and integrability)**.** As in the previous sections, we will never attempt to be precise with the assumptions on the regularity of the fields. Even assuming "smooth fields", meaning C^∞, may not be enough. For instance, the function $f(t) = \frac{1}{t}$ is perfectly smooth in the open interval $(0, 1)$, but its integral $\int_0^1 f(t)\,\mathrm{d}t$ is not bounded (i.e. is infinite). Roughly speaking, possible assumptions for the theorems we will prove are that all the derivatives involved in the formulas are continuous in the *closure* of the considered domains (even if we usually consider the domains to be *open* sets). However, we have already seen in Example 2.33

[31] You may think at the difference $f(b) - f(a)$ as the signed integral of f over the zero-dimensional set $\{a, b\}$, or as the integral of f over the "oriented set" $\{a, b\}$.

[32] These theorems receive their names from George Green (1793–1841) and Sir George Gabriel Stokes (1819–1903). The divergence theorem is sometimes called Gauss' theorem from Johann Carl Friedrich Gauss (1777–1855). Stokes' theorem is also known as Kelvin–Stokes' theorem (from William Thomson, 1st Baron Kelvin, 1824–1907) or curl theorem.

(the "smile domain", see also Figure 30) that we can easily integrate fields tending to infinity on the domain boundary. An even more complicated issue is related to the regularity of domains, curves and surfaces. We will ignore this problem and always implicitly assume that all the geometric object considered are sufficiently "well-behaved".

3.1 Green's theorem

In this section, we fix a two-dimensional, piecewise smooth, connected, bounded region $R \subset \mathbb{R}^2$. "**Connected**" means that for any two points $\vec{\mathbf{p}}, \vec{\mathbf{q}} \in R$ there exists a path Γ lying entirely in R with endpoints $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ (R "is made of only one piece"). "**Bounded**" means that $\sup\{|\vec{\mathbf{r}}|, \vec{\mathbf{r}} \in R\} < \infty$, i.e. the region is contained in some ball of finite radius (R "does not go to infinity"). The boundary ∂R of R is composed by one or more loops (closed paths); in the first case, the domain has no "holes" and is called **simply connected**[33].

We consider the boundary as an oriented path, with the **orientation induced by** R after setting the unit normal field on R to be equal to $\hat{\boldsymbol{k}}$ (see the last paragraph of Section 2.2.5). In other words, if we draw R on the ground and walk on the path ∂R according to its orientation, we will see the region R at our left. The external part of ∂R is run **anti-clockwise**, the boundary of every hole (if any is present) is run clockwise; see Figure 41.

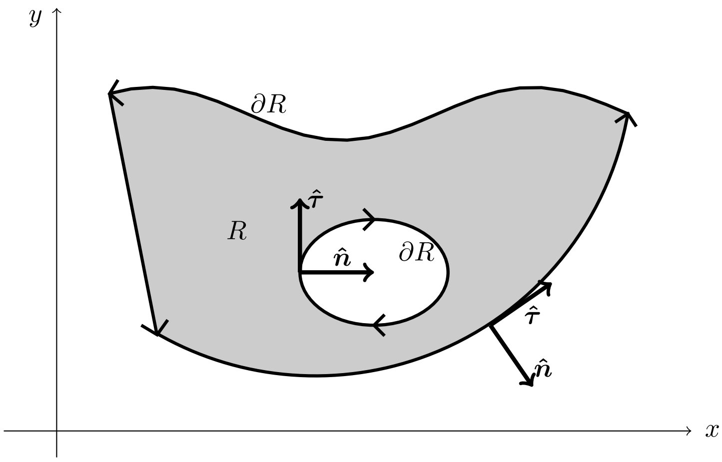


Figure 41: *The shaded area represents a two-dimensional region R. R is connected (composed of only one piece), piecewise smooth (its boundary is the union of four smooth paths), but not simply connected since it contains a hole (its boundary is composed by two loops). Its boundary ∂R is composed by two oriented paths which inherit the orientation from R: the external path is run anti-clockwise and the inner one clockwise. In both cases, if we proceed along the path according to its orientation we see the region R on our left. $\hat{\boldsymbol{\tau}}$ is the unit tangent vector to ∂R and $\hat{\boldsymbol{n}}$ is the outward-pointing unit normal vector.*

★ **Remark 3.2.** The line integrals we are using here are slightly more general than those seen in the previous sections. If R is not simply connected, its boundary is composed of two or more loops $\Gamma_1, \dots, \Gamma_n$. In this case the integrals on ∂R are meant as sums of integrals: $\int_{\partial R} = \int_{\Gamma_1 \cup \ldots \cup \Gamma_n} = \int_{\Gamma_1} + \cdots + \int_{\Gamma_n}$.

Since all boundaries are loops (i.e. paths starting and ending at the same points), the initial point of integration is irrelevant.

We prove an important lemma, which contains the essence of Green's theorem. We recall that the notation $\int_\Gamma f \, \mathrm{d}x$ and $\int_\Gamma f \, \mathrm{d}y$ for an oriented path Γ and a scalar field f was defined in equation (46).

Lemma 3.3. Consider a smooth scalar field f defined on a two-dimensional region $R \subset \mathbb{R}^2$. Then

$$\boxed{\iint_R \frac{\partial f}{\partial y} \, \mathrm{d}A = -\oint_{\partial R} f \, \mathrm{d}x,} \tag{92}$$

$$\boxed{\iint_R \frac{\partial f}{\partial x} \, \mathrm{d}A = \oint_{\partial R} f \, \mathrm{d}y.} \tag{93}$$

Note that in (92) the double integral of the derivative in y is associated to the line integral in $\mathrm{d}x$, as defined in (46); the roles of x and y are swapped in (93). The asymmetry in the sign is due to our choice of the anti-clockwise orientation of ∂R.

[33]Note that the definition of simply-connected domains in three dimensions is quite different from the two-dimensional definition we use here.

Proof of Lemma 3.3. We prove only the first identity (92); the second (93) follows in a similar way. The main ingredient of the proof is the use of the fundamental theorem of vector calculus in the y-direction to reduce the double integral to a line integral. We split the proof in three main steps.

Part 1. We first consider a y-simple domain R, as in formula (52):

$$R = \Big\{ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2, \text{ s.t. } x_L < x < x_R,\ a(x) < y < b(x) \Big\},$$

where $x_L < x_R$, $a, b : (x_L, x_R) \to \mathbb{R}$, $a(x) < b(x)$ for all $x \in (x_L, x_R)$. We compute the double integral at the left-hand side of (92) using the formula for the iterated integral (53) and the fundamental theorem of calculus (90) applied to the partial derivative of f in the y-direction:

$$\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A \overset{(53)}{=} \int_{x_L}^{x_R}\int_{a(x)}^{b(x)} \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \overset{(90)}{=} \int_{x_L}^{x_R} f(x,y)\Big|_{y=a(x)}^{b(x)}\,\mathrm{d}x = \int_{x_L}^{x_R}\Big(f\big(x,b(x)\big) - f\big(x,a(x)\big)\Big)\,\mathrm{d}x. \tag{94}$$

We now consider the boundary integral at the right-hand side of the assertion (92). We split it in four components, corresponding to the four oriented paths[34] in which the boundary ∂R is divided, as in the left plot of Figure 42:

$$\oint_{\partial R} f\,\mathrm{d}x = \int_{\Gamma_S} f\,\mathrm{d}x + \int_{\Gamma_E} f\,\mathrm{d}x + \int_{\Gamma_N} f\,\mathrm{d}x + \int_{\Gamma_W} f\,\mathrm{d}x.$$

We first note that the two lateral paths $\Gamma_E = \{x_R\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, a(x_R) \le y \le b(x_R)\}$ and $\Gamma_W = \{x_L\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, a(x_L) \le y \le b(x_L)\}$ are vertical. The integrals $\int \cdots \mathrm{d}x$ take into account only the horizontal component of the curve total derivative (see the definition in (46)) which is zero on these two segments, thus the two vertical paths do not give any contribution to the integral at the right-hand side of (92).[35] The x components of the parametrisations of the two remaining curvilinear sides can be chosen as affine functions ($\hat{\boldsymbol{\imath}} \cdot \vec{\mathbf{c}}_S(t) = x_L + t$ and $\hat{\boldsymbol{\imath}} \cdot \vec{\mathbf{c}}_N(t) = x_R - t$), while the y components depend on x through the boundary functions a and b ($y = b(x)$ on Γ_N and $y = a(x)$ on Γ_S), thus we can write the integral $\int \cdots \mathrm{d}x$ as in formula (48):

$$\begin{aligned}\oint_{\partial R} f\,\mathrm{d}x &= \int_{\Gamma_S} f\,\mathrm{d}x + \underbrace{\int_{\Gamma_E} f\,\mathrm{d}x}_{=0} + \int_{\Gamma_N} f\,\mathrm{d}x + \underbrace{\int_{\Gamma_W} f\,\mathrm{d}x}_{=0}\\ &\overset{(48)}{=} \int_{x_L}^{x_R}\Big(f\big(x,a(x)\big) - f\big(x,b(x)\big)\Big)\,\mathrm{d}x\\ &\overset{(94)}{=} -\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A,\end{aligned}$$

which is the assertion (92). We have concluded the proof for y-simple domains.

Part 2. We now consider a region R that is the union of two non-overlapping subregions R_1 and R_2 (i.e. $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \emptyset$, being quite imprecise with open and closed sets) such that on both R_1 and R_2 equation (92) holds true. Then we prove that the same equation holds also on the whole of R.

We denote by $\Gamma = \partial R_1 \cap \partial R_2$ the intersection of the boundaries of the two subregions. This is a path that cuts R in two; if the domain R is not simply-connected, i.e. it contains some holes, then the interface Γ might be composed of two or more disconnected paths. We fix on Γ the same orientation of ∂R_1 and we note that this is opposite to the orientation of ∂R_2; see the right plot of Figure 42. Combining the

[34] The path Γ_W collapses to a point if $a(x_L) = b(x_L)$ and similarly the path Γ_E if $a(x_R) = b(x_R)$; see e.g. Figure 25.

[35] To verify this in formulas, we can write explicitly the parametrisations of the four sides:

$$\begin{aligned}
&\Gamma_S: \vec{\mathbf{c}}_S(t) = (x_L + t)\hat{\boldsymbol{\imath}} + a(x_L + t)\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{c}}_S}{dt}(t) = \hat{\boldsymbol{\imath}} + a'(x_L + t)\hat{\boldsymbol{\jmath}}, && 0 < t < x_R - x_L,\\
&\Gamma_E: \vec{\mathbf{c}}_E(t) = x_R\hat{\boldsymbol{\imath}} + \big(a(x_R) + t\big)\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{c}}_E}{dt}(t) = \hat{\boldsymbol{\jmath}}, && 0 < t < b(x_R) - a(x_R),\\
&\Gamma_N: \vec{\mathbf{c}}_N(t) = (x_R - t)\hat{\boldsymbol{\imath}} + b(x_R - t)\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{c}}_N}{dt}(t) = -\hat{\boldsymbol{\imath}} - b'(x_R - t)\hat{\boldsymbol{\jmath}}, && 0 < t < x_R - x_L,\\
&\Gamma_W: \vec{\mathbf{c}}_W(t) = x_L\hat{\boldsymbol{\imath}} + \big(b(x_L) - t\big)\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{c}}_W}{dt}(t) = -\hat{\boldsymbol{\jmath}}, && 0 < t < b(x_L) - a(x_L).
\end{aligned}$$

We see that the two vertical paths Γ_E and Γ_W do not give any contribution to the integral at the right-hand side of (92):

$$\int_{\Gamma_E} f\,\mathrm{d}x \overset{(46)}{=} \int_{\Gamma_E} f\hat{\boldsymbol{\imath}}\cdot\mathrm{d}\vec{\mathbf{r}} \overset{(44)}{=} \int_0^{b(x_R)-a(x_R)} f\big(\vec{\mathbf{c}}_E(t)\big)\,\hat{\boldsymbol{\imath}}\cdot\frac{d\vec{\mathbf{c}}_E}{dt}(t)\,\mathrm{d}t = \int_0^{b(x_R)-a(x_R)} f\big(\vec{\mathbf{c}}_E(t)\big)\,\underbrace{\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\jmath}}}_{=0}\,\mathrm{d}t = 0;$$

and similarly on Γ_W.

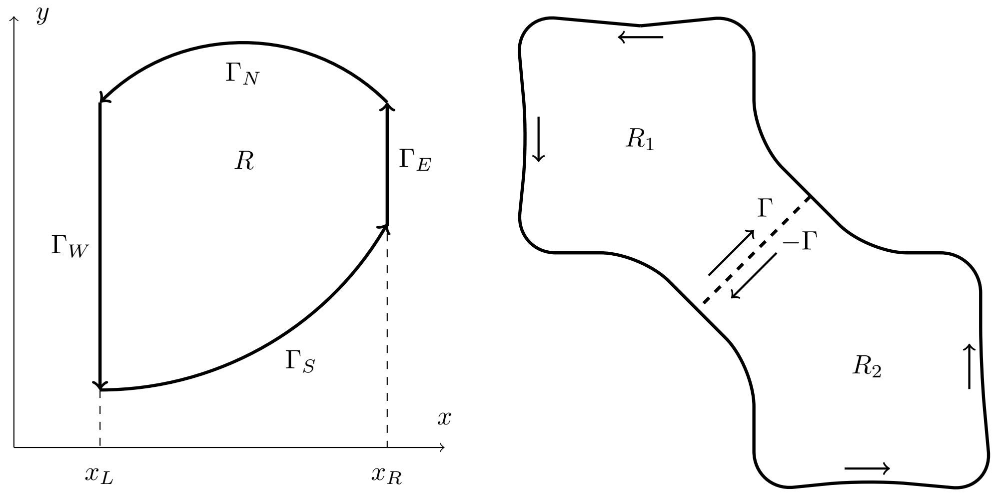


Figure 42: *Proof of Lemma 3.3: the boundary of a y-simple domain is decomposed in four paths (left plot, part 1 of the proof), and the region R is split in two subregions R_1 and R_2 (right plot, part 2 of the proof). The interface $\Gamma = \partial R_1 \cap \partial R_2$ has the same orientation of ∂R_1 and opposite to ∂R_2.*

integrals over different parts of the boundaries involved and using the relations between the sign of line integrals and path orientations we conclude:

$$\begin{aligned}
\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A &= \iint_{R_1} \frac{\partial f}{\partial y}\,\mathrm{d}A + \iint_{R_2} \frac{\partial f}{\partial y}\,\mathrm{d}A && \text{from } R = R_1 \cup R_2 \text{ and (40)}\\
&= -\oint_{\partial R_1} f\,\mathrm{d}x - \oint_{\partial R_2} f\,\mathrm{d}x && \text{since (92) holds in } R_1, R_2\\
&= -\left(\int_{\partial R_1\setminus\Gamma} f\,\mathrm{d}x + \int_\Gamma f\,\mathrm{d}x\right) - \left(\int_{\partial R_2\setminus\Gamma} f\,\mathrm{d}x + \int_{-\Gamma} f\,\mathrm{d}x\right)\\
&= -\left(\int_{\partial R_1\setminus\Gamma} f\,\mathrm{d}x + \int_{\partial R_2\setminus\Gamma} f\,\mathrm{d}x\right) - \underbrace{\left(\int_\Gamma f\,\mathrm{d}x + \int_{-\Gamma} f\,\mathrm{d}x\right)}_{=0}\\
&= -\int_{(\partial R_1\setminus\Gamma)\cup(\partial R_2\setminus\Gamma)} f\,\mathrm{d}x\\
&= -\int_{\partial R} f\,\mathrm{d}x && \text{since } \partial R = (\partial R_1\setminus\Gamma)\cup(\partial R_2\setminus\Gamma).
\end{aligned}$$

(To understand this part of the proof observe carefully in the right plot of Figure 42 how the different geometric objects are related to each other.)

Part 3. Every bounded, piecewise smooth region can be split in a finite number of y-simple subregions $R_1, \dots, R_N$ (this needs to be proved, we take it for granted here). We proved in part 1 that in each of these regions equation (92) holds true. From part 2 we see that this equation holds true in $Q_2 := R_1 \cup R_2$. Applying repeatedly part 2 of the proof we see that the result is true for $Q_j := Q_{j-1} \cup R_j = R_1 \cup \cdots \cup R_j$, for all $3 \le j \le n$ and this concludes the proof because $Q_N = R$. □

Green's theorem immediately follows from Lemma 3.3.

Theorem 3.4 (Green's theorem)**.** Consider a smooth two-dimensional vector field $\vec{\mathbf{F}}(x,y) = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath}$ defined on a region $R \subset \mathbb{R}^2$. Then

$$\boxed{\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}A = \oint_{\partial R} \big(F_1\,\mathrm{d}x + F_2\,\mathrm{d}y\big),} \tag{95}$$

which can also be written as

$$\boxed{\iint_R \big(\vec{\nabla}\times\vec{\mathbf{F}}\big)_3\,\mathrm{d}A = \oint_{\partial R} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}.} \tag{96}$$

Proof. Equation (95) follows from choosing $f = F_1$ in (92), $f = F_2$ in (93) and summing the two results. Equation (96) is obtained from the definition of the curl (23) and the expansion of the line integral (47). □

Remark 3.5 (Green's theorem for three-dimensional fields in two-dimensional domains)**.** We have stated Green's theorem for a two-dimensional vector field $\vec{\mathbf{F}}(x,y) = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}}$. If $\vec{\mathbf{F}}$ is any three-dimensional, smooth vector field, i.e. $\vec{\mathbf{F}}(x,y,z) = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}} + F_3\hat{\boldsymbol{k}}$ may depend also on the z coordinate and have three non-zero components, formula (96) still holds true. In this case, we think at $R \subset \mathbb{R}^2$ as lying in the xy-plane of $\mathbb{R}^3$ (the plane $\{z = 0\}$). Indeed, the left-hand side of the equation is not affected by this modification as (from the definition of the curl (23)) the third component of the curl $(\vec{\nabla} \times \vec{\mathbf{F}})_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ does not involve neither F_3 nor the partial derivative in z. Similarly, the right-hand side of (96) is not modified because the circulation of $\vec{\mathbf{F}}$ along ∂R is the integral of the scalar product of $\vec{\mathbf{F}}$ and the total derivative of the curve defining ∂R (see (44)), whose z-component vanish, thus the component F_3 does not contribute to the line integral.

Example 3.6. Use Green's theorem to compute the circulation of $\vec{\mathbf{F}} = (2xy + y^2)\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}}$ along the boundary of the bounded domain R delimited by the line $y = x$ and the parabola $y = x^2$.

The domain R can be written as $R = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ 0 < x < 1,\ x^2 < y < x\}$; see the left plot in Figure 43. The curl of the given field is $\vec{\nabla} \times \vec{\mathbf{F}} = (2x - 2x - 2y)\hat{\boldsymbol{k}} = -2y\hat{\boldsymbol{k}}$. Then, Green's theorem gives

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(96)}{=} \iint_R (\vec{\nabla} \times \vec{\mathbf{F}})_3 \,\mathrm{d}A \overset{(53)}{=} \int_0^1 \left(\int_{x^2}^x (-2y)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^1 (-x^2 + x^4)\,\mathrm{d}y = \frac{1}{5} - \frac{1}{3} = -\frac{2}{15}.$$

Of course, it is possible to directly compute the circulation, but the calculation is a bit longer, as we have to split the boundary in the straight segment Γ_S (e.g. parametrised by $\vec{\mathbf{a}}(t) = (1-t)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$, $0 < t < 1$) and the parabolic arc Γ_P (e.g. parametrised by $\vec{\mathbf{b}}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}$, $0 < t < 1$). (Recall the parametrisations of Remark 1.24 and that the boundary must be oriented anti-clockwise.) Then we have

$$\begin{aligned}
\oint_{\partial R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_P} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \\
&= \int_0^1 \big(3(1-t)^2\hat{\boldsymbol{\imath}} + (1-t)^2\hat{\boldsymbol{\jmath}}\big) \cdot (-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}})\,\mathrm{d}t + \int_0^1 \big((2t^3 + t^4)\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}\big) \cdot (\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t \\
&= -4\int_0^1 (1-t)^2\,\mathrm{d}t + \int_0^1 (4t^3 + t^4)\,\mathrm{d}t = -\frac{4}{3} + 1 + \frac{1}{5} = -\frac{2}{15}.
\end{aligned}$$

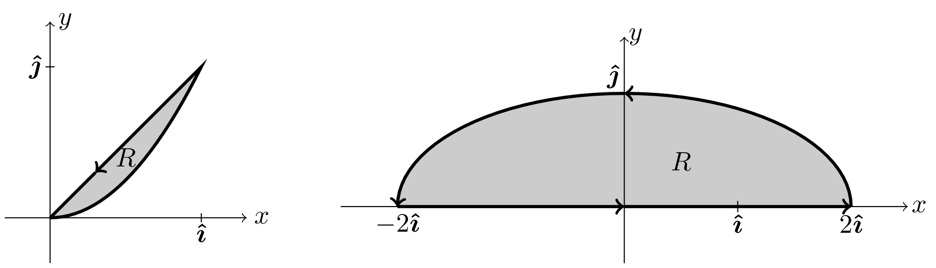


Figure 43: *Left plot: the domain $R = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ 0 < x < 1,\ x^2 < y < x\}$ in Example 3.6.*
Right plot: the half ellipse $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ \frac{1}{4}x^2 + y^2 < 1,\ y > 0\}$ in Exercise 3.8.

Exercise 3.7. ▶ Demonstrate Green's theorem for the vector field $\vec{\mathbf{F}} = (x + 2y)(\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})$ and the disc $\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},$ such that $x^2 + y^2 < 4\}$.

Exercise 3.8. ▶ Demonstrate Green's theorem for $\vec{\mathbf{F}} = y^2\hat{\boldsymbol{\imath}}$ on the half-ellipse $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ \frac{1}{4}x^2 + y^2 < 1,\ y > 0\}$ (right plot in Figure 43).

Example 3.9 (Measuring areas with line integrals)**.** A (perhaps) surprising application of Green's theorem is the possibility of measuring the area of a planar region R by computing only line integrals on its boundary[36]. To do this, we need a vector field $\vec{\mathbf{F}}$ such that the integrand on R in (95) is everywhere equal to one, namely $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. Several choices are possible, for instance $\vec{\mathbf{F}} = -y\hat{\boldsymbol{\imath}}$, $\vec{\mathbf{F}} = x\hat{\boldsymbol{\jmath}}$, or $\vec{\mathbf{F}} = \frac{1}{2}(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$. Each of these gives rise to a line integral that computes the area of R:

$$\mathrm{Area}(R) = \iint_R \mathrm{d}A = -\oint_{\partial R} y\,\mathrm{d}x = \oint_{\partial R} x\,\mathrm{d}y = \frac{1}{2}\oint_{\partial R} (-y\,\mathrm{d}x + \mathrm{d}y).$$

[36]The possibility of computing areas with line integrals should not be a surprise: we already know that the area of a y-simple domain (52), for example, can be computed as

$$\mathrm{Area}(R) = \iint_R \mathrm{d}A = \int_{x_L}^{x_R} \left(\int_{a(x)}^{b(x)} \mathrm{d}y \right) \mathrm{d}x = \int_{x_L}^{x_R} \big(b(x) - a(x)\big)\,\mathrm{d}x$$

which is nothing else than $-\oint_{\partial R} y\,\mathrm{d}x$. This is also the formula for the area of the region comprised between two graphs you already know from first year.

In Section 2.1.2, we defined the unit tangent vector field $\hat{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{a}}}{dt}/\left|\frac{d\vec{\mathbf{a}}}{dt}\right|$ of an oriented path Γ defined by the curve $\vec{\mathbf{a}}$. If $\Gamma = \partial R$ is the boundary of R (with the orientation convention described above), then the outward pointing unit normal vector $\hat{\boldsymbol{n}}$ on ∂R is related to the unit tangent $\hat{\boldsymbol{\tau}}$ by the formulas

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{k}} = t_2\hat{\boldsymbol{\imath}} - t_1\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\tau}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = -n_2\hat{\boldsymbol{\imath}} + n_1\hat{\boldsymbol{\jmath}}, \qquad (n_1 = t_2,\ n_2 = -t_1); \tag{97}$$

see Figure 41. (Note that, if R contains a hole, then on the the corresponding part of the boundary the vector field $\hat{\boldsymbol{n}}$ points out of R and into the hole.) For a vector field $\vec{\mathbf{F}} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}}$, choosing $f = F_1$ in (93), $f = F_2$ in (92), and summing the resulting equations we obtain

$$\begin{aligned}
\iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) \mathrm{d}A \overset{(92),(93)}{=} & \oint_{\partial R} \big(F_1\,\mathrm{d}y - F_2\,\mathrm{d}x\big) \\
\overset{(46)}{=} & \oint_{\partial R} \big(F_1\hat{\boldsymbol{\jmath}} - F_2\hat{\boldsymbol{\imath}}\big)\cdot \mathrm{d}\vec{\mathbf{r}} \\
\overset{(2)}{=} & \oint_{\partial R} \big(\hat{\boldsymbol{k}} \times \vec{\mathbf{F}}\big)\cdot \mathrm{d}\vec{\mathbf{r}} \\
\overset{(45)}{=} & \oint_{\partial R} \big((\hat{\boldsymbol{k}} \times \vec{\mathbf{F}})\cdot\hat{\boldsymbol{\tau}}\big)\,\mathrm{d}s \\
\overset{\text{Ex. } 1.15}{=} & \oint_{\partial R} \big((\hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{k}})\cdot\vec{\mathbf{F}}\big)\,\mathrm{d}s \\
\overset{(97)}{=} & \oint_{\partial R} \big(\vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}\big)\,\mathrm{d}s.
\end{aligned}$$

This is the proof of the divergence theorem in two dimensions.

Theorem 3.10 (Divergence theorem in two-dimensions). Let $\vec{\mathbf{F}} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}}$ be a smooth vector field defined in a region $R \subset \mathbb{R}^2$ with outward pointing unit normal vector field $\hat{\boldsymbol{n}}$. Then

$$\boxed{\iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) \mathrm{d}A = \oint_{\partial R} \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s.} \tag{98}$$

The integrand in the double integral at the left-hand side of (98) is the two-dimensional divergence $\vec{\nabla}\cdot\vec{\mathbf{F}}$ of $\vec{\mathbf{F}}$.

Example 3.11. As in Example 3.6, consider the vector field $\vec{\mathbf{F}} = (2xy + y^2)\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}}$ and the bounded domain R delimited by the line $y = x$ and the parabola $y = x^2$. Compute the "flux"[37] $\oint_{\partial R} \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,\mathrm{d}s$ of $\vec{\mathbf{F}}$ through ∂R.

We simply apply the divergence theorem (98):

$$\oint_{\partial R} \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,\mathrm{d}s = \iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)\mathrm{d}A = \iint_R 2y\,\mathrm{d}A = \int_0^1 \left(\int_{x^2}^{x} 2y\,\mathrm{d}y\right)\mathrm{d}x = \frac{2}{15},$$

with the same computation as in the previous example. The direct computation of the contour integral is slightly more complicated in this case than in Example 3.6, in that it requires the calculation of the outward-pointing unit normal vector field.

Exercise 3.12. ▶ Prove the second formula in (97) without using the expansion of the vectors in components. You can find a helpful formula in the exercises in Section 1.1.

3.2 The divergence theorem

In this section we consider a three-dimensional domain $D \subset \mathbb{R}^3$ and we assume it to be piecewise smooth, bounded and connected. As in the previous section, we start with a preliminary lemma.

Lemma 3.13. Let f be a smooth scalar field on D. Then we have

$$\boxed{\iiint_D \frac{\partial f}{\partial z}\,\mathrm{d}V = \oiint_{\partial D} f\,\hat{\boldsymbol{k}}\cdot \mathrm{d}\vec{\mathbf{S}}.} \tag{99}$$

Here the surface ∂D is equipped with outward-pointing unit normal vector field $\hat{\boldsymbol{n}}$.

[37] Note that here the word "flux" is used to denote the contour integral of the normal component of a two-dimensional field, as opposed to the surface integral of the normal component in three dimensions.

Proof. The proof is very similar to that of Lemma 3.3 (in a sense, it is its extension to three dimensions). As in that case, we divide the proof in three parts: the first is devoted to simple domains, the second extends the result to the unions of two such simple domains, the third completes the extension to general domains by partitioning these in simpler parts. Once more, the basic instrument in the proof is the fundamental theorem of calculus.

Part 1. We consider a z-simple domain as that introduced at the beginning of Section 2.2.3:

$$D = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \text{ s. t. } x_L < x < x_R, \quad a(x) < y < b(x), \quad \alpha(x,y) < z < \beta(x,y)\Big\},$$

where $x_L < x_R$ are two real numbers, $a, b : [x_L, x_R] \to \mathbb{R}$ two real functions (with $a < b$), and $\alpha, \beta : R \to \mathbb{R}$ two scalar fields in two variables (with $\alpha < \beta$). We call R the planar region $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \text{ s. t. } x_L < x < x_R,\ a(x) < y < b(x)\}$, the projection of D on the xy-plane. See a representation of D and R in Figure 44.

The boundary ∂D is composed of six (curvilinear) faces[38]. Four of these faces are vertical, as they lie straight above the four (curvilinear) sides of R. Thus on these faces the outward-pointing unit normal vector field $\hat{\boldsymbol{n}}$ has zero vertical component $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}$, and the corresponding terms $\iint f\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}}$ in the surface integral at the right-hand side of (99) vanish.[39]

On the other hand, the top face S_T is the graph of the two-dimensional field $\beta : R \to \mathbb{R}$. The contribution to the surface integral at the right-hand side of (99) given by S_T is the flux of the vector field $f\hat{\boldsymbol{k}}$ (whose first two components are constantly zero) through S_T itself, and can be reduced to a double integral on R using formula (74):

$$\iint_{S_T} f(x,y,z)\, \hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_R f\big(x,y,\beta(x,y)\big) \,\mathrm{d}A.$$

Similarly, the bottom face S_B is the graph of the two-dimensional field $\alpha : R \to \mathbb{R}$. However, on this face, the outward-pointing unit normal vector field $\hat{\boldsymbol{n}}$ points downward, opposite to the convention stipulated in Section 2.2.5 for graph surfaces. Thus the sign in (74) needs to be reversed and the contribution of S_B to the right-hand side of (99) reads

$$\iint_{S_T} f(x,y,z)\, \hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} = -\iint_R f\big(x,y,\alpha(x,y)\big) \,\mathrm{d}A.$$

To conclude the proof of the first part we expand the right-hand side of the identity in the assertion in a sum over the faces of D and use the fundamental theorem of calculus to transform it into a triple integral:

$$\begin{aligned}
\oiint_{\partial D} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} &= \iint_{S_T} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_B} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} \\
&\quad + \underbrace{\iint_{S_W} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_S} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_E} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_N} f\,\hat{\boldsymbol{k}} \cdot \mathrm{d}\vec{\mathbf{S}}}_{=0 \quad (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}=0)} \\
&= \iint_R f\big(x,y,\beta(x,y)\big)\,\mathrm{d}A - \iint_R f\big(x,y,\alpha(x,y)\big)\,\mathrm{d}A \\
&= \iint_R \Big(f\big(x,y,\beta(x,y)\big) - f\big(x,y,\alpha(x,y)\big)\Big)\,\mathrm{d}A \\
&\overset{(90)}{=} \iint_R \left(\int_{\alpha(x,y)}^{\beta(x,y)} \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}z\right)\mathrm{d}A \qquad \text{(fundamental theorem of calculus)} \\
&\overset{(62)}{=} \iiint_D \frac{\partial f}{\partial z}(x,y,z)\,\mathrm{d}V \qquad \text{(triple integral as an iterated integral).}
\end{aligned}$$

---

[38] Actually, if $a(x_L) = b(x_L)$ or $a(x_R) = b(x_R)$, then the left and right faces are collapsed to a segment and the total number of faces is reduced to 5 or 4. The number of non-empty faces can be even lower, consider the following example: $\{-1 < x < 1, -\sqrt{x^2+y^2} < y < \sqrt{x^2+y^2}, \sqrt{1-x^2-y^2} < z < 2\sqrt{1-x^2-y^2}\}$. How many non-empty faces does this z-simple domain have?

[39] More explicitly, the faces and the corresponding unit normals can be written as:

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \begin{cases}
\hat{\boldsymbol{n}}_W = -\hat{\boldsymbol{\imath}} & \text{on } \big\{x = x_L,\ a(x_L) < y < b(x_L),\ \alpha(x_L, y) < z < \beta(x_L, y)\big\}, \\
\hat{\boldsymbol{n}}_S = \frac{a'(x)\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}}{\sqrt{1+(a'(x))^2}} & \text{on } \big\{x_L < x < x_R,\ y = a(x),\ \alpha\big(x, a(x)\big) < z < \beta\big(x, a(x)\big)\big\}, \\
\hat{\boldsymbol{n}}_E = \hat{\boldsymbol{\imath}} & \text{on } \big\{x = x_R,\ a(x_R) < y < b(x_R),\ \alpha(x_R, y) < z < \beta(x_R, y)\big\}, \\
\hat{\boldsymbol{n}}_N = \frac{-b'(x)\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{1+(b'(x))^2}} & \text{on } \big\{x_L < x < x_R,\ y = b(x),\ \alpha\big(x, b(x)\big) < z < \beta\big(x, b(x)\big)\big\}.
\end{cases}$$

You can verify these expressions for $\hat{\boldsymbol{n}}$ by finding two tangent vectors and checking that they are orthogonal to $\hat{\boldsymbol{n}}$.

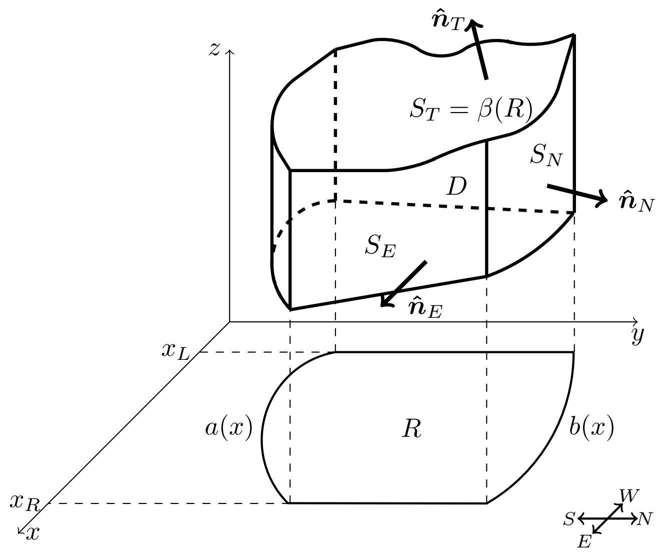


Figure 44: *A representation of the the z-simple domain in the first part of the proof of Lemma 3.13. The bottom face S_B is that underneath, the west face S_W is behind D and the south face S_S is on the left of the figure. The outward-pointing unit vector field $\hat{\boldsymbol{n}}$ on S_W, S_S, S_E and S_N, has zero vertical component $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}$. The fundamental theorem of calculus is applied along all vertical segments connecting the bottom face with the top face.*

Part 2. We now show the following: if a domain D is disjoint union of two subdomains D_1 and D_2, and if identity (99) holds in both of them, then the same identity holds in D.

We denote by $\Sigma = \partial D_1 \cap \partial D_2$ the interface between the subdomains and with $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2$ the outward-pointing unit normal vector fields on ∂D_1 and ∂D_2, respectively. Then $\partial D_1 = \Sigma \cup (\partial D_1 \cap \partial D)$ (and $\partial D_2 = \Sigma \cup (\partial D_2 \cap \partial D)$), $\hat{\boldsymbol{n}}_1 = -\hat{\boldsymbol{n}}_2$ on Σ and $\hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{n}}$ on $\partial D_1 \cap \partial D$ (similarly $\hat{\boldsymbol{n}}_2 = \hat{\boldsymbol{n}}$ on $\partial D_2 \cap \partial D$). Combining everything we have the assertion:

$$\begin{aligned}
\iiint_D \frac{\partial f}{\partial z} \,\mathrm{d}V &= \iiint_{D_1} \frac{\partial f}{\partial z} \,\mathrm{d}V + \iiint_{D_2} \frac{\partial f}{\partial z} \,\mathrm{d}V \\
&= \oiint_{\partial D_1} f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_1 \,\mathrm{d}S + \oiint_{\partial D_2} f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_2 \,\mathrm{d}S \\
&= \iint_{\partial D_1 \cap \partial D} f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_1 \,\mathrm{d}S + \iint_\Sigma f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_1 \,\mathrm{d}S + \iint_{\partial D_2 \cap \partial D} f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_2 \,\mathrm{d}S + \iint_\Sigma f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_2 \,\mathrm{d}S \\
&= \iint_{(\partial D_1 \cap \partial D) \cup (\partial D_2 \cap \partial D)} f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S + \iint_\Sigma \underbrace{(f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_1 + f\,\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}_2)}_{=0} \,\mathrm{d}S \\
&= \iint_{\partial D} f\,\hat{\boldsymbol{k}} \,\mathrm{d}\vec{\mathbf{S}}.
\end{aligned}$$

Part 3. Any "regular" domain can be decomposed in a finite union of disjoint z-simple domains (of course, this should be proved rigorously, here we only assume it since we have not even described in detail the regularity of the domains). Thus, to conclude the proof of the lemma, we can proceed by induction, exactly as we did in part 3 of Lemma 3.3. □

Proceeding as in Lemma 3.13, we can prove that

$$\iiint_D \frac{\partial f}{\partial x} \,\mathrm{d}V = \oiint_{\partial D} f\,\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{S}}, \qquad \iiint_D \frac{\partial f}{\partial y} \,\mathrm{d}V = \oiint_{\partial D} f\,\hat{\boldsymbol{\jmath}} \cdot \mathrm{d}\vec{\mathbf{S}}. \tag{100}$$

From identities (99) and (100) we obtain the following fundamental theorem.

Theorem 3.14 (Divergence theorem in three-dimensions)**.** Let $\vec{\mathbf{F}}$ be a smooth vector field defined on D. Then

$$\boxed{\iiint_D \vec{\nabla} \cdot \vec{\mathbf{F}} \,\mathrm{d}V = \oiint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}.} \tag{101}$$

In words: the triple integral over D of the divergence of $\vec{\mathbf{F}}$ equals the flux of $\vec{\mathbf{F}}$ through the boundary of D.

Proof. We simply sum (99) and (100) applied to the three components of $\vec{\mathbf{F}}$:

$$\begin{aligned}\iiint_D \vec{\nabla}\cdot\vec{\mathbf{F}}\,\mathrm{d}V &= \iiint_D \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)\mathrm{d}V\\ &= \iiint_D \frac{\partial F_1}{\partial x}\,\mathrm{d}V + \iiint_D \frac{\partial F_2}{\partial y}\,\mathrm{d}V + \iiint_D \frac{\partial F_3}{\partial z}\,\mathrm{d}V\\ &= \oiint_{\partial D} F_1\hat{\boldsymbol{\imath}}\cdot\mathrm{d}\vec{\mathbf{S}} + \oiint_{\partial D} F_2\hat{\boldsymbol{\jmath}}\cdot\mathrm{d}\vec{\mathbf{S}} + \oiint_{\partial D} F_3\hat{\boldsymbol{k}}\cdot\mathrm{d}\vec{\mathbf{S}}\\ &= \oiint_{\partial D}(F_1\hat{\boldsymbol{\imath}}+F_2\hat{\boldsymbol{\jmath}}+F_3\hat{\boldsymbol{k}})\cdot\mathrm{d}\vec{\mathbf{S}}\\ &= \oiint_{\partial D}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}}.\end{aligned}$$

□

The divergence theorem, together with its corollaries, is one of the most fundamental results in analysis and in calculus. It holds in any dimensions, appears in every sort of theoretical and applied setting (PDEs, differential geometry, fluid dynamics, electromagnetism, ...), has a deep physical meaning, is the starting point for the design of many numerical methods, and its extension to extremely irregular fields and domains is very challenging (and indeed it is currently an object of research after two centuries from the first discovery of the theorem!).

Remark 3.15 (Intuitive physical interpretation of the divergence theorem)**.** If the vector field $\vec{\mathbf{F}}$ represents the velocity of a fluid, its divergence might be interpreted as a measure of the sources and sinks in the fluid. Thus the divergence theorem may be interpreted as follows: the total net production of a quantity in a domain equals the amount of that quantity that exits the domain through its boundary (per unit of time). For this reason, the divergence theorem is used to formulate many "conservation laws" or "continuity equations" describing the evolution and the conservation of certain physical quantities (e.g. mass, energy, charge,...).

Exercise 3.16. ▶ Use the divergence theorem to compute the flux of the field $\vec{\mathbf{F}} = \mathrm{e}^x\hat{\boldsymbol{\imath}}+\cos^2 x\hat{\boldsymbol{\jmath}}+(x+y+z)\hat{\boldsymbol{k}}$ through the boundary of the box $D=(0,1)\times(-1,1)\times(0,10)$.

Exercise 3.17. ▶ (Difficult!) Use the divergence theorem to compute $\oiint_S(x^2+y^2)\,\mathrm{d}S$, where $S=\{|\vec{\mathbf{r}}|=R\}$ is the sphere of radius R centred at the origin. Hint: you need to define a suitable vector field.

Exercise 3.18. ▶ Consider the cone $C=\{0<z<1,\ x^2+y^2<z^2\}$. Use the divergence theorem to compute $\iiint_C |\vec{\mathbf{r}}|^2\,\mathrm{d}V$.

Exercise 3.19. ▶ *(i)* Use the divergence theorem to compute the flux of the vector field

$$\vec{\mathbf{F}} = \frac{1}{3}(x^3+y^3+z^3)\hat{\boldsymbol{\imath}} + yz(z\hat{\boldsymbol{\jmath}}+y\hat{\boldsymbol{k}})$$

through the boundary of the unit ball $B=\{\vec{\mathbf{r}}\in\mathbb{R}^3, |\vec{\mathbf{r}}|<1\}$. Hint: use spherical coordinates.

(ii) Find $L>0$ such that the flux of $\vec{\mathbf{F}}$ through the boundary of the cube $C_L=(0,L)^3=\{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}}, 0<x,y,z<L\}$ is equal to the flux of the same field through the boundary of the unit ball B.

Exercise 3.20. ▶ [★ Very difficult!] Let $D\subset\mathbb{R}^3$ be a bounded domain. Prove that

$$\oiint_{\partial D}\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3}\cdot\mathrm{d}\vec{\mathbf{S}} = \begin{cases}4\pi & \text{if the origin lies in } D,\\ 0 & \text{if the origin lies outside } D.\end{cases}$$

In the exercises above we used the divergence theorem for different purposes: to compute a surface integral over a boundary (by finding a vector field whose normal component on the boundary is given, as in 3.17) and to compute a volume integral (by finding a vector field whose divergence is given, as in 3.18).

★ **Remark 3.21** (The divergence theorem in one dimension is the fundamental theorem of calculus)**.** The divergence theorems 3.10 and 3.14 directly generalise the fundamental theorem of calculus to higher dimensions. Indeed, a bounded, connected (open) domain in $\mathbb{R}$ must be an interval $I=(a,b)$, whose boundary is composed by two points $\partial I=\{a,b\}$ with outward-pointing normal vector $n(a)=-1$ and $n(b)=1$ (a one-dimensional vector is simply a scalar). The "zero-dimensional integral" of a function f on ∂I reduces to the sum of the two values of f in b and a, and the flux to their difference. A "one-dimensional vector field" is a real function and its divergence is its derivative. So we have

$$\int_I f'(x)\,\mathrm{d}x = \int_I \vec{\nabla}\cdot f(x)\,\mathrm{d}x = \int_{\partial I} f(x)n(x)\,\mathrm{d}x = f(b)n(b)+f(a)n(a) = f(b)-f(a).$$

In the next corollary we show several other identities which follow from equations (99) and (100). On the boundary ∂D (or more generally on any oriented surface), we call **normal derivative** the scalar product of the gradient of a scalar field f and the unit normal field $\hat{\boldsymbol{n}}$ of f, and we denote it by

$$\boxed{\frac{\partial f}{\partial n} := \hat{\boldsymbol{n}} \cdot \vec{\nabla} f.}$$

Corollary 3.22. Let f and g be scalar fields and $\vec{\mathbf{F}}$ be a vector field, all defined on a domain D. Then

$$\iiint_D \vec{\nabla} f \, \mathrm{d}V = \oiint_{\partial D} f \hat{\boldsymbol{n}} \, \mathrm{d}S, \tag{102}$$

$$\iiint_D \vec{\nabla} \times \vec{\mathbf{F}} \, \mathrm{d}V = \oiint_{\partial D} \hat{\boldsymbol{n}} \times \vec{\mathbf{F}} \, \mathrm{d}S, \tag{103}$$

$$\iiint_D (f\Delta g + \vec{\nabla} g \cdot \vec{\nabla} f) \, \mathrm{d}V = \oiint_{\partial D} f \frac{\partial g}{\partial n} \, \mathrm{d}S \qquad \text{(Green's 1st identity)}, \tag{104}$$

$$\iiint_D (f\Delta g - g\Delta f) \, \mathrm{d}V = \oiint_{\partial D} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \mathrm{d}S \qquad \text{(Green's 2nd identity)}. \tag{105}$$

Note that identities (102) and (103) are vectorial (the values at the two sides of the equal sign are vectors).

Proof. The proof of identity (102) is very easy:

$$\begin{aligned}
\iiint_D \vec{\nabla} f \, \mathrm{d}V &= \iiint_D \left(\frac{\partial f}{\partial x} \hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y} \hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z} \hat{\boldsymbol{k}} \right) \mathrm{d}V \\
&\overset{(99),(100)}{=} \oiint_{\partial D} \left(f(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{\imath}} + f(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{\jmath}} + f(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{k}} \right) \mathrm{d}S = \oiint_{\partial D} f \hat{\boldsymbol{n}} \, \mathrm{d}S.
\end{aligned}$$

We prove identity (103) for the first component:

$$\begin{aligned}
\iiint_D (\vec{\nabla} \times \vec{\mathbf{F}})_1 \, \mathrm{d}V &\overset{(23)}{=} \iiint_D \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathrm{d}V \\
&\overset{(99),(100)}{=} \oiint_{\partial D} \left(F_3 \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}} - F_2 \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}} \right) \mathrm{d}S \\
&= \oiint_{\partial D} \left(F_3 n_2 - F_2 n_3 \right) \mathrm{d}S \overset{(2)}{=} \oiint_{\partial D} (\hat{\boldsymbol{n}} \times \vec{\mathbf{F}})_1 \, \mathrm{d}S;
\end{aligned}$$

and the same holds for the second and third components.

Using the vector identities of Section 1.4, we have

$$\vec{\nabla} \cdot (f \vec{\nabla} g) \overset{(29)}{=} \vec{\nabla} f \cdot \vec{\nabla} g + f \vec{\nabla} \cdot (\vec{\nabla} g) \overset{(24)}{=} \vec{\nabla} f \cdot \vec{\nabla} g + f \Delta g.$$

Then, Green's first identity (104) follows immediately from the application of the divergence theorem 3.14 to the field $\vec{\mathbf{F}} = f \vec{\nabla} g$. Green's second identity (105) follows from subtracting from (104) the same identity with f and g interchanged. □

Exercise 3.23. ▶ Demonstrate identity (102) for the field $f = xyz$ and the unit cube $D = (0,1)^3$.

Exercise 3.24. ▶ Show that if a smooth scalar field f is harmonic in a domain D and its normal derivative vanishes everywhere on the boundary ∂D, then f is constant.

★ **Remark 3.25.** The two Green's identities (104) and (105) are particularly important in the theory of partial differential equations (PDEs). In particular, several second order PDEs involving the Laplacian operator can be rewritten using (104) in a "variational form" or "weak form", which involves a volume integral and a boundary one. This is the starting point for the definition of the most common algorithms for the numerical approximation of the solutions on a computer, in particular for the finite element method (FEM, you might see it in a future class). Another common numerical scheme, the boundary element method (BEM), arises from the second Green identity (105).

Comparison with scalar calculus 3.26 (Integration by parts in more dimensions)**.** In the first calculus class you learned how to "integrate by parts":

$$\int_a^b f'(t)g(t) \, \mathrm{d}t + \int_a^b f(t)g'(t) \, \mathrm{d}t = f(b)g(b) - f(a)g(a),$$

for real functions $f, g : [a,b] \to \mathbb{R}$. This is a straightforward consequence of the fundamental theorem of calculus (90) and the product rule $(fg)' = f'g + fg'$. How does this extend to higher dimensions? The product rule was extended to partial derivatives in (8) and to differential operators in Proposition 1.55; we saw in Remark 3.21 that the divergence theorem extends the fundamental theorem of calculus. Several formulas arising from different combinations of these ingredients are usually termed "multidimensional integration by parts", for example

$$\iiint_D \vec{\nabla} f \cdot \vec{\mathbf{G}} \,\mathrm{d}V + \iiint_D f \vec{\nabla} \cdot \vec{\mathbf{G}} \,\mathrm{d}V = \oint\!\!\!\!\oint_{\partial D} f \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{S}},$$
$$\iiint_D (\vec{\nabla} f) g \,\mathrm{d}V + \iiint_D f \vec{\nabla} g \,\mathrm{d}V = \oint\!\!\!\!\oint_{\partial D} f g \,\hat{\boldsymbol{n}} \,\mathrm{d}S,$$

where f, g are scalar fields and $\vec{\mathbf{G}}$ is a vector field on D. For exercise, prove these two identities.

★ **Remark 3.27** (Differential and integral form of physical laws)**.** The main reason of the importance and ubiquity of the divergence theorem in physics and engineering is that it constitutes the relation between the two possible formulation of many physical laws: the "differential form", expressed by a partial differential equation, and the "integral form", expressed by an integral equation. The integral form usually better describes the main physical concepts and contains the quantities that can be measured experimentally, while the differential form allows an easier mathematical manipulation. The two forms may lead to different numerical algorithms for the approximation of the solutions of the equation.

For example, we consider Gauss' law of electrostatics, you might have already encountered elsewhere. Its integral form states that the flux of an electric field $\vec{\mathbf{E}}$ (which is a vector field) through a closed surface ∂D (in vacuum) is proportional to the total electrical charge in the volume D bounded by that surface:

$$\oint\!\!\!\!\oint_{\partial D} \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iiint_D \frac{\rho}{\varepsilon_0} \,\mathrm{d}V,$$

where ρ is the charge density (whose integral gives the total charge) and ε_0 is a constant of proportionality (the "vacuum permittivity"). Its differential form reads $\vec{\nabla} \cdot \vec{\mathbf{E}} = \rho/\varepsilon_0$. The divergence theorem immediately allows to deduce the integral form of Gauss law from the differential one; since the former holds for *any* domain D, also the converse implication can be proved. Gauss' law is one of the four celebrated Maxwell's equations, the fundamental laws of electromagnetism: all of them have a differential and an integral form, related to each other either by the divergence or Stokes' theorem.

3.3 Stokes' theorem

Stokes' theorem extends Green's theorem to general oriented surfaces. It states that the flux of the curl of a vector field through an oriented surface equals the circulation along the boundary of the same surface.

Theorem 3.28 (Stokes' theorem)**.** Let $S \subset \mathbb{R}^3$ be a piecewise smooth, bounded, oriented surface with unit normal field $\hat{\boldsymbol{n}}$, and let $\vec{\mathbf{F}}$ be a smooth vector field defined on S. Then

$$\boxed{\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} = \oint_{\partial S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}},} \tag{106}$$

where the boundary ∂S is understood as the oriented path with the orientation inherited from $(S, \hat{\boldsymbol{n}})$.

Proof. We prove the theorem only for the special class of surfaces describe in Remark 2.44: we assume S to be the graph of a two-dimensional field g, i.e. $S = \{\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \in \mathbb{R}^3 \text{ s.t. } x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in R, z = g(x,y)\}$ where $R \subset \mathbb{R}^2$ is a planar region as in Section 3.1. The proof of the theorem in its generality requires pasting together two or more simpler surfaces.

The main idea of the proof is to reduce the two integrals in (106) to similar integrals on the flat region R and its boundary, using the fact that S is the graph of g, so the variables on S are related to each other by the equation $z = g(x,y)$. Then one applies Green's theorem on R.

The surface integral (flux) at the left-hand side of the assertion (106) can easily be transformed into a double integral over the planar region R by using formula (74):

$$\begin{aligned}\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} &\overset{(74)}{=} \iint_R \Big(-(\vec{\nabla} \times \vec{\mathbf{F}})_1 \frac{\partial g}{\partial x} - (\vec{\nabla} \times \vec{\mathbf{F}})_2 \frac{\partial g}{\partial y} + (\vec{\nabla} \times \vec{\mathbf{F}})_3 \Big) \,\mathrm{d}A \\ &\overset{(23)}{=} \iint_R \Big(-\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) \frac{\partial g}{\partial x} - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) \frac{\partial g}{\partial y} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) \Big) \,\mathrm{d}A.\end{aligned} \tag{107}$$

We denote by $\vec{\mathbf{a}} : [t_I, t_F] \to \partial S$ a curve that parametrises ∂S. The surface S is the graph of g, thus the components of $\vec{\mathbf{a}}$ are related one another by the relation $a_3(t) = g(a_1(t), a_2(t))$. From the chain rule (37) we have $\frac{da_3}{dt} = \frac{\partial g}{\partial x}\frac{da_1}{dt} + \frac{\partial g}{\partial y}\frac{da_2}{dt}$. Moreover, the planar curve $a_1(t)\hat{\boldsymbol{\imath}} + a_2(t)\hat{\boldsymbol{\jmath}}$ is a parametrisation of ∂R, thus $\oint_{\partial S} f(x,y,z)\,\mathrm{d}x = \oint_{\partial R} f(x,y,g(x,y))\,\mathrm{d}x$ for any scalar field f (and similarly for the integral in $\mathrm{d}y$). Putting together all the pieces:

$$\begin{aligned}
\oint_{\partial S} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} &\overset{(44)}{=} \int_{t_I}^{t_F} \Big(F_1\big(\vec{\mathbf{a}}(t)\big)\frac{da_1(t)}{dt} + F_2\big(\vec{\mathbf{a}}(t)\big)\frac{da_2(t)}{dt} + F_3\big(\vec{\mathbf{a}}(t)\big)\frac{da_3(t)}{dt}\Big)\,\mathrm{d}t\\
&= \int_{t_I}^{t_F} \Big(F_1\big(\vec{\mathbf{a}}(t)\big)\frac{da_1(t)}{dt} + F_2\big(\vec{\mathbf{a}}(t)\big)\frac{da_2(t)}{dt} + F_3\big(\vec{\mathbf{a}}(t)\big)\underbrace{\Big(\frac{\partial g}{\partial x}\frac{da_1(t)}{dt} + \frac{\partial g}{\partial y}\frac{da_2(t)}{dt}\Big)}_{\text{chain rule (37) on } a_3(t)=g(a_1(t),a_2(t))}\Big)\,\mathrm{d}t\\
&= \int_{t_I}^{t_F} \Big(\Big(F_1\big(\vec{\mathbf{a}}(t)\big) + F_3\big(\vec{\mathbf{a}}(t)\big)\frac{\partial g}{\partial x}\Big)\frac{da_1(t)}{dt} + \Big(F_2\big(\vec{\mathbf{a}}(t)\big) + F_3\big(\vec{\mathbf{a}}(t)\big)\frac{\partial g}{\partial y}\Big)\frac{da_2(t)}{dt}\Big)\,\mathrm{d}t\\
&\overset{(46)}{=} \oint_{\partial S} \Big(\Big(F_1(x,y,z) + F_3(x,y,z)\frac{\partial g}{\partial x}(x,y)\Big)\,\mathrm{d}x + \Big(F_2(x,y,z) + F_3(x,y,z)\frac{\partial g}{\partial y}(x,y)\Big)\,\mathrm{d}y\Big)\\
&= \oint_{\partial R} \Big(\Big(F_1\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big)\frac{\partial g}{\partial x}\Big)\,\mathrm{d}x + \Big(F_2\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big)\frac{\partial g}{\partial y}\Big)\,\mathrm{d}y\Big)\\
&\overset{(92),(93)}{=} \iint_R \Big(-\frac{\partial}{\partial y}\Big(F_1\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big)\frac{\partial g}{\partial x}(x,y)\Big)\\
&\qquad + \frac{\partial}{\partial x}\Big(F_2\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big)\frac{\partial g}{\partial y}(x,y)\Big)\Big)\,\mathrm{d}A \qquad \text{(Lemma 3.3 / Green's theorem)}\\
&= \iint_R \Big(-\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial F_3}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial y}\frac{\partial g}{\partial x} - F_3\frac{\partial^2 g}{\partial x\partial y}\\
&\qquad + \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial g}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + F_3\frac{\partial^2 g}{\partial x\partial y}\Big)\,\mathrm{d}A \qquad \text{(chain and product rules)}\\
&= \iint_R \Big(\Big(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\Big)\frac{\partial g}{\partial x} + \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\frac{\partial g}{\partial y} + \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\,\mathrm{d}A\\
&\overset{(107)}{=} \iint_S (\vec\nabla\times\vec{\mathbf{F}})\cdot \mathrm{d}\vec{\mathbf{S}}.
\end{aligned}$$

Therefore, the right-hand side of (106) equals the left-hand side and we have proved the assertion. □

★ **Remark 3.29** (Only for brave students!)**.** We give also a more general proof of Stokes' theorem for a parametric surface S (with connected boundary) parametrised by a chart $\vec{\mathbf{X}}: R\to S$ as in Definition 2.38. The main tools are the same as in the proof above, namely repeated use of the chain rule to reduce to Green's theorem on the region R, but this proof is a bit more technical. We denote by $\vec{\mathbf{b}}:[t_I,t_F]\to\partial R$ a parametrisation of ∂R and fix $\vec{\mathbf{a}}(t) = \vec{\mathbf{X}}(\vec{\mathbf{b}}(t))$, the corresponding parametrisation of ∂S. In the following chain of equalities we do not write all the obvious dependences of the various integrands, e.g. $\vec{\mathbf{F}}$ stands for $\vec{\mathbf{F}}(\vec{\mathbf{a}}(t)) = \vec{\mathbf{F}}(\vec{\mathbf{X}}(\vec{\mathbf{b}}(t)))$ in the integrals in t and stands for $\vec{\mathbf{F}}(\vec{\mathbf{X}}(u,w))$ in the integrals in u and w. Then

$$\begin{aligned}
\oint_{\partial S}\vec{\mathbf{F}}\cdot \mathrm{d}r &= \int_{t_I}^{t_F}\vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{a}}}{dt}(t)\,\mathrm{d}t = \int_{t_I}^{t_F}\vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{X}}(\vec{\mathbf{b}})}{dt}(t)\,\mathrm{d}t\\
&\overset{(37)}{=} \int_{t_I}^{t_F}\Big(F_1\frac{\partial X_1}{\partial u}\big(\vec{\mathbf{b}}(t)\big)\frac{db_1}{dt}(t) + F_1\frac{\partial X_1}{\partial w}\big(\vec{\mathbf{b}}(t)\big)\frac{db_2}{dt}(t) + F_2\frac{\partial X_2}{\partial u}\big(\vec{\mathbf{b}}(t)\big)\frac{db_1}{dt}(t)\\
&\qquad + F_2\frac{\partial X_2}{\partial w}\big(\vec{\mathbf{b}}(t)\big)\frac{db_2}{dt}(t) + F_3\frac{\partial X_3}{\partial u}\big(\vec{\mathbf{b}}(t)\big)\frac{db_1}{dt}(t) + F_3\frac{\partial X_3}{\partial w}\big(\vec{\mathbf{b}}(t)\big)\frac{db_2}{dt}(t)\Big)\,\mathrm{d}t\\
&= \int_{t_I}^{t_F}\Big(\vec{\mathbf{F}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\frac{db_1}{dt} + \vec{\mathbf{F}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w}\frac{db_2}{dt}\Big)\,\mathrm{d}t\\
&\overset{(46)}{=} \oint_{\partial R}\Big(\vec{\mathbf{F}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\,\mathrm{d}u + \vec{\mathbf{F}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w}\,\mathrm{d}w\Big) \qquad \text{(using notation of Rem. 2.8 in } uw\text{-plane)}\\
&\overset{(96)}{=} \iint_R\Big(-\frac{\partial}{\partial w}\Big(\vec{\mathbf{F}}\cdot\frac{\partial X}{\partial u}\Big) + \frac{\partial}{\partial u}\Big(\vec{\mathbf{F}}\cdot\frac{\partial X}{\partial w}\Big)\Big)\,\mathrm{d}A \qquad \text{(Green's theorem)}\\
&\overset{(8)}{=} \iint_R\Big(-\frac{\partial\vec{\mathbf{F}}}{\partial w}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u} - \vec{\mathbf{F}}\cdot\frac{\partial^2\vec{\mathbf{X}}}{\partial w\partial u} + \frac{\partial\vec{\mathbf{F}}}{\partial u}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w} + \vec{\mathbf{F}}\cdot\frac{\partial^2\vec{\mathbf{X}}}{\partial u\partial w}\Big)\,\mathrm{d}A \qquad \text{(product rule for partial der.)}\\
&\overset{(38)}{=} \iint_R\Big(-\vec\nabla F_1\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w}\frac{\partial X_1}{\partial u} - \vec\nabla F_2\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w}\frac{\partial X_2}{\partial u} - \vec\nabla F_3\cdot\frac{\partial\vec{\mathbf{X}}}{\partial w}\frac{\partial X_3}{\partial u} \qquad \text{(Clairault theorem } \frac{\partial^2\vec{\mathbf{X}}}{\partial w\partial u} = \frac{\partial^2\vec{\mathbf{X}}}{\partial u\partial w}\\
&\qquad + \vec\nabla F_1\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\frac{\partial X_1}{\partial w} + \vec\nabla F_2\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\frac{\partial X_2}{\partial w} + \vec\nabla F_3\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\frac{\partial X_3}{\partial w}\Big)\,\mathrm{d}A \qquad \text{and chain rule for } \vec{\mathbf{F}}(\vec{\mathbf{X}}(u,w)))
\end{aligned}$$

$$\overset{(16)}{=} \iint_R \Big(\big(J\vec{\mathbf{F}} - (J\vec{\mathbf{F}})^T\big) \frac{\partial \vec{\mathbf{X}}}{\partial u} \Big) \cdot \frac{\partial \vec{\mathbf{X}}}{\partial w} \,\mathrm{d}A \qquad \text{(collecting terms, tricky!)}$$

$$= \iint_R \Big((\vec{\nabla} \times \vec{\mathbf{F}}) \times \frac{\partial \vec{\mathbf{X}}}{\partial u} \Big) \cdot \frac{\partial \vec{\mathbf{X}}}{\partial w} \,\mathrm{d}A \qquad \text{(Remark 1.64)}$$

$$= \iint_R (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \Big(\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} \Big) \,\mathrm{d}A \qquad \text{(Exercise 1.15)}$$

$$\overset{(67)}{=} \iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} \qquad \text{(surface integral formula).}$$

We note an immediate consequence of Stokes' theorem. If the surface S is the boundary of a three-dimensional domain D, then ∂S is empty (see Remark 2.59). Thus, (106) implies that, for every bounded, connected, piecewise smooth domain D and for every smooth vector field $\vec{\mathbf{F}}$ defined on ∂D, the flux of $\vec{\nabla} \times \vec{\mathbf{F}}$ on ∂D vanishes:

$$\oiint_{\partial D} (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} = 0.$$

Note that using the divergence theorem 3.14 and the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0$ (25) we could have derived the same identity, but only in the case when $\vec{\mathbf{F}}$ was defined (and smooth) in the whole domain D, and not only on its boundary.

Exercise 3.30. ▶ Demonstrate Stokes' theorem for the upper half sphere $S = \{|\vec{\mathbf{r}}| = 1, z > 0\}$ and the field $\vec{\mathbf{F}} = (2x - y)\hat{\boldsymbol{\imath}} - yz^2\hat{\boldsymbol{\jmath}} - y^2z\hat{\boldsymbol{k}}$.

Exercise 3.31. ▶ Compute $\oint_{\partial S} |\vec{\mathbf{r}}|^2 \,\mathrm{d}x$, with $S = \{z = x^2 - y^2,\ 0 < x, y < 1\}$ (cf. Exercise 2.55).

Exercise 3.32. ▶ Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. Prove that the line integral of the product $f\vec{\nabla} f$ along the boundary of an orientable surface S is zero.

Remark 3.33 (An alternative definition of divergence and curl)**.** The divergence and the curl of a vector field $\vec{\mathbf{F}}$ are often defined as the following limits (whenever the limits exist):

$$\vec{\nabla} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}_0) = \lim_{R\to 0} \frac{1}{\mathrm{Vol}(B_R)} \oiint_{\partial B_R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}},$$

$$\hat{\boldsymbol{a}} \cdot \big(\vec{\nabla} \times \vec{\mathbf{F}}(\vec{\mathbf{r}}_0)\big) = \lim_{R\to 0} \frac{1}{\mathrm{Area}(Q_R(\hat{\boldsymbol{a}}))} \oint_{\partial Q_R(\hat{\boldsymbol{a}})} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}},$$

where $B_R = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ |\vec{\mathbf{r}} - \vec{\mathbf{r}}_0| < R\}$ is the ball of centre $\vec{\mathbf{r}}_0$ and radius R, $\hat{\boldsymbol{a}}$ is a unit vector, and $Q_R(\hat{\boldsymbol{a}})$ is the disc of centre $\vec{\mathbf{r}}_0$, radius R and perpendicular to $\hat{\boldsymbol{a}}$. (Note that the curl is defined by its scalar product with all unit vectors.) Then one proves[40] that, if $\vec{\mathbf{F}}$ is differentiable, these definitions coincide with those given in Section 1.3. This also provides rigorous, geometric, coordinate-free interpretations of divergence and curl as "spreading" and "rotation" of a vector field, respectively:

The divergence of a vector field $\vec{\mathbf{F}}$ is the average of the flux of $\vec{\mathbf{F}}$ through an infinitesimal sphere. The curl of $\vec{\mathbf{F}}$ is the average of the circulation of $\vec{\mathbf{F}}$ around an infinitesimal circle.

Note that also the usual derivative of a real function $G : \mathbb{R} \to \mathbb{R}$ can be defined in a completely analogous way as $G'(t) = \lim_{R\to 0} \frac{1}{\mathrm{Length}([t-R,t+R])}(G(t+R) - G(t-R))$. Here, the interpretation is that the derivative of G is the average of the increment of G in an infinitesimal interval.

★ **Remark 3.34.** With the fundamental theorem of vector calculus we have seen that the line integral from a point $\vec{\mathbf{p}}$ to a point $\vec{\mathbf{q}}$ of a gradient is independent of the integration path. In other words, $\int_\Gamma \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\mathbf{r}}$ depends only on the scalar field φ and the endpoints of Γ, i.e. its boundary $\partial\Gamma$ (as opposed to the entire path Γ).

Stokes' theorem may be interpreted similarly: the flux of the curl of a vector field through a surface S only depends on the field itself and the *boundary* of S (as opposed to the entire surface S). If two surfaces share the boundary (e.g. the north and the south hemispheres of the same sphere), the flux of a curl through them will give the same value. The equivalences of (50) can be translated to this setting:

[40] If $\vec{\nabla}\cdot\vec{\mathbf{F}}$ or $\vec{\nabla}\times\vec{\mathbf{F}}$ are constant in a neighbourhood of $\vec{\mathbf{r}}_0$, these proofs are immediate from divergence and Stokes' theorem:

$$\lim_{R\to 0} \frac{\oiint_{\partial B_R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}}{\mathrm{Vol}(B_R)} \overset{(101)}{=} \lim_{R\to 0} \frac{\iiint_{B_R} \vec{\nabla} \cdot \vec{\mathbf{F}} \,\mathrm{d}V}{\mathrm{Vol}(B_R)} \overset{(40)}{=} \lim_{R\to 0} \frac{\mathrm{Vol}(B_R)\vec{\nabla} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}_0)}{\mathrm{Vol}(B_R)} = \vec{\nabla} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}_0),$$

$$\lim_{R\to 0} \frac{\oint_{\partial Q_R(\hat{\boldsymbol{a}})} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}}{\mathrm{Area}(Q_R(\hat{\boldsymbol{a}}))} \overset{(106)}{=} \lim_{R\to 0} \frac{\iint_{Q_R(\hat{\boldsymbol{a}})} (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}}}{\mathrm{Area}(Q_R(\hat{\boldsymbol{a}}))} \overset{(40)}{=} \lim_{R\to 0} \frac{\mathrm{Area}(Q_R(\hat{\boldsymbol{a}}))(\vec{\nabla} \times \vec{\mathbf{F}}(\vec{\mathbf{r}}_0)) \cdot \hat{\boldsymbol{n}}}{\mathrm{Area}(Q_R(\hat{\boldsymbol{a}}))} = (\vec{\nabla} \times \vec{\mathbf{F}}(\vec{\mathbf{r}}_0)) \cdot \hat{\boldsymbol{a}},$$

(since $\hat{\boldsymbol{n}} = \hat{\boldsymbol{a}}$ on $Q_R(\hat{\boldsymbol{a}})$); in the general case the proofs require a suitable version of the mean value theorem (attempt to write the proofs!).

$\vec{\mathbf{G}} = \vec{\nabla} \times \vec{\mathbf{A}}$ (∃ vector potential for $\vec{\mathbf{G}}$)	$\Longleftrightarrow$	$\iint_S \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}$ is independent of S	$\Longleftrightarrow$	$\oiint_{\partial D} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = 0$ ∀ closed surfaces ∂D	$\Longrightarrow$	$\vec{\nabla} \cdot \vec{\mathbf{G}} = 0$ ($\vec{\mathbf{G}}$ solenoidal)

(where, with "independent of S", we mean that $\iint_{S_A} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}} = \iint_{S_B} \vec{\mathbf{G}} \cdot d\vec{\mathbf{S}}$ if S_A and S_B are two surfaces with $\partial S_A = \partial S_B$). The converse of the last implication is true if the domain of $\vec{\mathbf{G}}$ does not contain "holes".

★ **Remark 3.35.** Often the name "Stokes' theorem" is referred to a much more general version of it, coming from the branch of mathematics known as "differential geometry". This involves the integration of a differential operator called "exterior derivative", simply denoted by "d". This acts on "differential forms", which are generalisations of scalar and vector fields defined on a "manifold" Ω, which in turn is an object that generalises domains, paths and surfaces to any dimension. This extremely general and deep theorem is stated with a simple and elegant formula: $\int_\Omega d\omega = \int_{\partial\Omega} \omega$. The fundamental theorem of (vector) calculus, Green's, divergence and Stokes' theorems are special instances of this result. If you are interested in deepening this topic, take a look at Chapter 10 of W. Rudin's book "Principles of mathematical analysis" (this is not an easy read!).

1D	(90)	Fundamental theorem of calculus	$\int_a^b \frac{d\phi}{dt}\,dt = \phi(b) - \phi(a)$
3D*	(49) (91)	Fundamental theorem of vector calculus	$\int_\Gamma \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{q}}) - f(\vec{\mathbf{p}})$
2D	(92)	Lemma 3.3	$\iint_R \frac{\partial f}{\partial y}\,dA = -\oint_{\partial R} f\,dx$
2D	(93)	Lemma 3.3	$\iint_R \frac{\partial f}{\partial x}\,dA = \oint_{\partial R} f\,dy$
2D	(95)	Green's theorem	$\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_{\partial R} (F_1\,dx + F_2\,dy)$
2D	(96)	Green's theorem	$\iint_R (\vec{\nabla} \times \vec{\mathbf{F}})_3\,dA = \iint_R (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \hat{\boldsymbol{k}}\,dA = \oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
2D	(98)	Divergence theorem	$\iint_R (\vec{\nabla} \cdot \vec{\mathbf{F}})\,dA = \oint_{\partial R} (F_1\,dy - F_2\,dx) = \oint_{\partial R} \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}\,ds$
3D	(99) (100)	Lemma 3.13	$\iiint_D \frac{\partial f}{\partial z}\,dV = \oiint_{\partial D} f\,\hat{\boldsymbol{k}} \cdot d\vec{\mathbf{S}}$
3D	(101)	Divergence theorem	$\iiint_D (\vec{\nabla} \cdot \vec{\mathbf{F}})\,dV = \oiint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$
3D	(102)	Corollary 3.22	$\iiint_D \vec{\nabla} f\,dV = \oiint_{\partial D} f\,\hat{\boldsymbol{n}}\,dS$
3D	(103)	Corollary 3.22	$\iiint_D \vec{\nabla} \times \vec{\mathbf{F}}\,dV = \oiint_{\partial D} \vec{\mathbf{F}} \times \hat{\boldsymbol{n}}\,dS$
3D	(104)	Green's 1st identity	$\iiint_D (f\Delta g + \vec{\nabla} g \cdot \vec{\nabla} f)\,dV = \oiint_{\partial D} f \frac{\partial g}{\partial n}\,dS$
3D	(105)	Green's 2nd identity	$\iiint_D (f\Delta g - g\Delta f)\,dV = \oiint_{\partial D} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS$
3D*	(106)	Stokes' theorem	$\iint_S \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

Table 3: A summary of the important integro-differential identities proved in Section 3. Here:
Γ is a path with starting point $\vec{\mathbf{p}}$ and end point $\vec{\mathbf{q}}$;
$R \subset \mathbb{R}^2$ is a two-dimensional, bounded, connected, piecewise smooth region;
$D \subset \mathbb{R}^3$ is a three-dimensional, bounded, connected, piecewise smooth domain;
$(S, \hat{\boldsymbol{n}})$ is a piecewise smooth, bounded, oriented surface;
ϕ is a real function;
f and g are smooth scalar fields;
$\vec{\mathbf{F}}$ is a smooth vector field.
(* Note that the first column of the table denotes the dimension of the space in which the domain of integration is defined; the intrinsic dimension of the domain of integration for the fundamental theorem of vector calculus is one (oriented path), and for Stokes' theorem is two (oriented surface).)

A General overview of the notes

The three main sections of these notes discuss the following general topics:

1. differential vector calculus (differentiation of vector quantities);
2. integral vector calculus (integration of vector quantities);
3. generalisations of the fundamental theorem of calculus (relation between differentiation and integration of vector quantities).

Section 1.1 recalls some *operations* involving vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ and scalars $\lambda \in \mathbb{R}$:

	inputs		output
Vector addition:	2 vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$	$\mapsto$	vector $\vec{\mathbf{u}} + \vec{\mathbf{w}}$
Scalar-vector multiplication:	a vector $\vec{\mathbf{u}}$ and a scalar λ	$\mapsto$	vector $\lambda\vec{\mathbf{u}}$
Scalar product:	2 vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$	$\mapsto$	scalar $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
Vector product:	2 vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$	$\mapsto$	vector $\vec{\mathbf{u}} \times \vec{\mathbf{w}}$
Triple product:	3 vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$	$\mapsto$	scalar $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

Sections 1.2 introduces different kinds of *functions* (or fields), taking as input and returning as output either scalar values (in $\mathbb{R}$) or vectors (in $\mathbb{R}^3$):

Real functions (of real variable)	$f : \mathbb{R} \to \mathbb{R}$	$t \mapsto f(t)$
Curves	$\vec{\mathbf{a}} : \mathbb{R} \to \mathbb{R}^3$	$t \mapsto \vec{\mathbf{a}}(t)$
Scalar fields	$f : \mathbb{R}^3 \to \mathbb{R}$	$\vec{\mathbf{r}} \mapsto f(\vec{\mathbf{r}})$
Vector fields	$\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$	$\vec{\mathbf{r}} \mapsto \vec{\mathbf{F}}(\vec{\mathbf{r}})$

The fundamental differential operators for scalar and vector fields are the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. They can be combined to construct several *vector differential operators*, as described in Section 1.3. The most important are:

Partial derivatives	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$	scalar	$\mapsto$	scalar
Gradient	$\vec{\nabla}$	scalar	$\mapsto$	vector
Divergence	$\vec{\nabla}\cdot$ (or div)	vector	$\mapsto$	scalar
Curl	$\vec{\nabla}\times$ (or curl)	vector	$\mapsto$	vector
Laplacian	Δ	scalar	$\mapsto$	scalar

Proposition 1.52 describes the result of the *composition* of differential operators.

Proposition 1.55 describes the result of the application of differential operators to different *products* of scalar and vector fields (generalising the product rule).

Section 1.5 studies some relations between vector fields $\vec{\mathbf{F}}$ lying in the *kernel* (or nullspace) of the divergence and the curl operators (i.e. solenoidal fields $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$ and irrotational fields $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$) and those in the *image* of the gradient and the curl (i.e. fields admitting a scalar potential $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ or a vector potential $\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{A}}$).

Section 1.6 describes the total derivatives (in t) of curves, denoted $\frac{d\vec{\mathbf{a}}}{dt}$.

Section 1.7 investigates the differentiation of *composition* of fields and curves.

Sections 2.1 and 2.2 extends the definition of integrals of real functions to:

Type of integral	domain of integration	integrand	notation
Integrals of real functions	interval (a, b)	real function	$\int_a^b f\,\mathrm{d}t$
Line integrals of scalar fields	path Γ	scalar field	$\int_\Gamma f\,\mathrm{d}s$
Line integrals of vector fields	oriented path Γ	vector field	$\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$
Double integrals	planar domain $R \subset \mathbb{R}^2$	scalar field	$\iint_R f\,\mathrm{d}A$
Triple integrals	domain $D \subset \mathbb{R}^3$	scalar field	$\iiint_D f\,\mathrm{d}V$
Surface integrals	surface S	scalar field	$\iint_S f\,\mathrm{d}S$
Fluxes	oriented surface $(S, \hat{\boldsymbol{n}})$	vector field	$\iint_R \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$

Several theorems establish important relations between integration and differentiation, see Table 3. The most relevant are:

	The integral on a(n)	of (the)	of a	is equal to the	of	at/on the
FTC	interval (a, b)	derivative	function f	difference	f	endpoints
FTVC	oriented path Γ	$\hat{\boldsymbol{\tau}}\cdot$ gradient	scalar field f	difference	f	endpoints
Green	2D region R	$\hat{\boldsymbol{k}}\cdot$ curl	vector field $\vec{\mathbf{F}}$	circulation	$\vec{\mathbf{F}}$	boundary ∂R
Stokes	oriented surface $(S, \hat{\boldsymbol{n}})$	$\hat{\boldsymbol{n}}\cdot$ curl	vector field $\vec{\mathbf{F}}$	circulation	$\vec{\mathbf{F}}$	boundary ∂S
Divergence	3D domain D	divergence	vector field $\vec{\mathbf{F}}$	flux	$\vec{\mathbf{F}}$	boundary ∂D
	1st domain of integration	differential operator	function or field	type of integral		2nd domain of integration

B Solutions of the exercises

B.1 Exercises of Section 1

Solution of Exercise 1.6. $\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\imath}=\hat{\boldsymbol\jmath}\cdot\hat{\boldsymbol\jmath}=\hat{\mathbf k}\cdot\hat{\mathbf k}=1,\quad \hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\jmath}=\hat{\boldsymbol\jmath}\cdot\hat{\boldsymbol\imath}=\hat{\boldsymbol\imath}\cdot\hat{\mathbf k}=\hat{\mathbf k}\cdot\hat{\boldsymbol\imath}=\hat{\boldsymbol\jmath}\cdot\hat{\mathbf k}=\hat{\mathbf k}\cdot\hat{\boldsymbol\jmath}=0.$

Solution of Exercise 1.7. *Existence.* Since $\vec{\mathbf u}_{||}$ must be parallel to $\vec{\mathbf w}$ and $\vec{\mathbf w}\neq\vec{\mathbf 0}$, it must have the form $\vec{\mathbf u}_{||}=\alpha\vec{\mathbf w}$ for some $\alpha\in\mathbb R$. Thus $\vec{\mathbf u}_\perp=\vec{\mathbf u}-\vec{\mathbf u}_{||}=\vec{\mathbf u}-\alpha\vec{\mathbf w}$. The orthogonality condition gives $0=\vec{\mathbf w}\cdot\vec{\mathbf u}_\perp=\vec{\mathbf w}\cdot(\vec{\mathbf u}-\alpha\vec{\mathbf w})=\vec{\mathbf w}\cdot\vec{\mathbf u}-\alpha|\vec{\mathbf w}|^2$, i.e. $\alpha=\vec{\mathbf w}\cdot\vec{\mathbf u}/|\vec{\mathbf w}|^2$. Thus the pair

$$\vec{\mathbf u}_{||}=\frac{\vec{\mathbf w}\cdot\vec{\mathbf u}}{|\vec{\mathbf w}|^2}\vec{\mathbf w},\qquad \vec{\mathbf u}_\perp=\vec{\mathbf u}-\vec{\mathbf u}_{||}=\vec{\mathbf u}-\frac{\vec{\mathbf w}\cdot\vec{\mathbf u}}{|\vec{\mathbf w}|^2}\vec{\mathbf w}$$

satisfies all the requests.

Uniqueness. Assume by contradiction that two pairs $\vec{\mathbf u}_{||},\vec{\mathbf u}_\perp$ and $\vec{\mathbf v}_{||},\vec{\mathbf v}_\perp$ give an orthogonal decomposition of $\vec{\mathbf u}$ with respect to $\vec{\mathbf w}$. Since $\vec{\mathbf u}_{||}$ and $\vec{\mathbf v}_{||}$ are both parallel to $\vec{\mathbf w}$, they can be written as $\vec{\mathbf u}_{||}=\alpha\vec{\mathbf w}$ and $\vec{\mathbf v}_{||}=\beta\vec{\mathbf w}$. Reasoning as in the existence part of the exercise, we find a unique possible value of α and β, so $\vec{\mathbf u}_{||}=\vec{\mathbf v}_{||}$ and consequently $\vec{\mathbf u}_\perp=\vec{\mathbf u}-\vec{\mathbf u}_{||}=\vec{\mathbf u}-\vec{\mathbf v}_{||}=\vec{\mathbf v}_\perp$ and there exists only one admissible pair.

Solution of Exercise 1.8. We only need to verify that the two scalar products between $\vec{\mathbf u}\times\vec{\mathbf w}$ and the two vectors are zero:

$$\begin{aligned}(\vec{\mathbf u}\times\vec{\mathbf w})\cdot\vec{\mathbf u}&\overset{(2)}{=}\big((u_2w_3-u_3w_2)\hat{\boldsymbol\imath}+(u_3w_1-u_1w_3)\hat{\boldsymbol\jmath}+(u_1w_2-u_2w_1)\hat{\mathbf k}\big)\cdot\vec{\mathbf u}\\&=u_2w_3u_1-u_3w_2u_1+u_3w_1u_2-u_1w_3u_2+u_1w_2u_3-u_2w_1u_3\\&=w_1(u_3u_2-u_2u_3)+w_2(u_1u_3-u_3u_1)+w_3(u_2u_1-u_1u_2)=0,\\(\vec{\mathbf u}\times\vec{\mathbf w})\cdot\vec{\mathbf w}&=\big((u_2w_3-u_3w_2)\hat{\boldsymbol\imath}+(u_3w_1-u_1w_3)\hat{\boldsymbol\jmath}+(u_1w_2-u_2w_1)\hat{\mathbf k}\big)\cdot\vec{\mathbf w}\\&=u_2w_3w_1-u_3w_2w_1+u_3w_1w_2-u_1w_3w_2+u_1w_2w_3-u_2w_1w_3=0.\end{aligned}$$

Solution of Exercise 1.9. We write in detail only $\hat{\boldsymbol\imath}\times\hat{\boldsymbol\jmath}$ and $\hat{\boldsymbol\imath}\times\hat{\boldsymbol\imath}$, all the other cases are similar:

$$\hat{\boldsymbol\imath}\times\hat{\boldsymbol\jmath}=\begin{vmatrix}\hat{\boldsymbol\imath}&\hat{\boldsymbol\jmath}&\hat{\mathbf k}\\1&0&0\\0&1&0\end{vmatrix}=\hat{\boldsymbol\imath}\begin{vmatrix}0&0\\1&0\end{vmatrix}-\hat{\boldsymbol\jmath}\begin{vmatrix}1&0\\0&0\end{vmatrix}+\hat{\mathbf k}\begin{vmatrix}1&0\\0&1\end{vmatrix}=0\hat{\boldsymbol\imath}+0\hat{\boldsymbol\jmath}+1\hat{\mathbf k}=\hat{\mathbf k},$$

$$\hat{\boldsymbol\imath}\times\hat{\boldsymbol\imath}=\begin{vmatrix}\hat{\boldsymbol\imath}&\hat{\boldsymbol\jmath}&\hat{\mathbf k}\\1&0&0\\1&0&0\end{vmatrix}=\hat{\boldsymbol\imath}\begin{vmatrix}0&0\\0&0\end{vmatrix}-\hat{\boldsymbol\jmath}\begin{vmatrix}1&0\\1&0\end{vmatrix}+\hat{\mathbf k}\begin{vmatrix}1&0\\1&0\end{vmatrix}=0\hat{\boldsymbol\imath}+0\hat{\boldsymbol\jmath}+0\hat{\mathbf k}=\vec{\mathbf 0}.$$

Solution of Exercise 1.10. The anticommutativity $\vec{\mathbf w}\times\vec{\mathbf u}=-\vec{\mathbf u}\times\vec{\mathbf w}$ follows immediately by computing $\vec{\mathbf w}\times\vec{\mathbf u}$ with Formula (2) and swapping the u_j's with the w_j's. From this it follows that $\vec{\mathbf u}\times\vec{\mathbf u}=-\vec{\mathbf u}\times\vec{\mathbf u}$, so $2\vec{\mathbf u}\times\vec{\mathbf u}=\vec{\mathbf 0}$.

Solution of Exercise 1.11. We use Exercises 1.9 and 1.10: $(\hat{\boldsymbol\imath}\times\hat{\boldsymbol\jmath})\times\hat{\boldsymbol\jmath}=\hat{\mathbf k}\times\hat{\boldsymbol\jmath}=-\hat{\boldsymbol\imath}\neq\vec{\mathbf 0}=\hat{\boldsymbol\imath}\times\vec{\mathbf 0}=\hat{\boldsymbol\imath}\times(\hat{\boldsymbol\jmath}\times\hat{\boldsymbol\jmath})$.

Solution of Exercise 1.12. We prove identity (4) componentwise, namely we verify that the three components on the left-hand side and the right-hand side of the identity agree with each other. Consider the first component:

$$\begin{aligned}\big[\vec{\mathbf u}\times(\vec{\mathbf v}\times\vec{\mathbf w})\big]_1&\overset{(2)}{=}u_2[\vec{\mathbf v}\times\vec{\mathbf w}]_3-u_3[\vec{\mathbf v}\times\vec{\mathbf w}]_2\\&\overset{(2)}{=}u_2(v_1w_2-v_2w_1)-u_3(v_3w_1-v_1w_3)\\&=(u_2w_2+u_3w_3)v_1-(u_2v_2+u_3v_3)w_1\\&=(\vec{\mathbf u}\cdot\vec{\mathbf w}-u_1w_1)v_1-(\vec{\mathbf u}\cdot\vec{\mathbf v}-u_1v_1)w_1\\&=(\vec{\mathbf u}\cdot\vec{\mathbf w})v_1-(\vec{\mathbf u}\cdot\vec{\mathbf v})w_1\\&=\big[(\vec{\mathbf u}\cdot\vec{\mathbf w})\vec{\mathbf v}-(\vec{\mathbf u}\cdot\vec{\mathbf v})\vec{\mathbf w}\big]_1.\end{aligned}$$

A similar computation holds for the y- and the z-components, thus the assertion (4) follows.

Solution of Exercise 1.13. Jacobi identity follows from applying three times the identity (4), using the commutativity of the scalar product, and cancelling all terms:

$$\begin{aligned}&\vec{\mathbf{u}}\times(\vec{\mathbf{v}}\times\vec{\mathbf{w}})+\vec{\mathbf{v}}\times(\vec{\mathbf{w}}\times\vec{\mathbf{u}})+\vec{\mathbf{w}}\times(\vec{\mathbf{u}}\times\vec{\mathbf{v}})\\ &\overset{(4)}{=}\big(\vec{\mathbf{v}}(\vec{\mathbf{u}}\cdot\vec{\mathbf{w}})-\vec{\mathbf{w}}(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\big)+\big(\vec{\mathbf{w}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{u}})-\vec{\mathbf{u}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}})\big)+\big(\vec{\mathbf{u}}(\vec{\mathbf{w}}\cdot\vec{\mathbf{v}})-\vec{\mathbf{v}}(\vec{\mathbf{w}}\cdot\vec{\mathbf{u}})\big)\\ &=\vec{\mathbf{u}}\big((\vec{\mathbf{w}}\cdot\vec{\mathbf{v}})-(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}})\big)+\big(\vec{\mathbf{v}}(\vec{\mathbf{u}}\cdot\vec{\mathbf{w}})-(\vec{\mathbf{w}}\cdot\vec{\mathbf{u}})\big)+\big(\vec{\mathbf{w}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{u}})-(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})\big)=\vec{\mathbf{0}}.\end{aligned}$$

To prove Binet–Cauchy identity we expand the left-hand side, collect the positive terms, add and subtract the term $u_1w_1v_1p_1+u_2w_2v_2p_2+u_3w_3v_3p_3$:

$$\begin{aligned}&(\vec{\mathbf{u}}\times\vec{\mathbf{v}})\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{p}})\\ &=\Big((u_2v_3-u_3v_2)\hat{\boldsymbol{\imath}}+(u_3v_1-u_1v_3)\hat{\boldsymbol{\jmath}}+(u_1v_2-u_2v_1)\hat{\boldsymbol{k}}\Big)\cdot\Big((w_2p_3-w_3p_2)\hat{\boldsymbol{\imath}}+(w_3p_1-w_1p_3)\hat{\boldsymbol{\jmath}}+(w_1p_2-w_2p_1)\hat{\boldsymbol{k}}\Big)\\ &=u_2v_3w_2p_3+u_3v_2w_3p_2+u_3v_1w_3p_1+u_1v_3w_1p_3+u_1v_2w_1p_2+u_2v_1w_2p_1\\ &\quad-u_3v_2w_2p_3-u_2v_3w_3p_2-u_1v_3w_3p_1-u_3v_1w_1p_3-u_2v_1w_1p_2-u_1v_2w_2p_1\\ &=u_2v_3w_2p_3+u_3v_2w_3p_2+u_3v_1w_3p_1+u_1v_3w_1p_3+u_1v_2w_1p_2+u_2v_1w_2p_1+(u_1w_1v_1p_1+u_2w_2v_2p_2+u_3w_3v_3p_3)\\ &\quad-u_3v_2w_2p_3-u_2v_3w_3p_2-u_1v_3w_3p_1-u_3v_1w_1p_3-u_2v_1w_1p_2-u_1v_2w_2p_1-(u_1w_1v_1p_1+u_2w_2v_2p_2+u_3w_3v_3p_3)\\ &=(u_1w_1+u_2w_2+u_3w_3)(v_1p_1+v_2p_2+v_3p_3)-(u_1p_1+u_2p_2+u_3p_3)(w_1v_1+w_2v_2+w_3v_3)\\ &=(\vec{\mathbf{u}}\cdot\vec{\mathbf{w}})(\vec{\mathbf{v}}\cdot\vec{\mathbf{p}})-(\vec{\mathbf{u}}\cdot\vec{\mathbf{p}})(\vec{\mathbf{v}}\cdot\vec{\mathbf{w}}).\end{aligned}$$

Lagrange identity is obtained by choosing $\vec{\mathbf{w}}=\vec{\mathbf{u}}$ and $\vec{\mathbf{p}}=\vec{\mathbf{v}}$ in the Binet–Cauchy identity and moving the last term to the left-hand side.

Solution of Exercise 1.15. The formula $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})=\vec{\mathbf{v}}\cdot(\vec{\mathbf{w}}\times\vec{\mathbf{u}})=\vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}})$ follows immediately from (5). From this and the anticommutativity of the vector product (Exercise 1.10) we have

$$\vec{\mathbf{w}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{u}})=-\vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}})=-\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}),\qquad \vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{u}})=\vec{\mathbf{v}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{u}})=0.$$

Solution of Exercise 1.18. ★ (i) If $\lim_{j\to\infty}\vec{\mathbf{u}}_j=\vec{\mathbf{u}}$ then $\lim_{j\to\infty}|\vec{\mathbf{u}}_j-\vec{\mathbf{u}}|=0$ by definition. Since $0\le|(\vec{\mathbf{u}}_j)_1-u_1|\le|\vec{\mathbf{u}}_j-\vec{\mathbf{u}}|$ from the definition of vector magnitude, by the sandwich theorem (see Theorem 4.18 in the Real Analysis lecture notes) we have $\lim_{j\to\infty}|(\vec{\mathbf{u}}_j)_1-u_1|=0$, which is equivalent to $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_1=u_1$. Similarly, we can prove $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_2=u_2$ and $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_3=u_3$.

To prove the converse, assume $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_1=u_1$, $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_2=u_2$ and $\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_3=u_3$. Then, we obtain the desired convergence simply using the properties of limits and the definition of magnitude:

$$\begin{aligned}\lim_{j\to\infty}|\vec{\mathbf{u}}_j-\vec{\mathbf{u}}|&=\lim_{j\to\infty}\sqrt{\big((\vec{\mathbf{u}}_j)_1-u_1\big)^2+\big((\vec{\mathbf{u}}_j)_2-u_2\big)^2+\big((\vec{\mathbf{u}}_j)_3-u_3\big)^2}\\ &=\sqrt{\big(\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_1-u_1\big)^2+\big(\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_2-u_2\big)^2+\big(\lim_{j\to\infty}(\vec{\mathbf{u}}_j)_3-u_3\big)^2}=0.\end{aligned}$$

(ii) It suffices to take a sequence converging to some $\vec{\mathbf{w}}\neq\vec{\mathbf{u}}$ with $|\vec{\mathbf{w}}|=|\vec{\mathbf{u}}|$. For example, $\vec{\mathbf{u}}_j=\hat{\boldsymbol{\imath}}$ for all $j\in\mathbb{N}$ and $\vec{\mathbf{u}}=\hat{\boldsymbol{k}}$.

Solution of Exercise 1.19. ★ Let $D\subset\mathbb{R}^3$ be open and denote $C:=\mathbb{R}^3\setminus D$ its complement. We want to prove that C is closed. Consider a sequence $\{\vec{\mathbf{p}}_j\}_{j\in\mathbb{N}}\subset C$ such that $\lim_{j\to\infty}\vec{\mathbf{p}}_j=\vec{\mathbf{p}}$. To prove that C is closed we need to prove that $\vec{\mathbf{p}}\in C$, which is the same as $\vec{\mathbf{p}}\notin D$, by definition of set complement. We proceed by contradiction: if $\vec{\mathbf{p}}\in D$ was true, then there would exist $\epsilon>0$ such that all points at distance smaller than ϵ from $\vec{\mathbf{p}}$ were in D. But from $\lim_{j\to\infty}|\vec{\mathbf{p}}_j-\vec{\mathbf{p}}|=0$, there exists a $\vec{\mathbf{p}}_j$ in the sequence (actually, infinitely many) at distance smaller than ϵ from $\vec{\mathbf{p}}$ but belonging to C. So we have a contradiction: $\vec{\mathbf{p}}$ must belong to C, thus C is closed.

Now we prove the second statement (the vice versa), which is very similar. Let $C\subset\mathbb{R}^3$ be closed and denote $D:=\mathbb{R}^3\setminus C$ its complement. We want to prove that D is open. Again, we proceed by contradiction: if D were not open, there would be a point $\vec{\mathbf{p}}\in D$ such that for all $\epsilon>0$ we have a point $\vec{\mathbf{p}}_\epsilon\notin D$ (i.e. $\vec{\mathbf{p}}_\epsilon\in C$) such that $|\vec{\mathbf{p}}-\vec{\mathbf{p}}_\epsilon|<\epsilon$. We fix a real positive sequence ϵ_j converging to 0 (e.g. $\epsilon_j=1/j$). Then by the contradiction assumption there exists $\vec{\mathbf{p}}_j\in C$ such that $|\vec{\mathbf{p}}_j-\vec{\mathbf{p}}|<\epsilon_j\to0$. Thus $\lim_{j\to\infty}\vec{\mathbf{p}}_j=\vec{\mathbf{p}}$. As C is closed, this means that $\vec{\mathbf{p}}\in C$, but we assumed $\vec{\mathbf{p}}\in D$, so we have a contradiction and we conclude.

Solution of Exercise 1.29. (We expand $p(\vec{\mathbf{r}}) = \frac{|\vec{\mathbf{r}}|^2}{x^2} = \frac{x^2+y^2+z^2}{x^2} = 1 + \frac{y^2+z^2}{x^2}$.)

$$\begin{aligned}
&f(\vec{\mathbf{r}}) = xy\mathrm{e}^z, && \frac{\partial f}{\partial x} = y\mathrm{e}^z, && \frac{\partial f}{\partial y} = x\mathrm{e}^z, && \frac{\partial f}{\partial z} = xy\mathrm{e}^z = f;\\
&g(\vec{\mathbf{r}}) = \frac{xy}{y+z}, && \frac{\partial g}{\partial x} = \frac{y}{y+z}, && \frac{\partial g}{\partial y} = \frac{xz}{(y+z)^2}, && \frac{\partial g}{\partial z} = \frac{-xy}{(y+z)^2};\\
&h(\vec{\mathbf{r}}) = \log(1+z^2\mathrm{e}^{yz}), && \frac{\partial h}{\partial x} = 0, && \frac{\partial h}{\partial y} = \frac{z^3\mathrm{e}^{yz}}{1+z^2\mathrm{e}^{yz}}, && \frac{\partial h}{\partial z} = \frac{(2z+yz^2)\mathrm{e}^{yz}}{1+z^2\mathrm{e}^{yz}};\\
&\ell(\vec{\mathbf{r}}) = \sqrt{x^2+y^4+z^6}, && \frac{\partial \ell}{\partial x} = \frac{x}{\sqrt{x^2+y^4+z^6}}, && \frac{\partial \ell}{\partial y} = \frac{2y^3}{\sqrt{x^2+y^4+z^6}}, && \frac{\partial \ell}{\partial z} = \frac{3z^5}{\sqrt{x^2+y^4+z^6}};\\
&m(\vec{\mathbf{r}}) = x^y = \mathrm{e}^{(\log x)y}, && \frac{\partial m}{\partial x} = \frac{y}{x}\mathrm{e}^{(\log x)y} && \frac{\partial y}{\partial y} = \mathrm{e}^{(\log x)y}\log x && \frac{\partial m}{\partial z} = 0;\\
&p(\vec{\mathbf{r}}) = \frac{|\vec{\mathbf{r}}|^2}{x^2} && \frac{\partial m}{\partial x} = \frac{-2y^2-2z^2}{x^3}, && \frac{\partial y}{\partial y} = \frac{2y}{x^2}, && \frac{\partial m}{\partial z} = \frac{2z}{x^2}.
\end{aligned}$$

Solution of Exercise 1.34.

$$\vec{\nabla} f = y\mathrm{e}^z\hat{\boldsymbol{\imath}} + x\mathrm{e}^z\hat{\boldsymbol{\jmath}} + xy\mathrm{e}^z\hat{\boldsymbol{k}}, \qquad \vec{\nabla} g = \frac{y}{y+z}\hat{\boldsymbol{\imath}} + \frac{xz}{(y+z)^2}\hat{\boldsymbol{\jmath}} - \frac{xy}{(y+z)^2}\hat{\boldsymbol{k}},$$

$$\vec{\nabla} h = \frac{z^3\mathrm{e}^{yz}\hat{\boldsymbol{\jmath}} + (2z+yz^2)\mathrm{e}^{yz}\hat{\boldsymbol{k}}}{1+z^2\mathrm{e}^{yz}}, \qquad \vec{\nabla}\ell = \frac{x\hat{\boldsymbol{\imath}} + 2y^3\hat{\boldsymbol{\jmath}} + 3z^5\hat{\boldsymbol{k}}}{\sqrt{x^2+y^4+z^6}},$$

$$\vec{\nabla} m = \frac{y}{x}\mathrm{e}^{(\log x)y}\hat{\boldsymbol{\imath}} + \mathrm{e}^{(\log x)y}\log x\hat{\boldsymbol{\jmath}} = y\,x^{y-1}\hat{\boldsymbol{\imath}} + x^y\log x\hat{\boldsymbol{\jmath}}, \qquad \vec{\nabla} p = \frac{-2y^2-2z^2}{x^3}\hat{\boldsymbol{\imath}} + \frac{2y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}}}{x^2}.$$

Solution of Exercise 1.35. The field s is a polynomial in three variables, so its gradient is easy to compute:

$$\vec{\nabla} s(\vec{\mathbf{r}}) = \vec{\nabla}(|\vec{\mathbf{r}}|^2) = \vec{\nabla}(x^2+y^2+z^2) = 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}} = 2\vec{\mathbf{r}}.$$

To compute $\vec{\nabla} m$ we use a simple trick (otherwise we can compute it directly). For $\alpha \in \mathbb{R}$, define the real function $G_\alpha(t) := t^{\alpha/2}$, whose derivative is $G_\alpha{}'(t) = (\alpha/2)t^{\alpha/2-1}$. Using the chain rule (14) for the gradient we obtain the gradient of $|\vec{\mathbf{r}}|^\alpha$:

$$\vec{\nabla}(|\vec{\mathbf{r}}|^\alpha) = \vec{\nabla}\big(s(\vec{\mathbf{r}})^{\alpha/2}\big) = \vec{\nabla}\Big(G_\alpha\big(s(\vec{\mathbf{r}})\big)\Big) = G_\alpha{}'\big(s(\vec{\mathbf{r}})\big)\vec{\nabla} s(\vec{\mathbf{r}}) = \frac{\alpha}{2}\big(s(\vec{\mathbf{r}})\big)^{\alpha/2-1}2\vec{\mathbf{r}} = \alpha\big(|\vec{\mathbf{r}}|^2\big)^{\alpha/2-1}\vec{\mathbf{r}} = \alpha|\vec{\mathbf{r}}|^{\alpha-2}\vec{\mathbf{r}}.$$

The field $m(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|$ corresponds to the case $\alpha = 1$, so $\vec{\nabla} m(\vec{\mathbf{r}}) = \vec{\mathbf{r}}/|\vec{\mathbf{r}}|$, as required.

Solution of Exercise 1.37. The level surfaces of $f = (2x+y)^3$ are the sets $\{\vec{\mathbf{r}}, (2x+y)^3 = C\}$, or equivalently $\{\vec{\mathbf{r}}, 2x+y = C\}$. These are parallel planes. They are all perpendicular to the xy-plane, so $\hat{\boldsymbol{k}}$ is tangent to each one of these planes. They intersect the xy-plane in the lines $\{2x+y = C, z = 0\}$, so also the vector $\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ is tangent to all planes in all their points.

We have $\vec{\mathbf{G}} = 2\mathrm{e}^z\hat{\boldsymbol{\imath}} + \mathrm{e}^z\hat{\boldsymbol{\jmath}}$, so $\vec{\mathbf{G}} \cdot \hat{\boldsymbol{k}} = 0$ and $\vec{\mathbf{G}} \cdot (\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}) = 2\mathrm{e}^z - 2\mathrm{e}^z = 0$, thus $\vec{\mathbf{G}}$ is perpendicular to the level sets of $\vec{\mathbf{G}}$ in all points.

Alternatively one can check that $\vec{\nabla} f = 3(2x+y)^2(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = 3(2x+y)^2\mathrm{e}^{-z}\vec{\mathbf{G}}$, so $\vec{\nabla} f$ and $\vec{\mathbf{G}}$ are parallel and we conclude by Proposition 1.33(4).

Solution of Exercise 1.39. We compute the gradient of f and the unit vector in the direction of $\vec{\mathbf{F}}$:

$$\vec{\nabla} f(\vec{\mathbf{r}}) = (\mathrm{e}^x + z^2)\hat{\boldsymbol{\imath}} - \cos y\hat{\boldsymbol{\jmath}} + 2xz\hat{\boldsymbol{k}}, \qquad \frac{\vec{\mathbf{F}}(\vec{\mathbf{r}})}{|\vec{\mathbf{F}}(\vec{\mathbf{r}})|} = \frac{x\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{x^2+1}}.$$

The directional derivative is

$$\frac{\partial f}{\partial \hat{\boldsymbol{F}}}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{F}}(\vec{\mathbf{r}})}{|\vec{\mathbf{F}}(\vec{\mathbf{r}})|} \cdot \vec{\nabla} f(\vec{\mathbf{r}}) = \frac{x\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{x^2+1}} \cdot \big((\mathrm{e}^x + z^2)\hat{\boldsymbol{\imath}} - \cos y\hat{\boldsymbol{\jmath}} + 2xz\hat{\boldsymbol{k}}\big) = \frac{x(\mathrm{e}^x + z^2) - \cos y}{\sqrt{x^2+1}}.$$

Solution of Exercise 1.41. We simply compute all the partial derivatives:

$$J\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \begin{pmatrix} 2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad J\vec{\mathbf{G}}(\vec{\mathbf{r}}) = \begin{pmatrix} 0 & z & y\\ z & 0 & x\\ y & x & 0 \end{pmatrix}, \qquad J\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \begin{pmatrix} 2x & 2y & 2z\\ 0 & 0 & 0\\ 0 & -\sin y & 0 \end{pmatrix}.$$

Solution of Exercise 1.44. We first compute the gradients and then all the second derivatives (recall the identity $h(\vec{\mathbf{r}}) = (x^2+y^2+z^2)^2 = x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$):

$$\begin{aligned}
&\vec{\nabla} f = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}}, \qquad \vec{\nabla} g = y\mathrm{e}^z\hat{\boldsymbol{\imath}} + x\mathrm{e}^z\hat{\boldsymbol{\jmath}} + xy\mathrm{e}^z\hat{\boldsymbol{k}},\\
&\vec{\nabla} h = \vec{\nabla}(x^2+y^2+z^2)^2 = 2(x^2+y^2+z^2)(2x\hat{\boldsymbol{\imath}}+2y\hat{\boldsymbol{\jmath}}+2z\hat{\boldsymbol{k}}) = 4|\vec{\mathbf{r}}|^2\vec{\mathbf{r}},\\
&Hf = \begin{pmatrix} 2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad Hg = \begin{pmatrix} 0 & \mathrm{e}^z & y\mathrm{e}^z\\ \mathrm{e}^z & 0 & x\mathrm{e}^z\\ y\mathrm{e}^z & x\mathrm{e}^z & xy\mathrm{e}^z\end{pmatrix},\\
&Hh = \begin{pmatrix} 12x^2+4y^2+4z^2 & 8xy & 8xz\\ 8xy & 4x^2+12y^2+4z^2 & 8yz\\ 8xz & 8yz & 4x^2+4y^2+12z^2\end{pmatrix},\\
&\Delta f = 0, \qquad \Delta g = xy\mathrm{e}^z = g, \qquad \Delta h = 20|\vec{\mathbf{r}}|^2.
\end{aligned}$$

The field f is a "harmonic function", since its Laplacian vanishes everywhere.

Solution of Exercise 1.46. We only have to compute three derivatives for each field and sum them:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{\mathbf{F}} &= \frac{\partial(2x)}{\partial x} + \frac{\partial(-2y)}{\partial y} + \frac{\partial 0}{\partial z} = 2-2+0 = 0,\\
\vec{\nabla}\cdot\vec{\mathbf{G}} &= 0+0+0 = 0,\\
\vec{\nabla}\cdot\vec{\mathbf{H}} &= 2x+0+0 = 2x.
\end{aligned}$$

Solution of Exercise 1.50. For each field we have to evaluate six partial derivatives (the non-diagonal terms of the Jacobian matrix $J\vec{\mathbf{F}}$) and sum them appropriately:

$$\begin{aligned}
\vec{\nabla}\times\vec{\mathbf{F}} &= \vec{\mathbf{0}},\\
\vec{\nabla}\times\vec{\mathbf{G}} &= (x-x)\hat{\boldsymbol{\imath}} + (y-y)\hat{\boldsymbol{\jmath}} + (z-z)\hat{\boldsymbol{k}} = \vec{\mathbf{0}},\\
\vec{\nabla}\times\vec{\mathbf{H}} &= (-\sin y - 0)\hat{\boldsymbol{\imath}} + (2z-0)\hat{\boldsymbol{\jmath}} + (0-2y)\hat{\boldsymbol{k}} = -\sin y\,\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} - 2y\hat{\boldsymbol{k}}.
\end{aligned}$$

Solution of Exercise 1.61. We compute the left-hand side and the right-hand side, starting from the computation of the terms appearing in them, and we verify that they give the same result:

$$\begin{aligned}
\vec{\mathbf{F}}\times\vec{\mathbf{G}} &= x^2\mathrm{e}^{xy}\hat{\boldsymbol{\imath}} - z\mathrm{e}^{xy}\hat{\boldsymbol{\jmath}},\\
\text{left-hand side} &= \vec{\nabla}\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = \vec{\nabla}\cdot(x^2\mathrm{e}^{xy}\hat{\boldsymbol{\imath}} - z\mathrm{e}^{xy}\hat{\boldsymbol{\jmath}}) = 2x\mathrm{e}^{xy} + x^2y\mathrm{e}^{xy} - zx\mathrm{e}^{xy} = (2x+x^2y-zx)\mathrm{e}^{xy};\\
\vec{\nabla}\times\vec{\mathbf{F}} &= 3y^2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}}, \qquad \vec{\nabla}\times\vec{\mathbf{G}} = x\mathrm{e}^{xy}\hat{\boldsymbol{\imath}} - y\mathrm{e}^{xy}\hat{\boldsymbol{\jmath}},\\
\text{right-hand side} &= (\vec{\nabla}\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec{\nabla}\times\vec{\mathbf{G}})\\
&= (3y^2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}})\cdot(\mathrm{e}^{xy}\hat{\boldsymbol{k}}) - (z\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + y^3\hat{\boldsymbol{k}})\cdot(x\mathrm{e}^{xy}\hat{\boldsymbol{\imath}} - y\mathrm{e}^{xy}\hat{\boldsymbol{\jmath}})\\
&= 2x\mathrm{e}^{xy} - zx\mathrm{e}^{xy} + x^2y\mathrm{e}^{xy} = (2x - zx + x^2y)\mathrm{e}^{xy} = \text{left-hand side}.
\end{aligned}$$

Solution of Exercise 1.62. We first compute the divergence and the curl of the position vector:

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{\mathbf{r}}) &= \vec{\nabla}\cdot(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,\\
\vec{\nabla}\times(\vec{\mathbf{r}}) &= \vec{\nabla}\times(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) = \Big(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\Big)\hat{\boldsymbol{k}} = \vec{\mathbf{0}}.
\end{aligned}$$

(This solves the exercise for $\alpha = 0$.) To compute divergence and curl of the vector field $\vec{\mathbf{F}} = |\vec{\mathbf{r}}|^\alpha\vec{\mathbf{r}}$ we decompose it as the product between the scalar field $|\vec{\mathbf{r}}|^\alpha$ (for which we have already computed the gradient in Exercise 1.35) and the vector field $\vec{\mathbf{r}}$, so that we can apply the product rules (29) and (31):

$$\begin{aligned}
\vec{\nabla}\cdot(|\vec{\mathbf{r}}|^\alpha\vec{\mathbf{r}}) &\overset{(29)}{=} (\vec{\nabla}\cdot\vec{\mathbf{r}})|\vec{\mathbf{r}}|^\alpha + \vec{\mathbf{r}}\cdot\vec{\nabla}(|\vec{\mathbf{r}}|^\alpha)\\
&= 3|\vec{\mathbf{r}}|^\alpha + \vec{\mathbf{r}}\cdot\big(\alpha|\vec{\mathbf{r}}|^{\alpha-2}\vec{\mathbf{r}}\big) \qquad \text{from solution to Exercise 1.35}
\end{aligned}$$

$$= (3+\alpha)|\vec{\mathbf{r}}|^\alpha \qquad\qquad \text{from } \vec{\mathbf{r}}\cdot\vec{\mathbf{r}} = |\vec{\mathbf{r}}|^2,$$
$$\vec\nabla\times(|\vec{\mathbf{r}}|^\alpha\vec{\mathbf{r}}) \overset{(31)}{=} \vec\nabla(|\vec{\mathbf{r}}|^\alpha)\times\vec{\mathbf{r}} + |\vec{\mathbf{r}}|^\alpha \underbrace{\vec\nabla\times\vec{\mathbf{r}}}_{=\vec{\mathbf{0}}} = \alpha|\vec{\mathbf{r}}|^{\alpha-2}\underbrace{\vec{\mathbf{r}}\times\vec{\mathbf{r}}}_{=\vec{\mathbf{0}}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}.$$

Solution of Exercise 1.63. *(i)* Using that for all vectors $\vec{\mathbf{u}}\times\vec{\mathbf{u}} = \vec{\mathbf{0}}$, we have: $\vec\nabla\times(f\vec\nabla f) \overset{(31)}{=} (\vec\nabla f)\times(\vec\nabla f) + f\vec\nabla\times(\vec\nabla f) \overset{(26)}{=} \vec{\mathbf{0}}$.

(ii) Using vector identities, the chain rule (14) (with $G(u) = u^\ell$) and again $\vec{\mathbf{u}}\times\vec{\mathbf{u}} = \vec{\mathbf{0}}$ for all $\vec{\mathbf{u}}\in\mathbb{R}^3$:

$$\vec\nabla\times\big(f^n\vec\nabla(f^\ell)\big) \overset{(31)}{=} \vec\nabla(f^n)\times\vec\nabla(f^\ell) + f^n\underbrace{\vec\nabla\times(\vec\nabla(f^\ell))}_{=\vec{\mathbf{0}},(26)}$$
$$\overset{(14)}{=} (nf^{n-1}\vec\nabla f)\times(\ell f^{\ell-1}\vec\nabla f) = n\ell f^{n+\ell-2}\vec\nabla f\times\vec\nabla f \quad = \quad \vec{\mathbf{0}}.$$

Solution of Exercise 1.67. Proceeding as in the example, we obtain the potentials

$$\varphi_A = x^2+y^3+e^z, \qquad \varphi_C = \frac{x^4y^4}{4}+\frac{z^3}{3}, \qquad \vec{\mathbf{A}}_D = \Big(xy+\frac{y^2}{2}\Big)\hat{\boldsymbol{k}}.$$

Solution of Exercise 1.68. By direct computation we obtain $\vec\nabla\vec{\mathbf{r}} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3$, so the position vector is not solenoidal. Since the jth component of $\vec{\mathbf{r}}$ depends only on the jth component of its argument (which is itself!), by definition its curl vanishes, so the position vector field is irrotational. From equation (15) we have $\vec\nabla(|\vec{\mathbf{r}}|^2) = 2\vec{\mathbf{r}}$, so a scalar potential of $\vec{\mathbf{r}}$ is $\frac12|\vec{\mathbf{r}}|^2$.

Solution of Exercise 1.69. Simply take $\vec{\mathbf{H}} = \vec{\mathbf{F}}\times\vec{\mathbf{G}}$: by (30) we have $\vec\nabla\cdot\vec{\mathbf{H}} = \vec\nabla\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = (\vec\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec\nabla\times\vec{\mathbf{G}}) = \vec{\mathbf{0}}\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot\vec{\mathbf{0}} = 0$, so $\vec{\mathbf{H}}$ is solenoidal.

Solution of Exercise 1.73. If the field $\vec{\mathbf{F}}$ is conservative then its scalar potential φ satisfies $\vec{\mathbf{F}} = \vec\nabla\varphi$, so by identity (24) we have $\vec\nabla\cdot\vec{\mathbf{F}} = \vec\nabla\cdot(\vec\nabla\varphi) = \Delta\varphi$.

Solution of Exercise 1.74. If the gradient of a scalar field f admits a vector potential $\vec{\mathbf{A}}$, we have that $\vec\nabla f = \vec\nabla\times\vec{\mathbf{A}}$. If the vector potential itself is conservative, it means that $\vec{\mathbf{A}} = \vec\nabla\psi$ for some scalar field ψ. Thus $\vec\nabla f = \vec\nabla\times\vec{\mathbf{A}} = \vec\nabla\times\vec\nabla\psi = \vec{\mathbf{0}}$ by the identity (26). A scalar field whose gradient vanishes everywhere is constant, so the fields sought are simply the constant ones $f(\vec{\mathbf{r}}) = \lambda$ for all $\vec{\mathbf{r}}$.

Solution of Exercise 1.75. Using that the curl of a gradient vanishes, we immediately see that the product is solenoidal:

$$\vec\nabla\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) \overset{(30)}{=} (\vec\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec\nabla\times\vec{\mathbf{G}}) = (\underbrace{\vec\nabla\times\vec\nabla\varphi}_{=\vec{\mathbf{0}},\ (26)})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\underbrace{\vec\nabla\times\vec\nabla\psi}_{=\vec{\mathbf{0}},\ (26)}) = 0.$$

Two easy choices of the potential are possible:

$$\vec{\mathbf{A}} := \varphi\vec\nabla\psi \qquad \Rightarrow \qquad \vec\nabla\times\vec{\mathbf{A}} \overset{(31)}{=} (\vec\nabla\varphi)\times(\vec\nabla\psi) + \varphi\underbrace{\vec\nabla\times(\vec\nabla\psi)}_{=0,\ (26)} = \vec{\mathbf{F}}\times\vec{\mathbf{G}};$$
$$\vec{\mathbf{B}} := -\psi\vec\nabla\varphi \qquad \Rightarrow \qquad \vec\nabla\times\vec{\mathbf{B}} \overset{(31)}{=} -(\vec\nabla\psi)\times(\vec\nabla\varphi) - \psi\underbrace{\vec\nabla\times(\vec\nabla\varphi)}_{=0,\ (26)} = -\vec{\mathbf{G}}\times\vec{\mathbf{F}} \overset{(3)}{=} \vec{\mathbf{F}}\times\vec{\mathbf{G}}.$$

Solution of Exercise 1.78. We have $\frac{d\vec{\mathbf{a}}}{dt}(t) = -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}}$ and $\frac{d\vec{\mathbf{b}}}{dt}(t) = -2\sin 2\tau\,\hat{\boldsymbol{\imath}} + 2\cos 2\tau\,\hat{\boldsymbol{\jmath}}$.

The path of $\frac{d\vec{\mathbf{a}}}{dt}$ is again the unit circle, but the initial point is placed in $\hat{\boldsymbol{\jmath}}$ (as opposed to $\hat{\boldsymbol{\imath}}$ as for $\vec{\mathbf{a}}$). The path of $\frac{d\vec{\mathbf{b}}}{dt}$ is the circle centred at the origin with radius 2. Since $\vec{\mathbf{b}}$ moves more quickly than $\vec{\mathbf{a}}$ (it covers the unit circle in "time" π as opposed to 2π) the total derivative, representing its velocity, has greater magnitude.

To find a curve whose derivative equals that of $\vec{\mathbf{a}}$ it suffices to add a constant vector, e.g. $\vec{\mathbf{c}}(t) = (3+\cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$.

Solution of Exercise 1.79. We simply compute all the terms:

$$\frac{d\vec{\mathbf{a}}}{dt} = -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{b}}}{dt} = -2\sin 2t\,\hat{\boldsymbol{\imath}} + 2\cos 2t\,\hat{\boldsymbol{\jmath}},$$

$$(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) \overset{(1)}{=} \cos t\cos 2t + \sin t\sin 2t,$$

$$\text{LHS of (36)ii} \qquad (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})' = -\sin t\cos 2t - 2\cos t\sin 2t + \cos t\sin 2t + 2\sin t\cos 2t = \sin t\cos 2t - \cos t\sin 2t,$$

$$\text{RHS of (36)ii} \qquad \frac{d\vec{\mathbf{a}}}{dt}\cdot\vec{\mathbf{b}} + \vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{b}}}{dt} = (-\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}})\cdot(\cos 2t\,\hat{\boldsymbol{\imath}} + \sin 2t\,\hat{\boldsymbol{\jmath}}) + (\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}})\cdot(-2\sin 2t\,\hat{\boldsymbol{\imath}} + 2\cos 2t\,\hat{\boldsymbol{\jmath}}) \overset{(1)}{=} \sin t\cos 2t - \cos t\sin 2t,$$

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}}) \overset{(2)}{=} (\cos t\sin 2t - \sin t\cos 2t)\hat{\boldsymbol{k}},$$

$$\text{LHS of (36)iii} \qquad \frac{d(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}{dt} = (-\sin t\sin 2t + 2\cos t\cos 2t - \cos t\cos 2t + 2\sin t\sin 2t)\hat{\boldsymbol{k}} = (\cos t\cos 2t + \sin t\sin 2t)\hat{\boldsymbol{k}} = \cos(2t-t)\hat{\boldsymbol{k}} = \cos t\,\hat{\boldsymbol{k}},$$

$$\text{RHS of (36)iii} \qquad \frac{d\vec{\mathbf{a}}}{dt}\times\vec{\mathbf{b}} + \vec{\mathbf{a}}\times\frac{d\vec{\mathbf{b}}}{dt} = (-\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}})\times(\cos 2t\,\hat{\boldsymbol{\imath}} + \sin 2t\,\hat{\boldsymbol{\jmath}}) + (\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}})\times(-2\sin 2t\,\hat{\boldsymbol{\imath}} + 2\cos 2t\,\hat{\boldsymbol{\jmath}})$$
$$\overset{(2)}{=} (-\sin t\sin 2t - \cos t\cos 2t)\hat{\boldsymbol{k}} + (2\cos t\cos 2t + 2\sin t\sin 2t)\hat{\boldsymbol{k}} = (\cos t\cos 2t + \sin t\sin 2t)\hat{\boldsymbol{k}} = \cos t\,\hat{\boldsymbol{k}}.$$

Solution of Exercise 1.80. We simply use the definition of scalar and vector product, the product rule for real functions $(FG)' = F'G + FG'$, and rearrange some terms. Since the second equation is vectorial, we prove the identity only for its first component.

$$(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})' \overset{(1)}{=} \big(a_1b_1 + a_2b_2 + a_3b_3\big)' = \big(a_1'b_1 + a_1b_1' + a_2'b_2 + a_2b_2' + a_3'b_3 + a_3b_3'\big)$$
$$\overset{(1)}{=} \big(a_1'\hat{\boldsymbol{\imath}} + a_2'\hat{\boldsymbol{\jmath}} + a_3'\hat{\boldsymbol{k}}\big)\cdot\big(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}\big) + \big(a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}\big)\cdot\big(b_1'\hat{\boldsymbol{\imath}} + b_2'\hat{\boldsymbol{\jmath}} + b_3'\hat{\boldsymbol{k}}\big) = \frac{d\vec{\mathbf{a}}}{dt}\cdot\vec{\mathbf{b}} + \vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{b}}}{dt};$$

$$\left(\frac{d(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}{dt}\right)_1 = \frac{d(\vec{\mathbf{a}}\times\vec{\mathbf{b}})_1}{dt} \overset{(2)}{=} (a_2b_3 - a_3b_2)' = (a_2'b_3 + a_2b_3' - a_3'b_2 - a_3b_2') = (a_2'b_3 - a_3'b_2) + (a_2b_3' - a_3b_2') \overset{(2)}{=} \left(\frac{d\vec{\mathbf{a}}}{dt}\times\vec{\mathbf{b}}\right)_1 + \left(\vec{\mathbf{a}}\times\frac{d\vec{\mathbf{b}}}{dt}\right)_1.$$

Solution of Exercise 1.81. Saying that $\vec{\mathbf{a}}$ equal its own derivative means that $\frac{d\vec{\mathbf{a}}}{dt}(t) = \vec{\mathbf{a}}(t)$, which is a vector linear ordinary differential equation (ODE). This can be written in components as $\frac{da_1}{dt} = a_1$, $\frac{da_2}{dt} = a_2$ and $\frac{da_3}{dt} = a_3$, so we know how to solve it from first-year calculus. Using the initial condition $\vec{\mathbf{a}}(0) = \vec{\mathbf{u}}$ we have $a_1(t) = u_1\mathrm{e}^t$, $a_2(t) = u_2\mathrm{e}^t$ and $a_3(t) = u_3\mathrm{e}^t$, which can be written in vector form as $\vec{\mathbf{a}}(t) = \vec{\mathbf{u}}\mathrm{e}^t$. We know from ODE theory that this is the unique solution of the differential equation. If $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ the path of $\vec{\mathbf{a}}$ collapses to a point. If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$ the path of $\vec{\mathbf{a}}$ is the straight half-line starting at $\vec{\mathbf{u}}$ and pointing away from the origin. Be careful: despite the expression $\vec{\mathbf{a}}(t) = \vec{\mathbf{u}}\mathrm{e}^t$ contains an exponential, the corresponding path is a straight line, recall the difference between path and graph!

Solution of Exercise 1.83. *(i)* We compute the total derivative of $\vec{\mathbf{a}}$, the gradient of f evaluated in $\vec{\mathbf{a}}$, and their scalar product according to the chain rule (37):

$$\frac{d\vec{\mathbf{a}}}{dt} = \hat{\boldsymbol{\imath}} + 3t^2\hat{\boldsymbol{\jmath}}, \qquad \vec{\nabla} f = y\mathrm{e}^z\hat{\boldsymbol{\imath}} + x\mathrm{e}^z\hat{\boldsymbol{\jmath}} + xy\mathrm{e}^z\hat{\boldsymbol{k}} = t^3\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + t^4\hat{\boldsymbol{k}}$$

$$\Rightarrow \qquad \frac{df}{dt} = \vec{\nabla} f\cdot\frac{d\vec{\mathbf{a}}}{dt} = t^3 + 3t^3 = 4t^3.$$

(ii) Alternatively, we can compute the composition $f(\vec{\mathbf{a}}(t))$ and then derive it with respect to t:

$$f\big(\vec{\mathbf{a}}(t)\big) = t\,t^3\,\mathrm{e}^0 = t^4, \qquad \frac{df}{dt} = \frac{df\big(\vec{\mathbf{a}}(t)\big)}{dt} = 4t^3.$$

Solution of Exercise 1.87. We first compute all the compositions and then take the desired derivatives:

$$\begin{aligned}
(g\circ\vec{\mathbf{F}})(\vec{\mathbf{r}}) &= g(x\hat{\boldsymbol\imath}+z\hat{\boldsymbol\jmath}-y\hat{\mathbf{k}}) = x^2-z^2,\\
(g\circ\vec{\mathbf{a}})(t) &= g(\cos t\hat{\boldsymbol\imath}+\sin t\hat{\boldsymbol\jmath}) = \cos^2 t-\sin^2 t,\\
(g\circ\vec{\mathbf{F}}\circ\vec{\mathbf{a}})(t) &= g\big(\vec{\mathbf{F}}(\cos t\hat{\boldsymbol\imath}+\sin t\hat{\boldsymbol\jmath})\big) = g(\cos t\hat{\boldsymbol\imath}-\sin t\hat{\mathbf{k}}) = \cos^2 t,\\
\frac{\partial(g\circ\vec{\mathbf{F}})}{\partial x} &= \frac{\partial(x^2-z^2)}{\partial x} = 2x, \qquad \frac{\partial(g\circ\vec{\mathbf{F}})}{\partial y} = \frac{\partial(x^2-z^2)}{\partial y} = 0, \qquad \frac{\partial(g\circ\vec{\mathbf{F}})}{\partial z} = \frac{\partial(x^2-z^2)}{\partial z} = -2z,\\
\frac{d(g\circ\vec{\mathbf{a}})}{dt} &= \frac{d(\cos^2 t-\sin^2 t)}{dt} = -4\sin t\cos t,\\
\frac{d(g\circ\vec{\mathbf{F}}\circ\vec{\mathbf{a}})}{dt} &= \frac{d(\cos^2 t)}{dt} = -2\sin t\cos t.
\end{aligned}$$

Using the vector versions of the chain rule, the computations turn out to be slightly more complicated, in particular we have to be careful to evaluate the fields at the right values:

$$\vec\nabla g = 2x\hat{\boldsymbol\imath}-2y\hat{\boldsymbol\jmath}, \qquad \frac{\partial\vec{\mathbf{F}}}{\partial x} = \hat{\boldsymbol\imath}, \qquad \frac{\partial\vec{\mathbf{F}}}{\partial y} = -\hat{\mathbf{k}}, \qquad \frac{\partial\vec{\mathbf{F}}}{\partial z} = \hat{\boldsymbol\jmath},$$

$$\begin{aligned}
\frac{\partial(g\circ\vec{\mathbf{F}})}{\partial x} &\overset{(38)}{=} \vec\nabla g\cdot\frac{\partial\vec{\mathbf{F}}}{\partial x} = \big(2F_1(\vec{\mathbf{r}})\hat{\boldsymbol\imath}-2F_2(\vec{\mathbf{r}})\hat{\boldsymbol\jmath}\big)\cdot(\hat{\boldsymbol\imath}) = 2F_1(\vec{\mathbf{r}}) = 2x,\\
\frac{\partial(g\circ\vec{\mathbf{F}})}{\partial y} &\overset{(38)}{=} \vec\nabla g\cdot\frac{\partial\vec{\mathbf{F}}}{\partial y} = \big(2F_1(\vec{\mathbf{r}})\hat{\boldsymbol\imath}-2F_2(\vec{\mathbf{r}})\hat{\boldsymbol\jmath}\big)\cdot(-\hat{\mathbf{k}}) = 0,\\
\frac{\partial(g\circ\vec{\mathbf{F}})}{\partial z} &\overset{(38)}{=} \vec\nabla g\cdot\frac{\partial\vec{\mathbf{F}}}{\partial z} = \big(2F_1(\vec{\mathbf{r}})\hat{\boldsymbol\imath}-2F_2(\vec{\mathbf{r}})\hat{\boldsymbol\jmath}\big)\cdot(\hat{\boldsymbol\jmath}) = -2F_2(\vec{\mathbf{r}}) = -2z,\\
\frac{d(g\circ\vec{\mathbf{a}})}{dt} &\overset{(37)}{=} \vec\nabla g(\vec{\mathbf{a}})\cdot\frac{d(\vec{\mathbf{a}})}{dt} = \big(2a_1(t)\hat{\boldsymbol\imath}-2a_2(t)\hat{\boldsymbol\jmath}\big)\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\boldsymbol\jmath})\\
&= (2\cos t\hat{\boldsymbol\imath}-2\sin t\hat{\boldsymbol\jmath})\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\boldsymbol\jmath}) = -4\sin t\cos t,\\
\frac{d(g\circ\vec{\mathbf{F}}\circ\vec{\mathbf{a}})}{dt} &\overset{(37)}{=} \vec\nabla g\cdot\frac{d(\vec{\mathbf{F}}\circ\vec{\mathbf{a}})}{dt} = \vec\nabla g\cdot\Big(\frac{d(F_1\circ\vec{\mathbf{a}})}{dt}\hat{\boldsymbol\imath}+\frac{d(F_2\circ\vec{\mathbf{a}})}{dt}\hat{\boldsymbol\jmath}+\frac{d(F_3\circ\vec{\mathbf{a}})}{dt}\hat{\mathbf{k}}\Big)\\
&\overset{(37)}{=} \vec\nabla g\cdot\Big(\vec\nabla F_1\cdot\frac{d\vec{\mathbf{a}}}{dt}\hat{\boldsymbol\imath}+\vec\nabla F_2\cdot\frac{d\vec{\mathbf{a}}}{dt}\hat{\boldsymbol\jmath}+\vec\nabla F_3\cdot\frac{d\vec{\mathbf{a}}}{dt}\hat{\mathbf{k}}\Big)\\
&= \vec\nabla g\cdot\Big(\big(\hat{\boldsymbol\imath}\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\boldsymbol\jmath})\big)\hat{\boldsymbol\imath}+\big(-\hat{\mathbf{k}}\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\boldsymbol\jmath})\big)\hat{\boldsymbol\jmath}+\big(\hat{\boldsymbol\jmath}\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\boldsymbol\jmath})\big)\hat{\mathbf{k}}\Big)\\
&= \vec\nabla g\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\mathbf{k}}) = \big(2F_1(\vec{\mathbf{a}})\hat{\boldsymbol\imath}-2F_2(\vec{\mathbf{a}})\big)\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\mathbf{k}})\\
&= (2a_1\hat{\boldsymbol\imath}+a_3\hat{\boldsymbol\jmath})\cdot(-\sin t\hat{\boldsymbol\imath}+\cos t\hat{\mathbf{k}}) = -2a_1(t)\sin t = -2\sin^2 t.
\end{aligned}$$

Solution of review exercises in Section 1.8

1. *Compute gradient, Hessian and Laplacian of the five scalar fields.*

   We only need to compute several partial derivatives and combine them appropriately.

$$\nabla f = y^2z^3\hat{\boldsymbol\imath}+2xyz^3\hat{\boldsymbol\jmath}+3xy^2z^2\hat{\mathbf{k}} = \vec{\mathbf{M}},$$

$$Hf = \begin{pmatrix} 0 & 2yz^3 & 3y^2z^2\\ 2yz^3 & 2xz^3 & 6xyz^2\\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{pmatrix}, \qquad \Delta f = 2xz(z^2+3y^2),$$

$$\nabla g = \big(-\sin x+\cos(2y+x)\big)\hat{\boldsymbol\imath}+2\cos(2y+x)\hat{\boldsymbol\jmath},$$

$$Hg = \begin{pmatrix} -\cos x-\sin(2y+x) & -2\sin(2y+x) & 0\\ -2\sin(2y+x) & -4\sin(2y+x) & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta g = -\cos x-5\sin(2y+x),$$

$$\nabla h = e^x\cos y\hat{\boldsymbol\imath}-e^x\sin y\hat{\boldsymbol\jmath}, \qquad Hh = \begin{pmatrix} h & -e^x\sin y & 0\\ -e^x\sin y & -h & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta h = 0,$$

$$\nabla\ell = (3x^2+y^2+z^2)\hat{\boldsymbol\imath} + (2xy-3y^2)\hat{\boldsymbol\jmath} + 2xz\hat{\boldsymbol k},$$

$$H\ell = \begin{pmatrix} 6x & 2y & 2z \\ 2y & -6y & 0 \\ 2z & 0 & 2x \end{pmatrix}, \qquad \Delta\ell = 6(x-y),$$

$$\nabla m = yx^{y-1}\hat{\boldsymbol\imath} + x^y\log x\hat{\boldsymbol\jmath},$$

$$Hm = \begin{pmatrix} y(y-1)x^{y-2} & x^{y-1}(1+y\log x) & 0 \\ x^{y-1}(1+y\log x) & x^y\log^2 x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta m = y(y-1)x^{y-2} + x^y\log^2 x.$$

2. *Compute Jacobian, divergence, curl and vector Laplacian of the five vector fields.*

   We use $\boxed{\vec{\mathbf H} = \vec{\mathbf r}\times\hat{\boldsymbol\imath} = z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k}}$, $\quad\vec{\mathbf M} = \vec\nabla f$, $\quad\vec{\mathbf L} = \frac14\vec\nabla(|\vec{\mathbf r}^4|)$ (see also Exercise 1.44).

$$J\vec{\mathbf F} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \vec\nabla\cdot\vec{\mathbf F} = 0, \qquad \vec\nabla\times\vec{\mathbf F} = \vec{\mathbf 0}, \qquad \vec{\boldsymbol\Delta}\vec{\mathbf F} = \vec{\mathbf 0},$$

$$J\vec{\mathbf G} = \begin{pmatrix} 0 & 0 & -2ze^{-z^2} \\ 0 & 2y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \vec\nabla\cdot\vec{\mathbf G} = 2y,$$

$$\vec\nabla\times\vec{\mathbf G} = -2ze^{-z^2}\hat{\boldsymbol\jmath}, \qquad \vec{\boldsymbol\Delta}\vec{\mathbf G} = 2(2z^2-1)e^{-z^2}\hat{\boldsymbol\imath} + 2\hat{\boldsymbol\jmath},$$

$$J\vec{\mathbf H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \vec\nabla\cdot\vec{\mathbf H} = 0, \qquad \vec\nabla\times\vec{\mathbf H} = -2\hat{\boldsymbol\imath}, \qquad \vec{\boldsymbol\Delta}\vec{\mathbf H} = \vec{\mathbf 0},$$

$$J\vec{\mathbf L} = \begin{pmatrix} 3x^2+y^2+z^2 & 2xy & 2xz \\ 2xy & x^2+3y^2+z^2 & 2yz \\ 2xz & 2yz & x^2+y^2+z^2 \end{pmatrix}, \qquad \vec\nabla\cdot\vec{\mathbf L} = 5|\vec{\mathbf r}|^2,$$

$$\vec\nabla\times\vec{\mathbf L} = \vec{\mathbf 0}, \qquad \vec{\boldsymbol\Delta}\vec{\mathbf L} = 10\vec{\mathbf r},$$

$$J\vec{\mathbf M} = Hf, \qquad \vec\nabla\cdot\vec{\mathbf M} = \Delta f, \qquad \vec\nabla\times\vec{\mathbf M} = \vec\nabla\times\vec\nabla f = \vec{\mathbf 0},$$

$$\vec{\boldsymbol\Delta}\vec{\mathbf M} = \vec{\boldsymbol\Delta}\vec\nabla f = \vec\nabla\Delta f = (2z^3+6y^2z)\hat{\boldsymbol\imath} + 12xyz\hat{\boldsymbol\jmath} + (6xz^2+6xy^2)\hat{\boldsymbol k}.$$

3. *Which of the fields are solenoidal and which are irrotational? Which are harmonic?*

   The only scalar harmonic field is h.
   $\vec{\mathbf F}$ is solenoidal and irrotational.
   $\vec{\mathbf G}$ is neither solenoidal nor irrotational.
   $\vec{\mathbf H}$ is solenoidal only.
   $\vec{\mathbf L}$ is irrotational only.
   $\vec{\mathbf M}$ is irrotational only.
   (Moreover $\vec{\mathbf F}$ and $\vec{\mathbf H}$ are harmonic in the sense that all their components are harmonic.)

4. *Compute a scalar and a vector potential for $\vec{\mathbf F}$, a vector potential for $\vec{\mathbf H}$ (can (31) help you?) and a scalar potential for $\vec{\mathbf M}$. Can you guess a scalar potential for $\vec{\mathbf L}$?*

   Proceeding as in Example 1.67, we obtain the following potentials:

$$\vec{\mathbf F} = \vec\nabla(xy+xz+yz+\lambda) = \vec\nabla\times\frac12\Big((z^2-y^2)\hat{\boldsymbol\imath} + (x^2-z^2)\hat{\boldsymbol\jmath} + (y^2-x^2)\hat{\boldsymbol k}\Big),$$

$$\vec{\mathbf H} = \vec\nabla\times\Big(\frac{|\vec{\mathbf r}|^2}{2}\hat{\boldsymbol\imath}\Big), \qquad \vec{\mathbf L} = \vec\nabla\Big(\frac14|\vec{\mathbf r}|^4+\lambda\Big), \qquad \vec{\mathbf M} = \vec\nabla(f+\lambda) \qquad \text{for any scalar } \lambda.$$

5. *Show that $\vec{\mathbf G}$ does not admit neither scalar nor vector potential.*

   $\vec{\mathbf G}$ is neither irrotational nor solenoidal, therefore the existence of a scalar or a vector potential would entail a contradiction with the box in Section 1.5 (conservative $\Rightarrow$ irrotational, vector potential $\Rightarrow$ solenoidal) or with identities (24) and (25).

6. *Show that $\vec{\mathbf H}(\vec{\mathbf r})$ and $\vec{\mathbf L}(\vec{\mathbf r})$ are orthogonal to each other at every point $\vec{\mathbf r}\in\mathbb R^3$.*

$$\vec{\mathbf H}(\vec{\mathbf r})\cdot\vec{\mathbf L}(\vec{\mathbf r}) = (z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k})\cdot|\vec{\mathbf r}|^2\,\vec{\mathbf r} = |\vec{\mathbf r}|^2(0x+zy-yz) = 0.$$

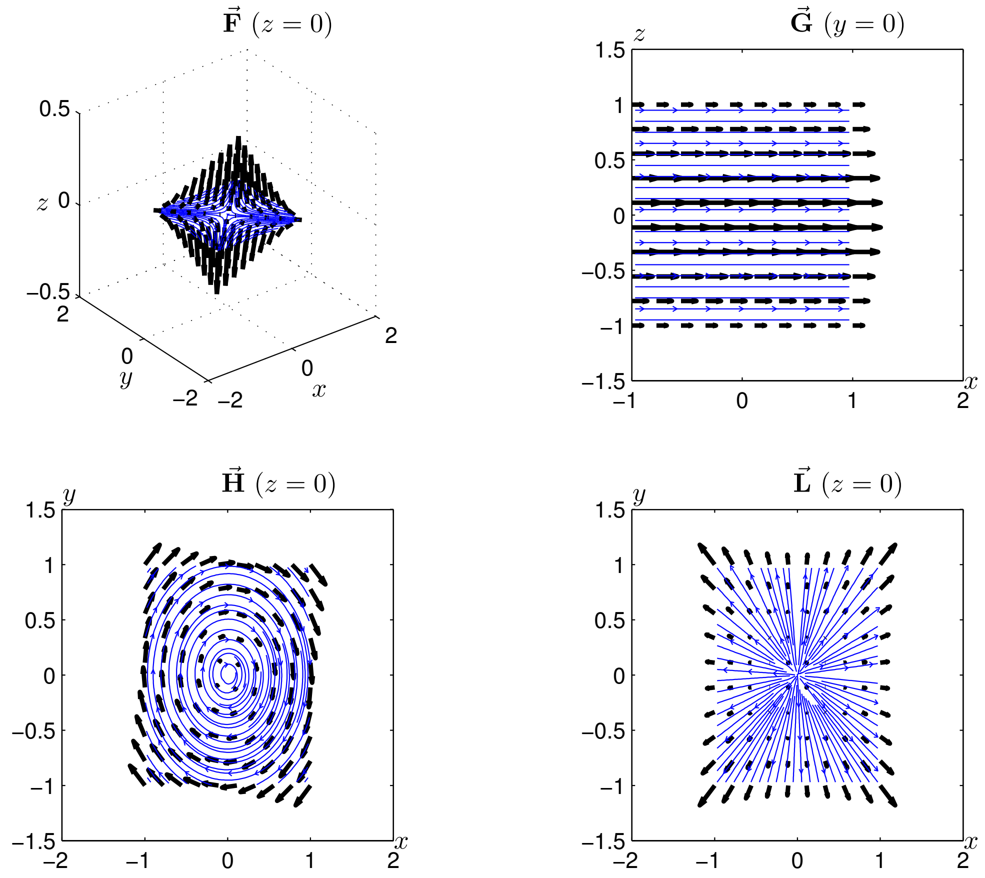


Figure 45: *Representation of some sections of the vector fields*

7. *Try to graphically represent the fields in* (39). *E.g. you can draw a qualitative plot like those of Section 1.2.2 for* $\vec{\mathbf{G}}$ *on the plane* $y = 0$*, for* $\vec{\mathbf{F}}$*,* $\vec{\mathbf{H}}$ *and* $\vec{\mathbf{L}}$ *on the plane* $z = 0$*.*

   The suggested sections are plotted in Figure 45. Note that only for $\vec{\mathbf{G}}$, $\vec{\mathbf{H}}$ and $\vec{\mathbf{L}}$ the plots are in the same form as those seen in the notes, since in these cases the fields are tangential to the considered planes. On the contrary, $\vec{\mathbf{F}}$ is not tangent to the plane $\{z = 0\}$, thus it can not be represented as a two-dimensional field. You might prove that $\vec{\mathbf{F}}$ is tangential to the plane $\{x + y + z = 0\}$.

8. *Demonstrate identities* (25) *and* (27) *for* $\vec{\mathbf{G}}$*;* (26) *for* f*;* (28) *for* f *and* h*;* (33) *for* $\vec{\mathbf{G}}$ *and* $\vec{\mathbf{H}}$*;* (29) *and* (31) *for* h *and* $\vec{\mathbf{H}}$*. (You can also demonstrate other identities of Propositions 1.52 and 1.55 for the various fields in* (39)*.)*

   We demonstrate the suggested identities. In some cases, the best strategy is to first expand the terms on both sides of the differential identities and then manipulate them, aiming at a common expression.

$$(25)\quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathbf{G}}) = \vec{\nabla}\cdot(-2ze^{-z^2}\hat{\boldsymbol{\jmath}}) = \frac{\partial}{\partial y}(-2ze^{-z^2})0,$$

$$\begin{aligned}(27)\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{G}}) &= \vec{\nabla}\times(-2ze^{-z^2}\hat{\boldsymbol{\jmath}})\\ &= -2(2z^2-1)e^{-z^2}\hat{\boldsymbol{\imath}}\\ &= 2\hat{\boldsymbol{\jmath}} - \Big(2\hat{\boldsymbol{\jmath}} + 2(2z^2-1)e^{-z^2}\hat{\boldsymbol{\imath}}\Big)\\ &= \vec{\nabla}(2y) - \vec{\boldsymbol{\Delta}}\vec{\mathbf{G}}\\ &= \vec{\nabla}(\vec{\nabla}\cdot\vec{\mathbf{G}}) - \vec{\boldsymbol{\Delta}}\vec{\mathbf{G}},\end{aligned}$$

$$\begin{aligned}(26)\quad \vec{\nabla}\times(\vec{\nabla}f) &= \vec{\nabla}\times(y^2z^3\hat{\boldsymbol{\imath}} + 2xyz^3\hat{\boldsymbol{\jmath}} + 3xy^2z^2\hat{\boldsymbol{k}})\\ &= (6xyz^2 - 6xyz^2)\hat{\boldsymbol{\imath}} + (3y^2z^2 - 3y^2z^2)\hat{\boldsymbol{\jmath}} + (2yz^3 - 2yz^3)\hat{\boldsymbol{k}} = \vec{\mathbf{0}},\end{aligned}$$

$$\begin{aligned}(28)\quad \vec{\nabla}(fh) &= \vec{\nabla}(xy^2z^3e^x\cos y)\\ &= (1+x)e^xy^2z^3\cos y\,\hat{\boldsymbol{\imath}} + xe^x(2y\cos y - y^2\sin y)z^3\hat{\boldsymbol{\jmath}} + 3xe^xy^2\cos y z^2\hat{\boldsymbol{k}}\end{aligned}$$

$$
\begin{aligned}
&= xy^2z^3(e^x\cos y\hat{\boldsymbol\imath} - e^x\sin y\hat{\boldsymbol\jmath}) + e^x\cos y(y^2z^3\hat{\boldsymbol\imath} + 2xyz^3\hat{\boldsymbol\jmath} + 3xy^2z^2\hat{\boldsymbol k})\\
&= f\vec\nabla h + h\vec\nabla f,
\end{aligned}
$$

(33)
$$
\begin{aligned}
\vec\nabla(\vec{\mathbf G}\cdot\vec{\mathbf H}) &= \vec\nabla\big((e^{-z^2}\hat{\boldsymbol\imath} + y^2\hat{\boldsymbol\jmath})\cdot(z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k})\big)\\
&= \vec\nabla(y^2z)\\
&= 2yz\hat{\boldsymbol\jmath} + y^2\hat{\boldsymbol k}\\
&= -y^2\hat{\mathbf k} + (2yz\hat{\boldsymbol\jmath} + 2yze^{-z^2}\hat{\boldsymbol\imath}) - 2yze^{-z^2}\hat{\boldsymbol\imath} + 2y^2\hat{\boldsymbol k}\\
&= \Big(e^{-z^2}\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\Big)(z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k}) + \Big(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\Big)(e^{-z^2}\hat{\boldsymbol\imath} + y^2\hat{\boldsymbol\jmath})\\
&\quad + (z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k})\times(-2ze^{-z^2}\hat{\boldsymbol\jmath}) + (e^{-z^2}\hat{\boldsymbol\imath} + y^2\hat{\boldsymbol\jmath})\times(-2\hat{\boldsymbol\imath})\\
&= (\vec{\mathbf G}\cdot\vec\nabla)\vec{\mathbf H} + (\vec{\mathbf H}\cdot\vec\nabla)\vec{\mathbf G} + \vec{\mathbf H}\times(\vec\nabla\times\vec{\mathbf G}) + \vec{\mathbf G}\times(\vec\nabla\times\vec{\mathbf H}),
\end{aligned}
$$

(29)
$$
\begin{aligned}
\vec\nabla\cdot(h\vec{\mathbf H}) &= \vec\nabla\cdot\big((e^x\cos y)z\hat{\boldsymbol\jmath} - (e^x\cos y)y\hat{\boldsymbol k}\big)\\
&= 0 - e^x\sin yz + 0\\
&= (e^x\cos y\hat{\boldsymbol\imath} - e^x\sin y\hat{\boldsymbol\jmath})\cdot(z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k}) + 0\\
&= (\vec\nabla h)\cdot\vec{\mathbf H} + h\vec\nabla\cdot\vec{\mathbf H},
\end{aligned}
$$

(31)
$$
\begin{aligned}
\vec\nabla\times(h\vec{\mathbf H}) &= \vec\nabla\times\big((e^x\cos y)z\hat{\boldsymbol\jmath} - (e^x\cos y)y\hat{\boldsymbol k}\big)\\
&= \big(-e^x(\cos y - y\sin y) - e^x\cos y\big)\hat{\boldsymbol\imath} + e^x\cos yy\hat{\boldsymbol\jmath} + e^x\cos yz\hat{\boldsymbol k}\\
&= ye^x\sin y\hat{\boldsymbol\imath} + ye^x\cos y\hat{\boldsymbol\jmath} + ze^x\cos y\hat{\boldsymbol k} - 2e^x\cos y\hat{\boldsymbol\imath}\\
&= (e^x\cos y\hat{\boldsymbol\imath} - e^x\sin y\hat{\boldsymbol\jmath})\times(z\hat{\boldsymbol\jmath} - y\hat{\boldsymbol k}) - 2e^x\cos y\hat{\boldsymbol\imath}\\
&= (\vec\nabla h)\times\vec{\mathbf H} + h\vec\nabla\times\vec{\mathbf H}.
\end{aligned}
$$

9. *Compute the (total) derivatives of the curves (i.e. $\frac{d\vec{\mathbf a}}{dt}$, $\frac{d\vec{\mathbf b}}{dt}$ and $\frac{d\vec{\mathbf c}}{dt}$) and try to draw them.*

$$
\begin{aligned}
\frac{d\vec{\mathbf a}}{dt} &= \frac{d(t^3-t)\hat{\boldsymbol\imath} + (1-t^2)\hat{\boldsymbol k}}{dt} = (3t^2-1)\hat{\boldsymbol\imath} - 2t\hat{\boldsymbol k},\\
\frac{d\vec{\mathbf b}}{dt} &= \frac{d(t^3\hat{\boldsymbol\imath} + t^2\hat{\boldsymbol\jmath} + \hat{\boldsymbol k})}{dt} = 3t^2\hat{\boldsymbol\imath} + 2t\hat{\boldsymbol\jmath},\\
\frac{d\vec{\mathbf c}}{dt} &= \frac{d\big(e^t\cos(2\pi t)\hat{\boldsymbol\imath} + e^t\sin(2\pi t)\hat{\boldsymbol\jmath}\big)}{dt}\\
&= e^t\Big(\big(\cos(2\pi t) - 2\pi\sin(2\pi t)\big)\hat{\boldsymbol\imath} + \big(\sin(2\pi t) + 2\pi\cos(2\pi t)\big)\hat{\boldsymbol\jmath}\Big)\\
&= e^t\Big((\hat{\mathbf i} + 2\pi\hat{\boldsymbol\jmath})\cos(2\pi t) + (\hat{\boldsymbol\jmath} - 2\pi\hat{\boldsymbol\imath})\sin(2\pi t)\Big).
\end{aligned}
$$

The curves and their derivatives are shown in Figure 46. From the plots of the curves we note that $\vec{\mathbf a}$ is a loop, $\vec{\mathbf b}$ is singular, $\vec{\mathbf c}$ is a (part of a) logarithmic spiral. The total derivatives of $\vec{\mathbf a}$ and $\vec{\mathbf b}$ (which are also curves) have almost the same plot, they can be transformed into each other by a rigid motion. Despite this fact, $\vec{\mathbf a}$ and $\vec{\mathbf b}$ have quite different shapes.

10. *Compute the following total derivatives of the scalar fields along the curves: $\frac{dh(\vec{\mathbf a})}{dt}$, $\frac{df(\vec{\mathbf b})}{dt}$ and $\frac{d\ell(\vec{\mathbf c})}{dt}$. (You can either use the chain rule (37) or first compute the composition.)*

Deriving the composition of the field with the curve:

$$
\begin{aligned}
\frac{d\big(h(\vec{\mathbf a})\big)}{dt} &= \frac{d(e^{t^3-t})}{dt} = e^{t^3-t}(3t^2-1) \qquad \text{(note that } \vec{\mathbf a} \text{ lies in plane } y=0),\\
\frac{d\big(f(\vec{\mathbf b})\big)}{dt} &= \frac{d(t^7)}{dt} = 7t^6,\\
\frac{d\big(\ell(\vec{\mathbf c})\big)}{dt} &= \frac{d\big(e^{3t}\cos(2\pi t) - e^{3t}\sin^3(2\pi t)\big)}{dt}\\
&= e^{3t}\Big(3\cos(2\pi t) - 3\sin^3(2\pi t) - 2\pi\sin(2\pi t) - 6\pi\sin^2(2\pi t)\cos(2\pi t)\Big).
\end{aligned}
$$

Alternatively, using the vector chain rule (37):

$$
\frac{d\big(h(\vec{\mathbf a})\big)}{dt} = \vec\nabla h(\vec{\mathbf a})\cdot\frac{d\vec{\mathbf a}}{dt} = (e^x\cos y\hat{\boldsymbol\imath} - e^x\sin y\hat{\boldsymbol\jmath})\cdot\Big((3t^2-1)\hat{\boldsymbol\imath} - 2t\hat{\boldsymbol k}\Big) = e^{t^3-t}(3t^2-1),
$$

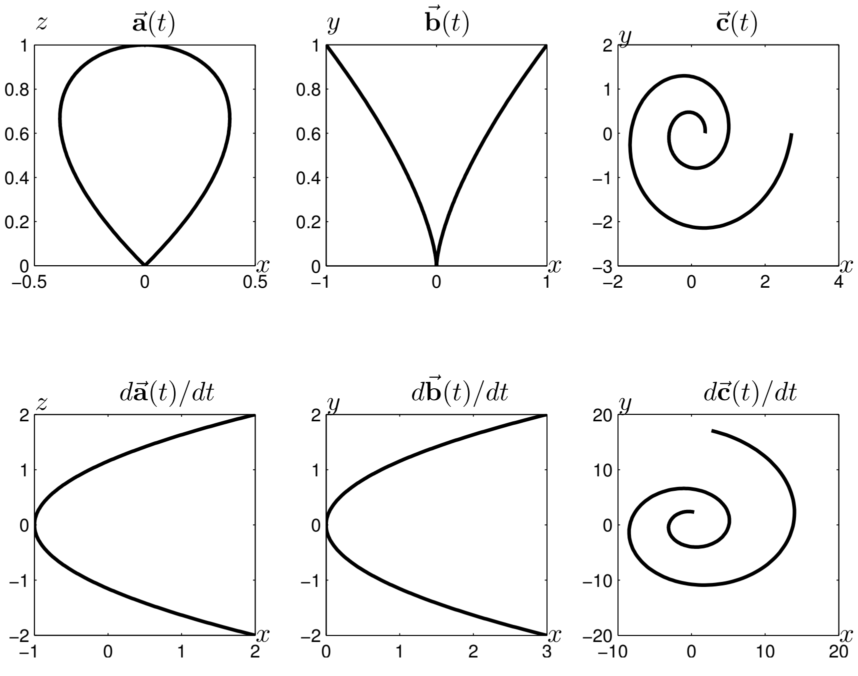


Figure 46: *The images of the three curves and of their total derivatives.*

$$\begin{aligned}
\frac{d\big(f(\vec{\mathbf{b}})\big)}{dt} &= \vec{\nabla} f(\vec{\mathbf{b}}) \cdot \frac{d\vec{\mathbf{b}}}{dt}\\
&= \Big(y^2z^3\hat{\boldsymbol{\imath}} + 2xyz^3\hat{\boldsymbol{\jmath}} + 3xy^2z^2\hat{\mathbf{k}}\Big)\cdot(3t^2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}})\\
&= \Big(t^4\hat{\boldsymbol{\imath}} + 2t^5\hat{\boldsymbol{\jmath}} + 3t^7\hat{\mathbf{k}}\Big)\cdot(3t^2\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}})\\
&= 7t^6,\\
\frac{d\big(\ell(\vec{\mathbf{c}})\big)}{dt} &= \vec{\nabla}\ell(\vec{\mathbf{c}})\cdot\frac{d\vec{\mathbf{c}}}{dt}\\
&= \Big((3x^2+y^2+z^2)\hat{\boldsymbol{\imath}} + (2xy-3y^2)\hat{\boldsymbol{\jmath}} + 2xz\hat{\mathbf{k}}\Big)\\
&\qquad\cdot e^t\Big(\big(\cos(2\pi t) - 2\pi\sin(2\pi t)\big)\hat{\boldsymbol{\imath}} + \big(\sin(2\pi t) + 2\pi\cos(2\pi t)\big)\hat{\boldsymbol{\jmath}}\Big)\\
&= e^{2t}\Big(\big(3\cos^2(2\pi t) + \sin^2(2\pi t)\big)\hat{\boldsymbol{\imath}} + \big(2\cos(2\pi t)\sin(2\pi t) - 3\sin^2(2\pi t)\big)\hat{\boldsymbol{\jmath}}\Big)\\
&\qquad\cdot e^t\Big(\big(\cos(2\pi t) - 2\pi\sin(2\pi t)\big)\hat{\boldsymbol{\imath}} + \big(\sin(2\pi t) + 2\pi\cos(2\pi t)\big)\hat{\boldsymbol{\jmath}}\Big)\\
&= e^{3t}\Big(3\cos(2\pi t) - 3\sin^3(2\pi t) - 2\pi\sin(2\pi t) - 6\pi\sin^2(2\pi t)\cos(2\pi t)\Big).
\end{aligned}$$

B.2 Exercises of Section 2

Solution of Exercise 2.4. We start by computing the total derivative of the curve and its magnitude:

$$\frac{d\vec{\mathbf{a}}_A}{dt} = -\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}, \qquad \left|\frac{d\vec{\mathbf{a}}_A}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}.$$

The value of the field $f_A(\vec{\mathbf{r}}) = x + y + z$ along the curve is

$$f_A\big(\vec{\mathbf{a}}_A(t)\big) = \cos t + \sin t + t.$$

To conclude, we simply apply formula (36) for the line integral of a scalar field:

$$\int_{\Gamma_A} f_A\,\mathrm{d}s = \int_0^{2\pi} f_A\big(\vec{\mathbf{a}}_A(t)\big)\left|\frac{d\vec{\mathbf{a}}_A}{dt}\right|\mathrm{d}t = \int_0^{2\pi}(\cos t + \sin t + t)\sqrt{2}\,\mathrm{d}t = \Big(0 + 0 + \frac{(2\pi)^2}{2}\Big)\sqrt{2} = 2\sqrt{2}\pi^2.$$

Similarly, for the two remaining integrals we have:

$$\left|\frac{d\vec{\mathbf{a}}_B}{dt}\right| = \sqrt{1+t^4}, \qquad \int_{\Gamma_B} f_B\,\mathrm{d}s = \int_0^1 \sqrt{(1+t^4)(t^6+t^2)}\,\mathrm{d}t = \int_0^1 (t^5 + t)\,\mathrm{d}t = \frac{2}{3},$$

$$\left|\frac{d\vec{\mathbf{a}}_C}{dt}\right| = \sqrt{1+t^2} = \sqrt{1+x^2+y^2}, \qquad \int_{\Gamma_C} f_C \,\mathrm{d}s = \int_0^{10} \sqrt{\frac{t^2}{1+t^2}}\sqrt{1+t^2}\,\mathrm{d}t = \int_0^{10} t\,\mathrm{d}t = 50.$$

Solution of Exercise 2.7. (1) We compute the length of the logarithmic spiral using the formula (43):

$$\frac{d\vec{\mathbf{a}}}{dt} = \mathrm{e}^{-t}(-\sin t - \cos t)\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}(\cos t - \sin t)\hat{\boldsymbol{\jmath}},$$
$$\left|\frac{d\vec{\mathbf{a}}}{dt}\right| = \mathrm{e}^{-t}\sqrt{\sin^2 t + \cos^2 t + 2\sin t\cos t + \sin^2 t + \cos^2 t - 2\sin t\cos t} = \sqrt{2}\mathrm{e}^{-t},$$
$$\mathrm{length}(\Gamma_a) \overset{(43)}{=} \int_{\Gamma_a} \mathrm{d}s = \int_0^\infty \left|\frac{d\vec{\mathbf{a}}}{dt}\right| \mathrm{d}t = \int_0^\infty \sqrt{2}\mathrm{e}^{-t}\,\mathrm{d}t = -\sqrt{2}\mathrm{e}^{-t}\Big|_0^\infty = \sqrt{2}.$$

(2) To compute the length of the second spiral we can proceed directly:

$$\frac{d\vec{\mathbf{b}}}{dt} = \frac{-\tau\sin\tau - \cos\tau}{\tau^2}\hat{\boldsymbol{\imath}} + \frac{\tau\cos\tau - \sin\tau}{\tau^2}\hat{\boldsymbol{\jmath}},$$
$$\left|\frac{d\vec{\mathbf{b}}}{dt}\right| = \frac{1}{\tau^2}\sqrt{\tau^2\sin^2\tau + \cos^2\tau + 2\tau\sin\tau\cos\tau + \tau^2\cos^2\tau + \sin^2\tau - 2\tau\sin\tau\cos\tau} = \frac{\sqrt{\tau^2+1}}{\tau^2},$$
$$\mathrm{length}(\Gamma_b) \overset{(43)}{=} \int_1^\infty \left|\frac{d\vec{\mathbf{b}}}{d\tau}\right| \mathrm{d}\tau = \int_1^\infty \frac{\sqrt{\tau^2+1}}{\tau^2}\,\mathrm{d}\tau = \Big(\mathrm{arsinh}\tau - \frac{\sqrt{\tau^2+1}}{\tau}\Big)\Big|_1^\infty = \infty - 1 - \mathrm{arsinh}1 + \sqrt{2} = \infty.$$

However the integral of $\frac{\sqrt{\tau^2+1}}{\tau^2}$, which we computed by parts, is not easy. A simpler way is to note that the integrand satisfies the lower bound $\frac{\sqrt{\tau^2+1}}{\tau^2} \geq \frac{\tau}{\tau^2} = \frac{1}{\tau}$, thus

$$\mathrm{length}(\Gamma_b) = \int_1^\infty \frac{\sqrt{\tau^2+1}}{\tau^2}\,\mathrm{d}\tau \geq \int_1^\infty \frac{1}{\tau}\,\mathrm{d}\tau = \log\tau\Big|_1^\infty = \infty.$$

Thus the length of the path cannot be shorter than ∞.

(3) Both curves are in the form $F(t)(\cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}})$ for some positive function F, so they rotate around the origin in anticlockwise direction with period 2π. Since both curves are defined for an unbounded parametrisation interval (i.e. $t \geq 0$ and $\tau \geq 1$), they rotate infinitely many times: for all values $t, \tau = 2n\pi$ with $n \in \mathbb{N}$ we have $a_2(t) = b_2(\tau) = 0$ and $a_1(t), b_1(\tau) > 0$ so the curves cross the positive x axis; for all values $t, \tau = (2n+1)\pi$ with $n \in \mathbb{N}$ we have $a_2(t) = b_2(\tau) = 0$ and $a_1(t), b_1(\tau) < 0$ so the curves cross the negative x axis. This happens for all $n \in \mathbb{N}$, so infinitely many times.

(4) We first note that ℓ must be at least 1, since no curve connecting $\hat{\boldsymbol{\imath}}$ and $\vec{\mathbf{0}}$ can be shorter than the straight segment, which has length 1. The simplest way to construct $\vec{\mathbf{c}}$ is to slightly modify the curve $\vec{\mathbf{a}}$. To guarantee infinitely many turns around the origin we keep the $\cos t$ and $\sin t$ terms. To modify the speed of convergence to the origin, thus the length of the path, we modify the exponential term as $\mathrm{e}^{-\lambda t}$, using a positive parameter λ. So we define a new curve $\vec{\mathbf{c}}$, depending on λ, and we compute its length:

$$\vec{\mathbf{c}}(t) = \mathrm{e}^{-\lambda t}\cos t\hat{\boldsymbol{\imath}} + \mathrm{e}^{-\lambda t}\sin t\hat{\boldsymbol{\jmath}}, \qquad t \in [0,\infty),$$
$$\frac{d\vec{\mathbf{c}}}{dt} = \mathrm{e}^{-\lambda t}(-\sin t - \lambda\cos t)\hat{\boldsymbol{\imath}} + \mathrm{e}^{-\lambda t}(\cos t - \lambda\sin t)\hat{\boldsymbol{\jmath}},$$
$$\left|\frac{d\vec{\mathbf{c}}}{dt}\right| = \mathrm{e}^{-\lambda t}\sqrt{\sin^2 t + \lambda^2\cos^2 t + 2\lambda\sin t\cos t + \lambda^2\sin^2 t + \cos^2 t - 2\lambda\sin t\cos t} = \sqrt{1+\lambda^2}\mathrm{e}^{-\lambda t},$$
$$\mathrm{length}(\Gamma_c) \overset{(43)}{=} \int_0^\infty \left|\frac{d\vec{\mathbf{c}}}{dt}\right| \mathrm{d}t = \int_0^\infty \sqrt{1+\lambda^2}\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t = -\frac{\sqrt{1+\lambda^2}}{\lambda}\,\mathrm{e}^{-\lambda t}\Big|_0^\infty = \frac{\sqrt{1+\lambda^2}}{\lambda} = \sqrt{1+1/\lambda^2}.$$

Thus to obtain a length ℓ we need to take $\sqrt{1+1/\lambda^2} = \ell$, which gives $\lambda = 1/\sqrt{\ell^2-1}$, which is acceptable for any $\ell > 1$. The desired curve is $\vec{\mathbf{c}}(t) = \mathrm{e}^{-t/\sqrt{\ell^2-1}}\cos t\,\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t/\sqrt{\ell^2-1}}\sin t\,\hat{\boldsymbol{\jmath}}$ for $t \in [0,\infty)$.

Solution of Exercise 2.11. Proceeding as in Example 2.10 we obtain

$$\int_{\Gamma_A} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}}_A = \int_{\Gamma_B} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}}_B = -\int_{\Gamma_C} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}}_C = \int_{\Gamma_D} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_{\Gamma_E} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = \frac{5}{6}.$$

Solution of Exercise 2.12. We first note that the position vector field evaluated on any curve equals the parametrisation of the curve itself: $\vec{\mathbf{F}}(\vec{\mathbf{a}}(t)) = \vec{\mathbf{a}}(t)$, thus $\int_{\Gamma_a} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_a} \vec{\mathbf{a}} \cdot \frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}s$. We report the parametrisation of the curves and their total derivatives, compute their scalar products and integrate:

$$\vec{\mathbf{a}}(t) = \mathrm{e}^{-t}\cos t\,\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}\sin t\,\hat{\boldsymbol{\jmath}}, \qquad t \in [0,\infty), \qquad\qquad \vec{\mathbf{b}}(\tau) = \frac{1}{\tau}\cos\tau\,\hat{\boldsymbol{\imath}} + \frac{1}{\tau}\sin\tau\,\hat{\boldsymbol{\jmath}}, \qquad \tau \in [1,\infty),$$

$$\frac{d\vec{\mathbf{a}}}{dt} = \mathrm{e}^{-t}(-\sin t - \cos t)\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}(\cos t - \sin t)\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{b}}}{dt} = \frac{-\tau\sin\tau - \cos\tau}{\tau^2}\hat{\boldsymbol{\imath}} + \frac{\tau\cos\tau - \sin\tau}{\tau^2}\hat{\boldsymbol{\jmath}},$$

$$\vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{a}}}{dt} = \mathrm{e}^{-2t}(-\sin t\cos t - \cos^2 t + \cos t\sin t - \sin t\sin t) = -\mathrm{e}^{-2t},$$

$$\vec{\mathbf{b}}\cdot\frac{d\vec{\mathbf{b}}}{dt} = \frac{1}{\tau^3}(-\tau\sin\tau\cos\tau - \cos^2\tau + \tau\cos\tau\sin\tau - \sin^2\tau) = -\frac{1}{\tau^3},$$

$$\int_{\Gamma_a}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_a}\vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}s = \int_0^\infty -\mathrm{e}^{-2t}\,\mathrm{d}t = \frac{\mathrm{e}^{-2t}}{2}\Big|_0^\infty = \boxed{-\frac{1}{2}},$$

$$\int_{\Gamma_b}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_a}\vec{\mathbf{b}}\cdot\frac{d\vec{\mathbf{b}}}{d\tau}\,\mathrm{d}s = \int_1^\infty -\frac{1}{\tau^3}\,\mathrm{d}\tau = \frac{1}{2\tau^2}\Big|_1^\infty = \boxed{-\frac{1}{2}}.$$

Here we see a fact that might be surprising: the second spiral Γ_b has infinite length, as we have verified in Exercise (2.7), but the line integral of $\vec{\mathbf{r}}$ along it is finite. This is because, when we approach the origin proceeding along Γ_b, the field $\vec{\mathbf{r}}$ becomes smaller and "more orthogonal" to the curve itself.

We could have guessed from the beginning the signs of the integrals. The position vector points in the radial direction away from the origin. The total derivatives of each curve instead point roughly towards the origin, as the magnitude of the curve points decreases monotonically with t. Thus the scalar products $\vec{\mathbf{F}}(\vec{\mathbf{a}})\cdot\frac{d\vec{\mathbf{a}}}{dt}$ and $\vec{\mathbf{F}}(\vec{\mathbf{b}})\cdot\frac{d\vec{\mathbf{b}}}{d\tau}$ are negative, and the integrals of negative integrands are negative.

Solution of Exercise 2.17. We know the scalar potentials of the two vector fields considered: $\vec{\mathbf{r}} = \vec{\nabla}(\frac{1}{2}|\vec{\mathbf{r}}|^2)$ and $z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{k}} = \vec{\nabla}(xz)$. Applying the fundamental theorem of vector calculus (49) we have:

$$\int_\Gamma \vec{\mathbf{r}}\cdot\mathrm{d}\vec{\mathbf{r}} = \frac{1}{2}|\vec{\mathbf{q}}|^2 - \frac{1}{2}|\vec{\mathbf{p}}|^2 = \frac{1}{2}(3-5) = -1, \qquad \text{and} \qquad \int_\Gamma (z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{k}})\cdot\mathrm{d}\vec{\mathbf{r}} = (-1)(-1) - (0)(2) = 1.$$

Solution of Exercise 2.20. We first need to find a parametrisation of the two paths[41]:

$$\vec{\mathbf{a}}(t) = (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}, \qquad 0\le t\le 1, \qquad\qquad \vec{\mathbf{b}}(\tau) = \cos\tau\,\hat{\boldsymbol{\imath}} + \sin\tau\,\hat{\boldsymbol{\jmath}}, \qquad 0\le\tau\le\pi/2.$$

The corresponding total derivatives are

$$\frac{d\vec{\mathbf{a}}(t)}{dt} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{b}}(\tau)}{d\tau} = -\sin\tau\,\hat{\boldsymbol{\imath}} + \cos\tau\,\hat{\boldsymbol{\jmath}}.$$

The line integrals are computed using the definition (44):

$$\begin{aligned}\int_{\Gamma_a}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} &= \int_0^1 \vec{\mathbf{F}}\big(a_1(t),a_2(t)\big)\cdot\frac{d\vec{\mathbf{a}}(t)}{dt}\,\mathrm{d}t = \int_0^1 \big(a_1^2(t) + a_2^2(t)\big)(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\cdot(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t\\ &= \int_0^1 \big(1 + t^2 - 2t + t^2\big)(-1+2)\,\mathrm{d}t = 1 + \frac{1}{3} - 1 + \frac{1}{3} = \frac{2}{3},\\ \int_{\Gamma_b}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} &= \int_0^{\pi/2}\vec{\mathbf{F}}\big(b_1(\tau),b_2(\tau)\big)\cdot\frac{d\vec{\mathbf{b}}(\tau)}{d\tau}\,\mathrm{d}\tau = \int_0^{\pi/2}(\cos^2\tau + \sin^2\tau)(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\cdot(-\sin\tau\,\hat{\boldsymbol{\imath}} + \cos\tau\,\hat{\boldsymbol{\jmath}})\,\mathrm{d}\tau\\ &= \int_0^{\pi/2}(-\sin\tau + 2\cos\tau)\,\mathrm{d}\tau\\ &= \cos\frac{\pi}{2} - \cos 0 + 2\sin\frac{\pi}{2} - 2\sin 0 = 0 - 1 + 2 - 0 = 1.\end{aligned}$$

Since the values of the line integrals of $\vec{\mathbf{F}}$ over two paths connecting the same endpoints are different from one another ($\frac{2}{3}\ne 1$), by Theorem 2.19 (or by the equivalence in box (50)) the field $\vec{\mathbf{F}}$ is not conservative.

Solution of Exercise 2.23.

$$\iint_R \mathrm{e}^{3y}\sin x\,\mathrm{d}x\,\mathrm{d}y = \Big(\int_0^\pi \sin x\,\mathrm{d}x\Big)\Big(\int_1^2 \mathrm{e}^{3y}\,\mathrm{d}y\Big) = \Big(-\cos x\Big|_0^\pi\Big)\Big(\frac{\mathrm{e}^{3y}}{3}\Big|_1^2\Big) = \frac{2}{3}(\mathrm{e}^6 - \mathrm{e}^3),$$

[41] How did we find these parametrisations? Many choices are possible, we look for the simplest one; see also Remark 1.24.
We first look for the first component $a_1(t)$ of $\vec{\mathbf{a}}(t)$. If we choose to parametrise Γ_a with the unit interval $[0,1]$, a_1 must be a scalar function such that $a_1(0) = 1$ and $a_1(1) = 0$: the easiest possible choice is $a_1(t) = 1 - t$. Since by definition of Γ_a we have $a_1(t) + a_2(t) = 1$, we have $1 - t + a_2(t) = 1$ and thus $a_2(t) = t$.
For the second path, we can proceed similarly (see Example 2.3), but we would obtain a complicated parametrisation. Instead, we note that we need two scalar functions b_1 and b_2 such that $b_1^2(\tau) + b_2^2(\tau) = 1$, so the most natural choice is $b_1(\tau) = \cos\tau$ and $b_2(\tau) = \sin\tau$. For this choice, we need to fix the parametrisation endpoints to be $\tau_I = 0$ and $\tau_F = \pi/2$, so that $\vec{\mathbf{b}}(\tau_I) = \hat{\boldsymbol{\imath}}$ and $\vec{\mathbf{b}}(\tau_F) = \hat{\boldsymbol{\jmath}}$.

$$\iint_Q y \,\mathrm{d}x\,\mathrm{d}y = \int_0^5 \left(\int_{\frac{x}{5}}^1 y\,\mathrm{d}y\right)\mathrm{d}x = \int_0^5 \left(\frac{1}{2} - \frac{x^2}{50}\right)\mathrm{d}x = \frac{5}{2} - \frac{125}{150} = \frac{5}{3},$$
$$\iint_S \cos x \,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \cos x\left(\int_0^{\sin x} \mathrm{d}y\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos x \sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2x)\,\mathrm{d}x = \frac{-\cos(2x)}{4}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}.$$

Solution of Exercise 2.29. Proceeding as in Example 2.28, we have to compute the Jacobian determinant $\frac{\partial(\xi_{LR},\eta_{LR})}{\partial(x,y)}$ and integrate it over the unit square $S = \{0 < x < 1, 0 < y < 1\}$:

$$J\vec{\mathbf{T}}_{LR} = \begin{pmatrix} y\mathrm{e}^{xy}\cos(2\pi y) & x\mathrm{e}^{xy}\cos(2\pi y) - 2\pi\mathrm{e}^{xy}\sin(2\pi y) \\ y\mathrm{e}^{xy}\sin(2\pi y) & x\mathrm{e}^{xy}\sin(2\pi y) + 2\pi\mathrm{e}^{xy}\cos(2\pi y)\end{pmatrix}, \qquad \frac{\partial(\xi_{LR},\eta_{LR})}{\partial(x,y)} = 2\pi y\mathrm{e}^{2xy},$$
$$\mathrm{Area}(R_{LR}) = 2\pi\int_0^1\int_0^1 y\mathrm{e}^{2xy}\,\mathrm{d}x\,\mathrm{d}y = \pi\int_0^1 \mathrm{e}^{2xy}\Big|_{x=0}^1 \mathrm{d}y = \pi\int_0^1 (\mathrm{e}^{2y}-1)\,\mathrm{d}y = \pi\Big(\frac{\mathrm{e}^{2y}}{2} - y\Big)\Big|_{y=0}^1 = \frac{\pi(\mathrm{e}^2-3)}{2}.$$

Solution of Exercise 2.30. *(i)* The domain is defined implicitly by the change of coordinates $Q \ni x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \mapsto \xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}} \in (0,1)^2$. We compute the Jacobian determinant and the integral:

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \det\begin{pmatrix}4-\eta & -\xi \\ -\eta & \pi-\xi\end{pmatrix} = (4-\eta)(\pi-\xi) - \xi\eta = 4\pi - 4\xi - \pi\eta;$$
$$\mathrm{Area}(Q) = \iint_Q 1\,\mathrm{d}A = \iint_{(0,1)^2}\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1\int_0^1 |4\pi - 4\xi - \pi\eta|\,\mathrm{d}\xi\,\mathrm{d}\eta = 4\pi - \frac{4}{2} - \frac{\pi}{2} = \frac{7}{2}\pi - 2.$$

You can visualise this domain in Matlab/Octave using the function `VCplotter.m` (available on the course web page), with the command:

```
VCplotter(6, @(x,y) x*(4-y), @(x,y) y*(pi-x),0,1,0,1);
```

(ii) We repeat the same computations done in part *(i)* with the general parameters:

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \det\begin{pmatrix}a-\eta & -\xi \\ -\eta & b-\xi\end{pmatrix} = (a-\eta)(b-\xi) - \xi\eta = ab - a\xi - b\eta;$$
$$\mathrm{Area}(Q_{ab}) = \iint_{Q_{ab}} 1\,\mathrm{d}A = \iint_{(0,1)^2}\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1\int_0^1 |ab - a\xi - b\eta|\,\mathrm{d}\xi\,\mathrm{d}\eta = ab - \frac{a+b}{2}.$$

Alternative solution: Q_{ab} is the quadrilateral with vertices $\vec{\mathbf{0}}, a\hat{\boldsymbol{\imath}}, (a-1)\hat{\boldsymbol{\imath}} + (b-1)\hat{\boldsymbol{\jmath}}, b\hat{\boldsymbol{\jmath}}$ and straight edges, thus computing its area by decomposing it in the rectangle $\vec{\mathbf{0}}, (a-1)\hat{\boldsymbol{\imath}}, (a-1)\hat{\boldsymbol{\imath}} + (b-1)\hat{\boldsymbol{\jmath}}, (b-1)\hat{\boldsymbol{\jmath}}$, the triangle $(a-1)\hat{\boldsymbol{\imath}}, a\hat{\boldsymbol{\imath}}, (a-1)\hat{\boldsymbol{\imath}} + (b-1)\hat{\boldsymbol{\jmath}}$ and the triangle $(b-1)\hat{\boldsymbol{\jmath}}, b\hat{\boldsymbol{\jmath}}, (a-1)\hat{\boldsymbol{\imath}} + (b-1)\hat{\boldsymbol{\jmath}}$ (so that the area is $(a-1)(b-1) + \frac{1}{2}(a-1+b-1) = ab - \frac{1}{2}(a+b)$) is also correct.

Solution of Exercise 2.31. We first compute the inverse change of variables, solving (60) for x and y: taking the sum and the difference of the two equations we have

$$\begin{cases}\xi + \eta = 2y - 1, \\ \xi - \eta = \frac{1}{2}x - 1,\end{cases} \qquad \Rightarrow \qquad \begin{cases} x = 2(\xi - \eta + 1), \\ y = \frac{\xi+\eta+1}{2}.\end{cases} \tag{108}$$

The Jacobian determinant we will need in the integral is

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \overset{(56)}{=} \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} = \frac{1}{4}1 - 1\Big(-\frac{1}{4}\Big) = \frac{1}{2}.$$

We call $\vec{\mathbf{T}}$ the change of variables and R the counterimage of P, i.e.:

$$\vec{\mathbf{T}} : R \to P, \qquad \vec{\mathbf{T}}(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) = \xi(x,y)\hat{\boldsymbol{\xi}} + \eta(x,y)\hat{\boldsymbol{\eta}}.$$

The domain P is a square, so we want to identify the four parts of the boundary of R corresponding to the four edges[42] of P. For this purpose, it is enough to insert the equations of the edges of P either in (60) or in (108):

$$\begin{aligned}
\text{line connecting } \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}: &\quad \xi + \eta = 1, \quad y = 1,\\
\text{line connecting } \hat{\boldsymbol{\eta}}, -\hat{\boldsymbol{\xi}}: &\quad \xi - \eta = -1, \quad x = 0,\\
\text{line connecting } -\hat{\boldsymbol{\xi}}, -\hat{\boldsymbol{\eta}}: &\quad \xi + \eta = -1, \quad y = 0,\\
\text{line connecting } -\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\xi}}: &\quad \xi - \eta = 1, \quad x = 4,
\end{aligned}$$

[42] If you don't remember how to compute the equations of the straight lines defining the edges of P, draw the domain and use four times the formula for the line through two points $\vec{\mathbf{p}} = \xi_p\hat{\boldsymbol{\xi}} + \eta_p\hat{\boldsymbol{\eta}}$ and $\vec{\mathbf{q}} = \xi_q\hat{\boldsymbol{\xi}} + \eta_q\hat{\boldsymbol{\eta}}$: $\frac{\eta-\eta_p}{\eta_q-\eta_p} = \frac{\xi-\xi_p}{\xi_q-\xi_p}$. You have to repeat this for each of the four edges. In this exercise all the coefficients are either 1, 0 or −1. See also Figure 47.

so all the four lines are parallel to the axes in the xy-plane and the domain R is the rectangle $R = (0,4)\times(0,1) = \{x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath}\in\mathbb{R}^2,\ 0<x<4, 0<y<1\}$. Now we can compute the desired integral using the change of variables formula (57):

$$\iint_P f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \overset{(57)}{=} \iint_R f\Big(\frac{x}{4}+y-1,\ y-\frac{x}{4}\Big)\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|\mathrm{d}x\,\mathrm{d}y$$
$$= \iint_R \mathrm{e}^{-\frac{x}{4}-y+1-y+\frac{x}{4}}\,\frac{1}{2}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1\Big(\int_0^4 \mathrm{e}^{-2y+1}\frac{1}{2}\,\mathrm{d}x\Big)\,\mathrm{d}y = \int_0^1 2\mathrm{e}^{-2y+1}\,\mathrm{d}y = 2\frac{\mathrm{e}^{-2y+1}}{-2}\Big|_0^1 = e-\mathrm{e}^{-1}\approx 2.35.$$

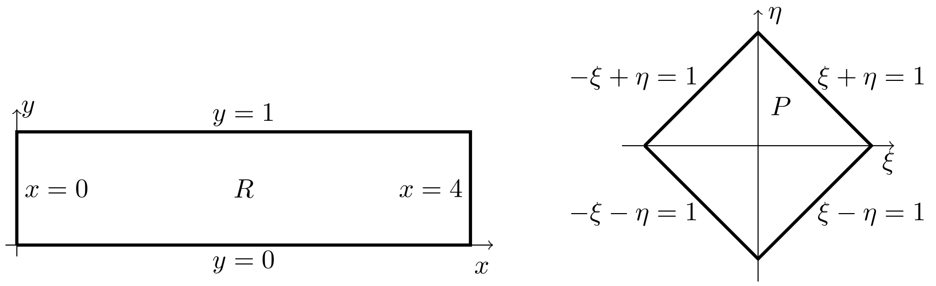


Figure 47: *The rectangle R and the square P in Exercise 2.31.*

Solution of Exercise 2.36. *(i)* We use the following change of variables (we write also the inverse):[43]

$$(I)\ \begin{cases}\xi = xy,\\ \eta = y,\end{cases}\qquad \begin{pmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{pmatrix} = \begin{pmatrix} y & x\\ 0 & 1\end{pmatrix},\qquad \frac{\partial(\xi,\eta)}{\partial(x,y)} = y,\qquad \begin{cases}x=\frac{\xi}{\eta},\\ y=\eta,\end{cases}\qquad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{\eta}.$$

(ii) The integral is computed as:

$$\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta \overset{(57)}{=} \iint_S \frac{\xi(x,y)}{\eta(x,y)}\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|\mathrm{d}x\,\mathrm{d}y = \iint_S \frac{xy}{y}\,y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 x\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y = \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

[43]How did we find this transformation? First we decide how to change the domain geometrically: since we are transforming a square into a triangle, have to "get rid of" one of the sides of S, we *choose* to maintain the upper and the left side, collapse the lower one and transform the right one into the diagonal (see below for other options). We then note that we can maintain the vertical coordinate of every point and modify only the horizontal one, so we fix $\eta = y$. We want to modify x by multiplying with a suitable function, so $\xi = xg(x,y)$. The function g must have value 1 on the horizontal line $y=1$ (so the upper side is preserved), and value 0 on the horizontal line $y=0$ (so the lower side is collapsed). The choice $g(x,y)=y$ is the easiest that satisfies these two conditions ($g(x,1)=1, g(x,0)=0$), so the candidate transformation is $\xi = xy$, $\eta = y$.

Then, we can immediately verify that, under this transformation, the upper side of S is preserved ($y=1\Rightarrow\eta=1$), the left side is preserved ($x=0\Rightarrow\xi=0$), the right side is mapped into the diagonal side of T ($x=1\Rightarrow\xi=\eta$), the lower side is collapsed to the origin ($y=0\Rightarrow\xi=\eta=0$). Thus the transformation maps correctly the boundaries: $\partial S\to\partial T$. It is surjective (for all $\xi\hat{\boldsymbol\xi}+\eta\hat{\boldsymbol\eta}\in T$, $\xi/\eta\hat{\boldsymbol\imath}+\eta\hat{\boldsymbol\jmath}$ belongs to S), and it is injective in the *interior* of S:

$$\vec{\mathbf{T}}(x_A,y_A) = x_Ay_A\hat{\boldsymbol\xi}+y_A\hat{\boldsymbol\eta} = x_By_B\hat{\boldsymbol\xi}+y_B\hat{\boldsymbol\eta} = \vec{\mathbf{T}}(x_B,y_B)\quad\Rightarrow\quad \begin{matrix}x_Ay_A = x_By_B,\\ y_A=y_B,\end{matrix}\quad \overset{y_A,y_B>0}{\Rightarrow}\quad \begin{matrix}x_A=x_B,\\ y_A=y_B.\end{matrix}$$

(The verification of the mapping of the boundary and the bijectivity was not required in the exercise, however it is what you should do to check if your transformation is correct.)

Another smart way to obtain and verify the transformation, is to rewrite the domains as

$$T = \{0<\xi<\eta<1\} = \{0<\xi/\eta<1,\ 0<\eta<1\},\qquad S=\{0<x<1,\ 0<y<1\},$$

from which the inverse transformation $x=\frac{\xi}{\eta}$, $y=\eta$ is immediate (this corresponds to the setting of Example 2.34).

Many other changes of variables are possible, some simple ones are

$$(II)\ \begin{cases}\xi = x,\\ \eta = x+y-xy,\end{cases}\qquad (III)\ \begin{cases}\begin{cases}\xi=\frac{1}{2}x,\\ \eta = y+\frac{1}{2}x,\end{cases} & \text{for } x+y\le 1,\\ \begin{cases}\xi = x+\frac{1}{2}(y-1),\\ \eta=\frac{1}{2}(y+1),\end{cases} & \text{for } x+y>1.\end{cases}$$

The transformation *(II)* corresponds to that described in Example 2.33 for general y-simple domain; *(III)* is "piecewise affine" (and continuous). See a representation of the three transformations in Figure 48; you can play around with these transformations using `VCplotter.m`. Note that the transformation of the unit cube into a tetrahedron in equation (64) is very similar to this exercise.

We can double check the computation above using the iterated integral on T:

$$\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1 \int_0^\eta \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1 \frac{\eta^2}{2\eta}\,\mathrm{d}\eta = \frac{1}{4}.$$

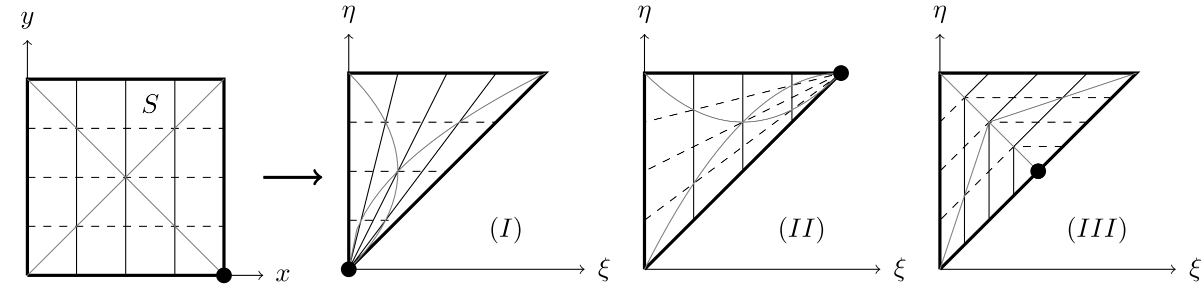


Figure 48: *The square S and its image T under the three transformation described in exercise **(1)**; many other options are possible. In each case, the point $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ are mapped into $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\eta}}$, respectively, while $\hat{\boldsymbol{\imath}}$ is mapped into the point denoted with the little circle. The continuous, dashed and grey lines represent the images of the vertical, horizontal and diagonal segments in the first plot. Note that the first two transformations (which are polynomials of degree two) map some straight lines into parabolas, while the third one (which is piecewise affine) maps straight lines into polygonal lines.*

Solution of Exercise 2.42. We only need to show that for all $u, w \in \mathbb{R}$ we have $\vec{\mathbf{X}}(u, w) \in S$, i.e. $X_1^2(u, w) + X_2^2(u, w) + X_3^2(u, w) = 1$ and $X_3(u, w) \neq 1$. We compute the magnitude of $\vec{\mathbf{X}}(u, w)$:

$$|\vec{\mathbf{X}}(u,w)|^2 = \frac{u^2 + w^2 + \left(\frac{u^2}{4} + \frac{w^2}{4} - 1\right)^2}{\left(1 + \frac{u^2}{4} + \frac{w^2}{4}\right)^2} = \frac{u^2 + w^2 + \frac{u^4}{16} + \frac{w^4}{16} + 1 - \frac{u^2}{2} - \frac{w^2}{2} + \frac{u^2w^2}{8}}{1 + \frac{u^4}{16} + \frac{w^4}{16} + \frac{u^2}{2} + \frac{w^2}{2} + \frac{u^2w^2}{8}} = 1,$$

so $\vec{\mathbf{X}}(u, w)$ belongs to the unit sphere. Since $X_3(u, w) = 1$ reads $\frac{u^2}{4} + \frac{w^2}{4} - 1 = 1 + \frac{u^2}{4} + \frac{w^2}{4}$, which has no solutions, no point is mapped to the north pole and we conclude that $\vec{\mathbf{X}}$ maps into S.

To compute the inverse of $\vec{\mathbf{X}}$, we have to solve $\vec{\mathbf{X}}(u, w) = \vec{\mathbf{r}}$ for u and w. Using the relation $x^2 + y^2 + z^2 = 1$ between the components of a point in S, after some computations we find $u = 2x/(1 - z)$ and $w = 2y/(1 - z)$.

Solution of Exercise 2.43. The surface is a torus, i.e. a "mathematical doughnut". It can be described as the set of points with distance b from a circle of radius a centred at the origin and lying in the xy-plane. You can see it in the left plot of Figure 49 for $a = 3$ and $b = 1$. To be more precise, S is the torus after the inner circle (corresponding to $w = \pm\pi$) and the circle in the $\{y = 0, x < 0\}$ half plane (corresponding to $u = \pm\pi$) have been deleted. The two deleted circles are highlighted in the figure.

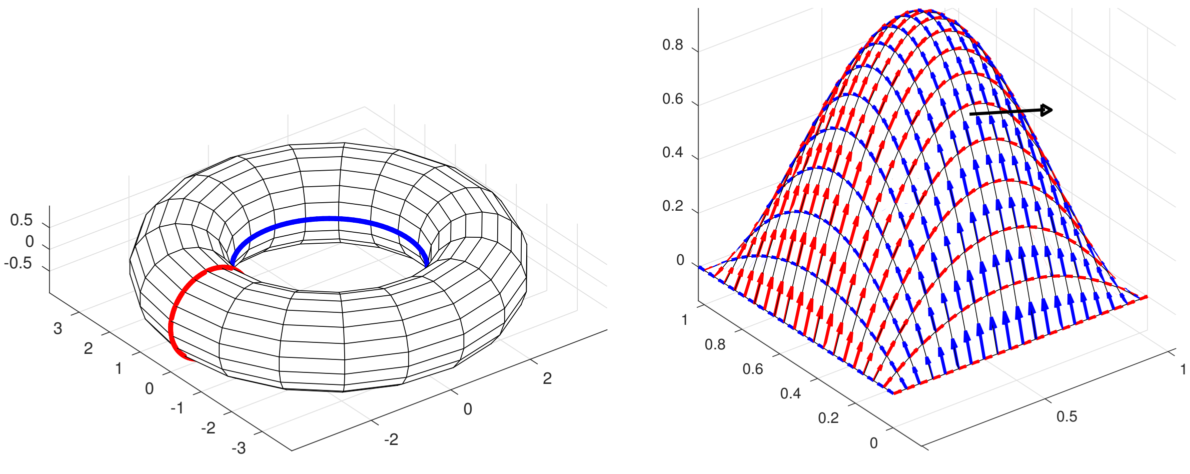


Figure 49: ***Left:*** *The torus described by the parametrisation in Exercise 2.43 with $a = 3$ and $b = 1$.* ***Right:*** *The "hump" graph surface of Exercise 2.45. The little coloured arrows represent the (scaled) tangent vectors. The black arrow is the (scaled) unit normal vector at $1/2\hat{\boldsymbol{\imath}} + 1/4\hat{\boldsymbol{\jmath}} + 3/4\hat{\boldsymbol{k}}$.*

Solution of Exercise 2.45. We have $0 \le g \le 1$ in R with $g(1/2, 1/2) = 1$ and $g = 0$ on the boundary of R, i.e. where either x or y are equal to 0 or 1. So its graph S_g looks like a "hump", see the right plot of Figure 49.

The gradient of g is $\vec{\nabla} g = 16(1-2x)y(1-y)\hat{\boldsymbol{\imath}} + 16x(1-x)(1-2y)\hat{\boldsymbol{\jmath}}$. We can see S_g as a parametric curve with chart $\vec{\mathbf{X}}_g = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + g(x,y)\hat{\boldsymbol{k}}$ defined on R, so $\frac{\partial \vec{\mathbf{X}}_g}{\partial x} = \hat{\boldsymbol{\imath}} + \frac{\partial g}{\partial x}\hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} + 16(1-2x)y(1-y)\hat{\boldsymbol{k}}$ and $\frac{\partial \vec{\mathbf{X}}_g}{\partial y} = \hat{\boldsymbol{\jmath}} + \frac{\partial g}{\partial y}\hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} + 16x(1-x)(1-2y)\hat{\boldsymbol{k}}$. These are tangent vector fields to S_g.

Solution of Exercise 2.49.

$$\frac{\partial \vec{\mathbf{X}}}{\partial u} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}, \qquad \frac{\partial \vec{\mathbf{X}}}{\partial w} = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}, \qquad \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}, \qquad \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| = \sqrt{6},$$

$$\text{Area}(S) = \iint_S \mathrm{d}S = \iint_R \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| \mathrm{d}A = \int_0^1 \int_0^1 \sqrt{6}\,\mathrm{d}u\,\mathrm{d}w = \sqrt{6}.$$

Since the chart is affine, S is the flat parallelogram with vertices $\vec{\mathbf{X}}(0,0) = \vec{\mathbf{0}}$, $\vec{\mathbf{X}}(1,0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$, $\vec{\mathbf{X}}(1,1) = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$, $\vec{\mathbf{X}}(0,1) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$. Moreover, since two sides of this are perpendicular (i.e. $(\vec{\mathbf{X}}(1,0) - \vec{\mathbf{X}}(0,0)) \cdot (\vec{\mathbf{X}}(0,1) - \vec{\mathbf{X}}(0,0)) = 0$), this is a rectangle. The sides of this rectangle have length $|\vec{\mathbf{X}}(1,0) - \vec{\mathbf{X}}(0,0)| = \sqrt{3}$ and $|\vec{\mathbf{X}}(0,1) - \vec{\mathbf{X}}(0,0)| = \sqrt{2}$, thus we have verified geometrically that the area of S is $\sqrt{6}$.

Solution of Exercise 2.50. We simply integrate the field $f = 1$ over S using formula (69). Here the region R is the unit square $(0,1) \times (0,1)$.

$$\begin{aligned}
\text{Area}(S) = \iint_S 1\,\mathrm{d}S &= \iint_{(0,1)\times(0,1)} \sqrt{1 + \Big(\frac{\partial(\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}y^{\frac{3}{2}})}{\partial x}\Big)^2 + \Big(\frac{\partial(\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}y^{\frac{3}{2}})}{\partial y}\Big)^2}\,\mathrm{d}A \\
&= \int_0^1 \int_0^1 \sqrt{1 + x + y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \frac{2}{3}(1 + x + y)^{\frac{3}{2}}\Big|_{y=0}^1 \mathrm{d}x \\
&= \frac{2}{3}\int_0^1 \Big((2+x)^{\frac{3}{2}} - (1+x)^{\frac{3}{2}}\Big)\,\mathrm{d}x \\
&= \frac{2}{3}\frac{2}{5}\Big((2+x)^{\frac{5}{2}} - (1+x)^{\frac{5}{2}}\Big)\Big|_{x=0}^1 \\
&= \frac{4}{15}\Big(3^{\frac{5}{2}} - 2^{\frac{5}{2}} - 2^{\frac{5}{2}} + 1\Big) = \frac{4}{15}\Big(\sqrt{243} - 2\sqrt{32} + 1\Big) \approx 1.407.
\end{aligned}$$

Solution of Exercise 2.51. *(i)* This surface is not a graph, so we cannot use formula (68) but we need to use the more general (67). We compute the area element, by taking the partial derivatives of the chart $\vec{\mathbf{X}}$, using the definition of the vector product $\times$, the definition of vector magnitude $|\cdot|$, and the trigonometric identity $\sin^2 u + \cos^2 = 1$:

$$\begin{aligned}
\vec{\mathbf{X}}(u,w) &= (a + b\cos w)\cos u\hat{\boldsymbol{\imath}} + (a + b\cos w)\sin u\hat{\boldsymbol{\jmath}} + b\sin w\hat{\boldsymbol{k}} \qquad -\pi < u < \pi, \quad -\pi < w < \pi, \\
\frac{\partial \vec{\mathbf{X}}}{\partial u} &= -(a + b\cos w)\sin u\hat{\boldsymbol{\imath}} + (a + b\cos w)\cos u\hat{\boldsymbol{\jmath}}, \\
\frac{\partial \vec{\mathbf{X}}}{\partial w} &= -b\sin w\cos u\hat{\boldsymbol{\imath}} - b\sin w\sin u\hat{\boldsymbol{\jmath}} + b\cos w\hat{\boldsymbol{k}}, \\
\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} &= b(a + b\cos w)\cos u\cos w\hat{\boldsymbol{\imath}} + b(a + b\cos w)\sin u\cos w\hat{\boldsymbol{\jmath}} + b(a + b\cos w)\sin w\hat{\boldsymbol{k}}, \\
\left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| &= b|a + b\cos w|(\cos^2 u\cos^2 w + \sin^2 u\cos^w + \sin^2 w)^{1/2} = b|a + b\cos w|.
\end{aligned}$$

Since $0 < b < a$, we have $a + b\cos w > 0$ for all values of w, so we can write $\left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| = b(a + b\cos w)$. The area of S is simply the surface integral of the constant field 1:

$$\text{Area}(S) = \iint_S \mathrm{d}S = \iint_R \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| \mathrm{d}A$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b(a+b\cos w)\,\mathrm{d}u\,\mathrm{d}w = \int_{-\pi}^{\pi}\mathrm{d}u\,\int_{-\pi}^{\pi} b(a+b\cos w)\,\mathrm{d}w = (2\pi)(2\pi ab) = 4\pi^2 ab.$$

Choosing $a = 2$ and $b = 1$ we have that the area is $8\pi^2$.

(ii) To compute the integral of the scalar field $f(\vec{\mathbf{r}}) = (x^2+y^2)^{-1/2}$ we need to evaluate f on S and write it in terms of u and w. We use the chart $\vec{\mathbf{X}}$:

$$f\big(\vec{\mathbf{X}}(u,w)\big) = \big((a+b\cos w)^2\cos^2 u + (a+b\cos w)^2\sin^2 u\big)^{-1/2} = |a+b\cos w|^{-1} = (a+b\cos w)^{-1},$$

$$\iint_S f\,\mathrm{d}S \overset{(67)}{=} \iint_R f(\vec{\mathbf{X}})\left|\frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w}\right|\mathrm{d}A = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(a+b\cos w)^{-1}\,b(a+b\cos w)\,\mathrm{d}u\,\mathrm{d}w = 4\pi^2 b.$$

<u>Solution of Exercise 2.55.</u> We use formula (74) together with the expression of $\vec{\mathbf{F}}$ and $g(x,y) = x^2-y^2$:

$$\iint_S \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} = \iint_{(0,1)\times(0,1)}\bigg(-\underbrace{0}_{=F_1}\frac{\partial(x^2-y^2)}{\partial x} - \underbrace{z(x,y)}_{=F_2}\frac{\partial(x^2-y^2)}{\partial y} + \underbrace{(-y)}_{=F_3}\bigg)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1\int_0^1\big((y^2-x^2)(-2y)-y\big)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\Big(-2y^3+\frac{2}{3}y-y\Big)\,\mathrm{d}y = -\frac{2}{4}+\frac{1}{3}-\frac{1}{2} = -\frac{2}{3}.$$

<u>Solution of Exercise 2.56.</u> Recall that in Exercise 2.45 we computed $\vec{\nabla}g = 16(1-2x)y(1-y)\hat{\boldsymbol{\imath}} + 16x(1-x)(1-2y)\hat{\boldsymbol{\jmath}}$. The fluxes are

$$\iint_{S_g}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} \overset{(74)}{=} \iint_R\Big(-\frac{\partial g}{\partial x}+0+0\Big)\,\mathrm{d}A = \int_0^1\int_0^1 16(2x-1)y(1-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= 16\int_0^1(2x-1)\,\mathrm{d}x\int_0^1(y-y^2)\,\mathrm{d}x\,\mathrm{d}y = 16(0)\frac{1}{6} = 0,$$

$$\iint_{S_g}\vec{\mathbf{G}}\cdot\mathrm{d}\vec{\mathbf{S}} \overset{(74)}{=} \iint_R\Big(-x\frac{\partial g}{\partial x}+0+0\Big)\,\mathrm{d}A = \int_0^1\int_0^1 16x(2x-1)y(1-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= 16\int_0^1(2x^2-x)\,\mathrm{d}x\int_0^1(y-y^2)\,\mathrm{d}x\,\mathrm{d}y = 16\frac{1}{6}\frac{1}{6} = \frac{4}{9}.$$

To guess the signs of the fluxes we observe the shape of S_g in the right plot of Figure 49. The flux of $\vec{\mathbf{F}}$ can be understood as the "amount of vector field" passing through the surface and is defined as the integral of the scalar product between $\vec{\mathbf{F}}$ and the unit normal vector $\hat{\boldsymbol{n}}$, which points upwards as in (71).

The first field $\vec{\mathbf{F}} = \hat{\boldsymbol{\imath}}$ is constant (i.e. has the same value everywhere) and points in the x direction, from the plot we see that $\hat{\boldsymbol{n}}$ has positive x component only for $x > 1/2$ so $\vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}$ is positive for $x > 1/2$ and negative for $x < 1/2$. Since S_g is symmetric with respect to the vertical plane $\{x = 1/2\}$, so it is $\hat{\boldsymbol{n}}$, and the contributions to the flux from $\{\vec{\mathbf{r}}\in S_g, x < 1/2\}$ and $\{\vec{\mathbf{r}}\in S_g, x > 1/2\}$ have the same absolute value and cancel each other. So the total flux is zero.

The second field $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}}$ points again in the x direction but is not constant. So the contribution to the flux from $\{\vec{\mathbf{r}}\in S_g, x < 1/2\}$ is again negative (because here $\vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}} < 0$) and that from $\{\vec{\mathbf{r}}\in S_g, x > 1/2\}$ is positive. But now $\vec{\mathbf{F}}$ is larger in this second portion of S_g, so the positive term is larger in absolute value than the negative one and the sum of the two terms is then positive.

What do you obtain if you make a similar reasoning for $\vec{\mathbf{H}} = y\hat{\boldsymbol{\imath}}$?

<u>Solution of Exercise 2.57.</u> We recall that we have:

$$\vec{\mathbf{X}} = (a+b\cos w)\cos u\,\hat{\boldsymbol{\imath}} + (a+b\cos w)\sin u\,\hat{\boldsymbol{\jmath}} + b\sin w\,\hat{\boldsymbol{k}} \qquad -\pi < u < \pi, \quad -\pi < w < \pi,$$

$$\frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w} = b(a+b\cos w)\cos u\cos w\,\hat{\boldsymbol{\imath}} + b(a+b\cos w)\sin u\cos w\,\hat{\boldsymbol{\jmath}} + b(a+b\cos w)\sin w\,\hat{\boldsymbol{k}}.$$

The position vector has value equal to its argument, so $\vec{\mathbf{r}}(\vec{\mathbf{X}}(u,w)) = \vec{\mathbf{X}}(u,w)$. The flux is

$$\iint_S\vec{\mathbf{r}}\cdot\mathrm{d}\vec{\mathbf{S}} \overset{(73)}{=} \iint_R\vec{\mathbf{r}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w}\,\mathrm{d}A = \iint_R\vec{\mathbf{X}}\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w}\,\mathrm{d}A$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(b(a+b\cos w)^2(\cos^2 u\cos w+\sin^2 u\cos w) + b^2(a+b\cos w)\sin^2 w\Big)\,\mathrm{d}u\,\mathrm{d}w$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b(a+b\cos w)\Big((a+b\cos w)\cos w + b\sin^2 w\Big)\,\mathrm{d}u\,\mathrm{d}w$$

$$= \int_{-\pi}^{\pi} \mathrm{d}u \int_{-\pi}^{\pi} b(a + b\cos w)(a\cos w + b)\,\mathrm{d}w$$
$$= 2\pi \int_{-\pi}^{\pi} \Big(ab^2 + (a^2 b + b^3)\cos w + ab^2\cos^2 w\Big)\,\mathrm{d}w = 2\pi\Big(2\pi ab^2 + \pi ab^2\Big) = 6\pi^2 ab^2.$$

Solution of Exercise 2.65. The ellipsoid $E = \{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } x^2 + y^2 + \frac{z^2}{c^2} < 1\}$, in cylindrical coordinates reads

$$E = \left\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } r^2 < 1 - \frac{z^2}{c^2}\right\} = \left\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ s.t. } r < \sqrt{1 - \frac{z^2}{c^2}}\right\}.$$

Thus, it is a solid of revolution as in equation (83), with $-c < z < c$ (the admissible interval for z corresponds to the largest interval which guarantees $1 - \frac{z^2}{c^2} \geq 0$). From formula (84),

$$\mathrm{Vol}(E) = \iiint_E \mathrm{d}V = \pi \int_{-c}^{c} \left(1 - \frac{z^2}{c^2}\right) \mathrm{d}z = \pi\left(z - \frac{z^3}{3c^2}\right)\Big|_{-c}^{c} = \frac{4}{3}\pi c.$$

Note that this result agrees with the formula for the volume of the sphere when $c = 1$.

Solution of Exercise 2.66. Using formula (84) it is easy to find $\mathrm{Vol}(B) = \frac{16}{15}\pi$ and $\mathrm{Vol}(F) = \frac{\pi}{2}$.

Solution of Exercise 2.67. Using the computations already done in Example 2.62 we have:

$$\mathrm{Vol}(D) = \iiint_D \mathrm{d}V = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \int_0^{(\cos z)(2+\sin 3\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos z)^2\,\mathrm{d}z \int_{-\pi}^{\pi} (2 + \sin 3\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\,\frac{\pi}{2}\,9\pi = \frac{9\pi^2}{4}.$$

You can plot the domain D in Matlab/Octave using the function `VCplotter.m` with the command:

```
VCplotter(7, @(theta,z) (cos(z))*(2+sin(3*theta)), -pi/2, pi/2);
```

Solution of Exercise 2.68. We simply write the triple integrals in cylindrical coordinates using the volume element (82). The Cartesian coordinates are written in the cylindrical system using (81).

$$\iiint_C z\,\mathrm{d}V = \int_0^1 \int_{-\pi}^{\pi} \int_0^z zr\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \left(\int_{-\pi}^{\pi} \mathrm{d}\theta\right) \int_0^1 \left(z \int_0^z r\,\mathrm{d}r\right) \mathrm{d}z = 2\pi \int_0^1 z\frac{z^2}{2}\,\mathrm{d}z = \frac{\pi}{4};$$

$$\iiint_C x\,\mathrm{d}V \overset{(81)}{=} \int_0^1 \int_{-\pi}^{\pi} \int_0^z (r\cos\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \left(\int_{-\pi}^{\pi} \cos\theta\,\mathrm{d}\theta\right) \int_0^1 \left(\int_0^z r^2\,\mathrm{d}r\right) \mathrm{d}z = 0\frac{1}{12} = 0;$$

$$\iiint_C (x^2 + y^2 + z^2)\,\mathrm{d}V \overset{(81)}{=} \int_0^1 \int_{-\pi}^{\pi} \int_0^z (r^2\cos^2\theta + r^2\sin^2\theta + z^2) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \int_0^1 \int_{-\pi}^{\pi} \int_0^z (r^2 + z^2) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \left(\int_{-\pi}^{\pi} \mathrm{d}\theta\right) \int_0^1 \left(\int_0^z (r^3 + z^2 r)\,\mathrm{d}r\right) \mathrm{d}z = 2\pi \int_0^1 \frac{3}{4} z^4\,\mathrm{d}z = \frac{3\pi}{10};$$

$$\iiint_C g(z)\,\mathrm{d}V = \int_0^1 \int_{-\pi}^{\pi} \int_0^z g(z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \left(\int_{-\pi}^{\pi} \mathrm{d}\theta\right) \int_0^1 \left(g(z) \int_0^z r\,\mathrm{d}r\right) \mathrm{d}z = 2\pi \int_0^1 g(z) \frac{z^2}{2}\,\mathrm{d}z.$$

Solution of Exercise 2.71. We expand $\vec{\mathbf{F}}$ using (81), (85) and the fundamental trigonometry equation:

$$\vec{\mathbf{F}} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} = -(r\sin\theta)(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}) + (r\cos\theta)(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}) + \hat{\boldsymbol{z}} = r(\sin^2\theta + \cos^2\theta)\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{z}} = r\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{z}}.$$

From (86), its divergence is $\vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial 0}{\partial r} + \frac{1}{r}0 + \frac{1}{r}\frac{\partial r}{\partial \theta} + \frac{\partial 1}{\partial z} = 0$, thus $\vec{\mathbf{F}}$ is solenoidal.

Solution of Exercise 2.77.

$$\begin{aligned}\iiint_B f\,\mathrm{d}V &= \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 \rho^4(\sin^2\phi+\cos 12\theta)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &= \Big(\int_0^1\rho^6\,\mathrm{d}\rho\Big)\int_0^{\pi}\int_{-\pi}^{\pi}(\sin^2\phi+\cos 12\theta)\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{1}{7}\int_0^{\pi}2\pi\sin^3\phi\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{2\pi}{7}\frac{\cos 3t-9\cos t}{12}\Big|_0^{\pi} = \frac{8\pi}{21}.\end{aligned}$$

Solution of Exercise 2.79. We first convert the coordinates from degrees to radians. Denoting φ the latitude, the colatitude is $\phi=\frac{\pi}{2}-\varphi$.

$$\begin{aligned}\varphi_S &= 37° = \frac{2\pi}{360}37\approx 0.646, & \varphi_N &= 41° = \frac{2\pi}{360}41\approx 0.716,\\ \phi_S &= \frac{\pi}{2}-\varphi_S = 0.925, & \phi_N &= \frac{\pi}{2}-\varphi_N = 0.855,\\ \theta_E &= -102°03' = -\frac{2\pi}{360}\Big(102+\frac{3}{60}\Big)\approx -1.781, & \theta_W &= -109°03' = -\frac{2\pi}{360}\Big(109+\frac{3}{60}\Big)\approx -1.903,\end{aligned}$$

(where the minus signs come from the fact that θ measures the longitude in the East direction).

Placing the origin of the axis in the Earth's centre, Colorado can be represented as the surface $S=\{\vec{\mathbf{r}},\rho=R=6371,\ \theta_W\le\theta\le\theta_E,\ \phi_S\le\phi\le\phi_N\}$, so its area is

$$\mathrm{Area}(S)=\iint_S 1\,\mathrm{d}S=\int_{\theta_W}^{\theta_E}\int_{\phi_N}^{\phi_S}R^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta=(\theta_E-\theta_W)R^2\big(-\cos(\phi_S)+\cos(\phi_N)\big)\approx 269{,}305\,\mathrm{km}^2,$$

which roughly agree with the value 269,837 km^2 found on Wikipedia.

B.3 Exercises of Section 3

Solution of Exercise 3.7. We compute left- and right-hand sides of (96), and verify they coincide:

$$\begin{aligned}\vec\nabla\times\vec{\mathbf{F}} &= \hat{\boldsymbol{k}}, \qquad \Rightarrow \text{left-hand side} = \iint_R(\vec\nabla\times\vec{\mathbf{F}})_3\,\mathrm{d}A = \iint_R 1\,\mathrm{d}A = \mathrm{Area}(R)=4\pi,\\ \vec{\mathbf{a}}(t) &= 2\cos t\hat{\boldsymbol{\imath}}+2\sin t\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{a}}}{dt}(t) = -2\sin t\hat{\boldsymbol{\imath}}+2\cos t\hat{\boldsymbol{\jmath}}, \qquad 0\le t\le 2\pi,\\ \vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big) &= (2\cos t+4\sin t)(\hat{\boldsymbol{\imath}}+3\hat{\boldsymbol{\jmath}}),\\ \vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big)\cdot\frac{d\vec{\mathbf{a}}}{dt}(t) &= 4(\cos t+2\sin t)(-\sin t+3\cos t) = 4(3\cos^2 t-2\sin^2 t+5\cos t\,\sin t),\\ &= 2\Big(3(\cos 2t+1)-2(1-\cos 2t)+5\sin 2t\Big) = 2\Big(5\cos 2t+1+5\sin 2t\Big),\\ \oint_{\partial R}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} &= \int_0^{2\pi}\vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big)\cdot\frac{d\vec{\mathbf{a}}}{dt}(t)\,\mathrm{d}t\\ &= \int_0^{2\pi}2\Big(5\cos 2t+1+5\sin 2t\Big)\,\mathrm{d}t = 2\bigg(\frac{5}{2}\sin 2t-\frac{5}{2}\cos 2t+t\bigg)\bigg|_0^{2\pi} = 4\pi.\end{aligned}$$

Solution of Exercise 3.8. Since the domain R is y-simple, we compute the double integral at the left-hand side of (96) as an iterated integral:

$$\begin{aligned}\iint_R(\vec\nabla\times\vec{\mathbf{F}})_3\,\mathrm{d}A &\overset{(23)}{=} \iint_R(-2y)\,\mathrm{d}A \overset{(53)}{=} \int_{-2}^{2}\bigg(\int_0^{\sqrt{1-\frac{x^2}{4}}}(-2y)\,\mathrm{d}y\bigg)\mathrm{d}x\\ &= \int_{-2}^{2}\Big(-1+\frac{x^2}{4}\Big)\mathrm{d}x = 2\Big(-2+\frac{8}{12}\Big) = -\frac{8}{3}.\end{aligned}$$

The boundary of R is composed of two parts. The horizontal segment $\{-2<x<2, y=0\}$ does not give any contribution to the line integral at the right-hand side of (96), since $\vec{\mathbf{F}}$ vanishes on the x axis. The upper arc is parametrised by the curve $\vec{\mathbf{a}}(t)=2\cos t\hat{\boldsymbol{\imath}}+\sin t\hat{\boldsymbol{\jmath}}$ for $0<t<\pi$ (recall that it needs to be oriented anti-clockwise), whose total derivative is $\frac{d\vec{\mathbf{a}}}{dt}(t)=-2\sin t\hat{\boldsymbol{\imath}}+\cos t\hat{\boldsymbol{\jmath}}$. So the line integral reads:

$$\oint_{\partial R}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}} = \int_{\partial R\cap\{y>0\}}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}$$

$$\overset{(44)}{=} \int_0^\pi y^2 \hat{\boldsymbol{\imath}} \cdot (-2\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t$$
$$= \int_0^\pi -2y^2 \sin t\,\mathrm{d}t \qquad \text{(now use } y = a_2(t) = \sin t \text{ on } \partial R \cap \{y > 0\})$$
$$= \int_0^\pi -2\sin^3 t\,\mathrm{d}t = \int_0^\pi (2\sin t\cos^2 t - 2\sin t)\,\mathrm{d}t = \Big(-\frac{2}{3}\cos^3 t + 2\cos t\Big)\Big|_0^\pi = -\frac{8}{3},$$

and we have proved that the right- and the left-hand sides of formula (96) are equal to each other, for this choice of domain R and field $\vec{\mathbf{F}}$.

Solution of Exercise 3.12. $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} \times (\hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{k}}) \overset{(4)}{=} \hat{\boldsymbol{\tau}}(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}) - \hat{\boldsymbol{k}}(\hat{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{k}}) = \hat{\boldsymbol{\tau}}1 - \hat{\boldsymbol{k}}0 = \hat{\boldsymbol{\tau}}$.

Solution of Exercise 3.16. We compute the divergence $\vec{\nabla} \cdot \vec{\mathbf{F}} = \mathrm{e}^x + 1$ and the iterated triple integral:

$$\oint\!\!\!\oint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} \overset{(101)}{=} \iiint_D \vec{\nabla} \cdot \vec{\mathbf{F}}\,\mathrm{d}V = \int_0^1 \int_{-1}^1 \int_0^{10} (\mathrm{e}^x + 1)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 20\,\mathrm{e}.$$

Solution of Exercise 3.17. We want to find a vector field $\vec{\mathbf{F}}$ defined in the ball $B = \{|\vec{\mathbf{r}}| \le R\}$ such that $\vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}} = x^2 + y^2$ on $S = \partial B$. The outward-pointing unit vector field $\hat{\boldsymbol{n}}$ on ∂B is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}} = \frac{1}{R}\vec{\mathbf{r}}$, so a suitable vector field is $\vec{\mathbf{F}} = Rx\hat{\boldsymbol{\imath}} + Ry\hat{\boldsymbol{\jmath}}$. Thus we conclude:

$$\oint\!\!\!\oint_S (x^2 + y^2)\,\mathrm{d}S = \oint\!\!\!\oint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} \overset{(101)}{=} \iiint_B \vec{\nabla} \cdot \vec{\mathbf{F}}\,\mathrm{d}V = \iiint_B 2R\,\mathrm{d}V = 2R\mathrm{Vol}(B) = \frac{8}{3}\pi R^4.$$

Solution of Exercise 3.18. We first decompose the boundary ∂C in the disc $B = \{x^2 + y^2 < 1, z = 1\}$ and in the lateral part $L = \{x^2 + y^2 = z^2, 0 < z < 1\}$. We need to find a vector field $\vec{\mathbf{F}}$ such that $\vec{\nabla} \cdot \vec{\mathbf{F}} = |\vec{\mathbf{r}}|^2$ and $\vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}$ is "easy to integrate" on ∂D. Since the divergence must be a polynomial of degree two, the easiest choice is to look for a field whose components are polynomials of degree three. Moreover, the position field $\vec{\mathbf{r}}$ has zero normal component on the lateral boundary L of C (draw a sketch to figure out why), so a good $\vec{\mathbf{F}}$ may be a multiple of $\vec{\mathbf{r}}$. It is easy to figure out that a possible candidate is $\vec{\mathbf{F}} = \frac{1}{5}|\vec{\mathbf{r}}|^2\vec{\mathbf{r}}$, whose divergence is $|\vec{\mathbf{r}}|^2$, as desired. From the divergence theorem we have:

$$\iiint_C |\vec{\mathbf{r}}|^2\,\mathrm{d}V = \iiint_C \vec{\nabla} \cdot \vec{\mathbf{F}}\,\mathrm{d}V \overset{(101)}{=} \oint\!\!\!\oint_{\partial C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$$
$$= \iint_B \vec{\mathbf{F}} \cdot \underbrace{\hat{\boldsymbol{n}}}_{=\hat{\boldsymbol{k}}}\,\mathrm{d}S + \iint_L \underbrace{\vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}}_{=0}\,\mathrm{d}S$$
$$= \iint_B \frac{1}{5}(x^2 + y^2 + z^2)z\,\mathrm{d}S$$
$$= \iint_B \frac{1}{5}(r^2 + 1)\,\mathrm{d}S \qquad \text{using } z = 1 \text{ on } B \text{ and cylindrical coordinates}$$
$$= \frac{1}{5}\int_{-\pi}^{\pi} \Big(\int_0^1 (r^2 + 1)r\,\mathrm{d}r\Big)\,\mathrm{d}\theta = \frac{1}{5}2\pi\Big(\frac{1}{4} + \frac{1}{2}\Big) = \frac{3\pi}{10},$$

which agrees with the result found in Exercise 2.68.

Solution of Exercise 3.19. *(i)* We compute the divergence and integrate in spherical coordinates:

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = x^2 + z^2 + y^2 = |\vec{\mathbf{r}}|^2 = \rho^2,$$
$$\oint\!\!\!\oint_{\partial B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iiint_B \vec{\nabla} \cdot \vec{\mathbf{F}}\,\mathrm{d}V = \iiint_B \rho^2\,\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^\pi \sin\phi\,\mathrm{d}\phi \int_0^1 \rho^4\,\mathrm{d}\rho = 2\pi\,2\,\frac{1}{5} = \frac{4}{5}\pi.$$

(ii) Similarly, we find the flux through the boundary of $C_L = (0, L)^3$ integrating in Cartesian coordinates:

$$\oint\!\!\!\oint_{\partial C_L} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iiint_{C_L} \vec{\nabla} \cdot \vec{\mathbf{F}}\,\mathrm{d}V = \int_0^L \int_0^L \int_0^L (x^2 + y^2 + z^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^L \int_0^L \Big(Lx^2 + Ly^2 + \frac{1}{3}L^3\Big)\,\mathrm{d}y\,\mathrm{d}x = \int_0^L \Big(L^2x^2 + \frac{1}{3}L^4 + \frac{1}{3}L^4\Big)\,\mathrm{d}x = \frac{1}{3}(L^5 + L^5 + L^5) = L^5.$$

Thus, the two fluxes are equal if $L^5 = \frac{4}{5}\pi$, i.e. if $L = (\frac{4}{5}\pi)^{\frac{1}{5}} \approx 1.202$.

Solution of Exercise 3.23. The gradient of the field is $\vec{\nabla} f = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$, from which we compute the triple integral

$$\iiint_D \vec{\nabla} f \,\mathrm{d}V = \int_0^1 \int_0^1 \int_0^1 (yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$
$$= \hat{\imath} \int_0^1 1 \,\mathrm{d}x \int_0^1 y \,\mathrm{d}y \int_0^1 z \,\mathrm{d}z + \hat{\jmath} \int_0^1 x \,\mathrm{d}x \int_0^1 1 \,\mathrm{d}y \int_0^1 z \,\mathrm{d}z + \hat{k} \int_0^1 x \,\mathrm{d}x \int_0^1 y \,\mathrm{d}y \int_0^1 1 \,\mathrm{d}z = \frac{1}{4}(\hat{\imath} + \hat{\jmath} + \hat{k}).$$

The field f vanishes on the three faces of D incident to the origin (for example the lower face belongs to the plane $\{z = 0\}$, so $f(\vec{\mathbf{r}}) = xy0 = 0$). In the upper face $z = 1$ and $\hat{\boldsymbol{n}} = \hat{k}$, thus the contribution of this face to the surface integral at the right-hand side of (102) is

$$\int_0^1 \int_0^1 xy1\hat{k} \,\mathrm{d}y \,\mathrm{d}x = \frac{1}{4}\hat{\imath}.$$

Similarly, from the faces lying in the planes $\{x = 1\}$ and $\{y = 1\}$ we have the contributions $\frac{1}{4}\hat{\imath}$ and $\frac{1}{4}\hat{\jmath}$, respectively. The sum of the three contributions give the value of the triple integral computed above.

Solution of Exercise 3.24. We recall that harmonic means that $\Delta f = 0$ (Section 1.3.4), and the normal derivative is $\frac{\partial f}{\partial n} = \hat{\boldsymbol{n}} \cdot \vec{\nabla} f$. Taking $g = f$ in Green's first identity, we see that

$$\iiint_D |\vec{\nabla} f|^2 \,\mathrm{d}V = \iiint_D \vec{\nabla} f \cdot \vec{\nabla} f \,\mathrm{d}V = -\iiint_D f \underbrace{\Delta f}_{=0} \,\mathrm{d}V + \oiint_{\partial D} f \underbrace{\frac{\partial f}{\partial n}}_{=0} \,\mathrm{d}S = 0.$$

Since the scalar field $|\vec{\nabla} f|^2$ is non-negative ($|\vec{\nabla} f(\vec{\mathbf{r}})|^2 \geq 0$ for all $\vec{\mathbf{r}} \in D$), this equation implies that $\vec{\nabla} f = \vec{\mathbf{0}}$ everywhere in D. (More in detail, if $\vec{\nabla} f \neq \vec{\mathbf{0}}$ was true in a portion of D, this would imply that $|\vec{\nabla} f|^2 > 0$ gave a positive contribution to the integral $\iiint_D |\vec{\nabla} f|^2 \,\mathrm{d}V$ that can not be cancelled by negative contributions from other parts of the domain as $|\vec{\nabla} f|^2$ is never negative, so the integral in the formula above could not vanish.) Since the gradient of f vanishes, all the partial derivatives of f are zero, so f can not depend on any of the variables x, y and z, which means it is constant.

Solution of Exercise 3.20. We first note that the vector field $\vec{\mathbf{F}} := \vec{\mathbf{r}}/|\vec{\mathbf{r}}|^3$ is well-defined in all of $\mathbb{R}^3$ except at the origin. We also know from Exercise 1.62 that $\vec{\nabla} \cdot \vec{\mathbf{F}} = \vec{\mathbf{0}}$ in all points $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$. Thus, from divergence theorem we immediately conclude for the case $\vec{\mathbf{0}} \notin D$: $\oiint_{\partial D} \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3} \cdot \mathrm{d}\vec{\mathbf{S}} = \iiint_D \vec{\nabla} \cdot \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3} \,\mathrm{d}V = \iiint_D 0 \,\mathrm{d}V = 0$.

On the other hand, if $\vec{\mathbf{0}} \in D$ we cannot apply directly the divergence theorem to $\vec{\mathbf{F}}$ because this field is not defined at the origin. We denote by B a ball centred at $\vec{\mathbf{0}}$ and contained in D (it exists because D is open) and we call R its radius. The outward-pointing unit normal vector on ∂B is $\hat{\boldsymbol{n}}_B(\vec{\mathbf{r}}) = \vec{\mathbf{r}}/|\vec{\mathbf{r}}|$, so

$$\oiint_{\partial B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \oiint_{\partial B} \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}_B \,\mathrm{d}S = \oiint_{\partial B} \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|^3} \cdot \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} \,\mathrm{d}S = \oiint_{\partial B} \frac{1}{R^2} \,\mathrm{d}S = \frac{\mathrm{Area}(\partial B)}{R^2} = \frac{4\pi R^2}{R^2} = 4\pi.$$

We now apply the divergence theorem in $D \setminus B$, where $\vec{\mathbf{F}}$ is well-defined and $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$. So we have that the flux of $\vec{\mathbf{F}}$ through the boundary of $D \setminus B$ vanishes. The boundary of this set is made of two components: ∂D, which gives the integral we want to compute, and ∂B, for which we have just computed the flux of $\vec{\mathbf{F}}$. We need to be careful with the signs: on ∂B, the vector field $\hat{\boldsymbol{n}}_B$ we have used above is outward-pointing for B, so inward-pointing for $D \setminus B$, so the outward-pointing vector for $D \setminus B$ satisfies $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}_B$. In formulas:

$$0 = \iiint_{D \setminus B} \vec{\nabla} \cdot \vec{\mathbf{F}} \,\mathrm{d}V = \oiint_{\partial(D \setminus B)} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \oiint_{\partial D \cup \partial B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{\partial B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$$
$$= \iint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{\partial B} \vec{\mathbf{F}} \cdot (-\hat{\boldsymbol{n}}_B) \,\mathrm{d}S = \iint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} - 4\pi,$$

which gives $\iint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = 4\pi$. (This is sometimes called Gauss' theorem.)

Solution of Exercise 3.30. As usual, we parametrise the unit circumference ∂S with the curve $\vec{\mathbf{a}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}}$, $0 < t < 2\pi$. The circulation in (106) reads

$$\begin{aligned}\oint_{\partial S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_0^{2\pi} \big((2x-y)\hat{\boldsymbol{\imath}} - yz^2\hat{\boldsymbol{\jmath}} - y^2z\hat{\boldsymbol{k}}\big) \cdot (-\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t \\ &= \int_0^{2\pi} (-2\cos t \sin t + \sin^2 t)\,\mathrm{d}t \qquad \text{since on } \partial S\ x = \cos t, y = \sin t, z = 0, \\ &= \int_0^{2\pi} \Big(-\sin 2t + \frac{1}{2}(1-\cos 2t)\Big)\,\mathrm{d}t = \pi.\end{aligned}$$

The curl of $\vec{\mathbf{F}}$ is $\vec{\nabla} \times \vec{\mathbf{F}} = \hat{\boldsymbol{k}}$ and the surface S is the graph of the field $g(x,y) = \sqrt{1-x^2-y^2}$ over the disc $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, x^2+y^2 < 1\}$. Thus, using formula (74) for the flux through a graph surface, we have

$$\begin{aligned}\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} &\overset{(74)}{=} \iint_R \Big(-(\vec{\nabla} \times \vec{\mathbf{F}})_1 \frac{\partial g}{\partial x} - (\vec{\nabla} \times \vec{\mathbf{F}})_2 \frac{\partial g}{\partial y} + (\vec{\nabla} \times \vec{\mathbf{F}})_3\Big)\,\mathrm{d}A \\ &= \iint_R (0+0+1)\,\mathrm{d}A = \mathrm{Area}(R) = \pi,\end{aligned}$$

and both sides of (106) give the same value.

Solution of Exercise 3.31. The desired integral is the circulation of the field $|\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}}$ along the boundary of the paraboloid S, so we apply Stokes' theorem and exploit the computation already done in Exercise 2.55:

$$\oint_{\partial S} |\vec{\mathbf{r}}|^2\,\mathrm{d}x \overset{(46)}{=} \oint_{\partial S} |\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(106)}{=} \iint_S \big(\vec{\nabla} \times (|\vec{\mathbf{r}}|^2\hat{\boldsymbol{\imath}})\big) \cdot \mathrm{d}\vec{\mathbf{S}} \overset{(31)}{=} \iint_S 2(\vec{\mathbf{r}} \times \hat{\boldsymbol{\imath}}) \cdot \mathrm{d}\vec{\mathbf{S}} \overset{\text{Ex. } 2.55}{=} -\frac{4}{3}.$$

Solution of Exercise 3.32. We can prove the assertion in two different ways. The first option is to use Stokes' theorem to transform the line integral to a surface integral, then to use the product rule for the curl, and to verify that the terms obtained are zero:

$$\oint_{\partial S} f\vec{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(106)}{=} \iint_S \vec{\nabla} \times (f\vec{\nabla} f) \cdot \mathrm{d}\vec{\mathbf{S}} \overset{(31)}{=} \iint_S \big(\underbrace{\vec{\nabla} f \times \vec{\nabla} f}_{=\vec{\mathbf{0}},\ (\vec{\mathbf{u}}\times\vec{\mathbf{u}}=\vec{\mathbf{0}})} + f\underbrace{\vec{\nabla} \times \vec{\nabla} f}_{=\vec{\mathbf{0}},\ (26)}\big) \cdot \mathrm{d}\vec{\mathbf{S}} = 0.$$

Alternatively, we can use the chain rule to reduce the desired integral to the line integral of a gradient, which vanishes by the fundamental theorem of vector calculus since the path of integration is a loop:

$$\oint_{\partial S} f\vec{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(14)}{=} \frac{1}{2}\oint_{\partial S} \vec{\nabla}(f^2) \cdot \mathrm{d}\vec{\mathbf{r}} \overset{(49)}{=} 0.$$

C Further exercises for tutorials and personal revision

Some of these exercises will be solved in the tutorials. Which exercises you should try to solve before each tutorial will be announced in the lectures.

Exercises for Section 1.

Exercise C.1. (Exercise 4 in assignment 1 of MA2VC 2013–14.) Show that, for any unit vector $\hat{\boldsymbol{n}}$, and any vector $\vec{\mathbf{w}}$ perpendicular to $\hat{\boldsymbol{n}}$, the identity

$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{w}}$$

holds true. You can make use of the identities in Section 1.1. Demonstrate this identity for the vectors

$$\hat{\boldsymbol{n}} = \frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{w}} = 3\hat{\boldsymbol{\jmath}}.$$

Exercise C.2. Compute and draw the level sets of the scalar field $f(\vec{\mathbf{r}}) = x\mathrm{e}^y$.

Exercise C.3. Draw some paths of the curves

$$\vec{\mathbf{a}}(t) = (\lambda + t^2)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{b}}(t) = t\hat{\boldsymbol{\imath}} + (\lambda t - \lambda^2)\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{c}}(t) = t\cos(t+\lambda)\hat{\boldsymbol{\imath}} + t\sin(t+\lambda)\hat{\boldsymbol{\jmath}}$$

for different values of $\lambda \in \mathbb{R}$. Are they level sets of any scalar field?

Exercise C.4. Let $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{w}}$ be two unit vectors orthogonal to each other. Show that the curve $\vec{\mathbf{a}}(t) = \hat{\boldsymbol{u}}\cos t + \hat{\boldsymbol{w}}\sin t$ lies on the unit sphere for all $t \in \mathbb{R}$. Which curve is this?

Exercise C.5. Draw the path of the curve

$$\vec{\mathbf{a}}(t) = \frac{(\cos t - \sin t)\hat{\boldsymbol{\imath}} + (\cos t + \sin t)\hat{\boldsymbol{\jmath}}}{|\cos t| + |\sin t|}, \qquad 0 \le t \le 2\pi.$$

Exercise C.6. Find a quotient rule for the gradient, i.e. give a formula for $\vec{\nabla}(f/g)$ where f and $g \neq 0$ are scalar fields. Hint: use parts 2 and 3 of Proposition 1.33.

Exercise C.7.
- Compute the vector field $\hat{\boldsymbol{n}}$ of unit length defined on the sphere of radius $R > 0$, which is orthogonal to the sphere itself and points outwards. (This is called "outward-pointing unit normal vector field".) Hint: use part 4 of Proposition 1.33.
- Compute the outward-pointing unit normal vector field on the boundary of the parabolic "cup" $\{x^2 + y^2 < z < 1\}$.

Exercise C.8.
- Fix $\vec{\mathbf{F}} = \mathrm{e}^{xyz}\hat{\boldsymbol{\imath}}$. Show that $\vec{\mathbf{F}}$ and $\vec{\nabla} \times \vec{\mathbf{F}}$ are orthogonal to each other at each point of $\mathbb{R}^3$.
- Prove that any vector field with only one non-zero component is orthogonal to its own curl.
- Find a vector field that is not orthogonal to its own curl.

Exercise C.9. Find a scalar field f defined on the complement of the x axis, whose direction of maximal increase is $(\vec{\mathbf{r}} - x\hat{\boldsymbol{\imath}})/(y^2 + z^2)^{1/2}$.

Exercise C.10. (Exercises 1–2 in assignment 1 of MA2VC 2013–14.) Prove the following identity:

$$\vec{\nabla} \cdot (\vec{\mathbf{r}} fg) = 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g)f,$$

where $\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ is the position vector, and $f(\vec{\mathbf{r}})$ and $g(\vec{\mathbf{r}})$ are two scalar fields.
Hint: you can either use the vector differential identities in the boxes of Propositions 1.52 and 1.55, or the definitions of gradient and divergence.

Demonstrate the above identity for the scalar fields $f = \mathrm{e}^{xy}$ and $g = y^4 z$.

Exercise C.11. Prove identity (34) in the notes (curl of a vector product).

Exercise C.12.
- Find a vector field $\vec{\mathbf{F}}$ such that: $\vec{\mathbf{F}}$ is irrotational, $\vec{\mathbf{F}}$ is not solenoidal, all its streamlines (recall caption to Figure 7) are straight lines passing through the origin.

  This last condition means that $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ is parallel to $\vec{\mathbf{r}}$ in all points $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$.
- [Harder!] Find a vector field $\vec{\mathbf{G}}$ (defined on a suitable domain) such that: $\vec{\mathbf{G}}$ is solenoidal, $\vec{\mathbf{G}}$ is not irrotational, all its streamlines are straight lines passing through the origin.

Exercise C.13. Compute the total derivative of the field $f(\vec{\mathbf{r}}) = (x^2 + y^2)^\beta$, for $\beta \in \mathbb{R}$, evaluated along the curve $\vec{\mathbf{a}}(t) = \cos 2\pi t\hat{\boldsymbol{\imath}} + \sin 2\pi t\hat{\boldsymbol{\jmath}} + \mathrm{e}^{-t^2}\hat{\boldsymbol{k}}$.

Interpret the result geometrically.

Exercise C.14. Let $F : (0, \infty) \to \mathbb{R}$ be a smooth function. Let $f : \mathbb{R}^3 \setminus \{\vec{\mathbf{0}}\} \to \mathbb{R}$ be the scalar field defined by $f(\vec{\mathbf{r}}) = F(|\vec{\mathbf{r}}|)$.

- Compute $\vec{\nabla} f$.
- Prove that $\Delta f = F''(|\vec{\mathbf{r}}|) + \frac{2F'(|\vec{\mathbf{r}}|)}{|\vec{\mathbf{r}}|}$. Hint: use one of the identities of Section 1.4.

Exercise C.15. Let $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ be a *conservative force* field, i.e. a conservative field with $\vec{\mathbf{F}} = -\vec{\nabla}\psi$, where the scalar field ψ is the *potential energy*. Let a particle of *mass* m move with trajectory $\vec{\mathbf{a}}(t)$ according to Newton's law $\vec{\mathbf{F}}(\vec{\mathbf{a}}(t)) = m\frac{d^2\vec{\mathbf{a}}}{dt^2}(t)$ (force equals mass times acceleration). Define the *kinetic energy* of the particle $T(t) = \frac{1}{2}m\left|\frac{d\vec{\mathbf{a}}}{dt}(t)\right|^2$. Prove that the *total energy* $E(t) = \psi(\vec{\mathbf{a}}(t)) + T(t)$ of the particle is constant in time.

Hint: you only need to use chain and product rule for the total derivative.

Exercises for Section 2.

Exercise C.16. Consider the curve $\vec{\mathbf{a}}(t) = t\cos t\,\hat{\boldsymbol{\imath}} + t\sin t\,\hat{\boldsymbol{\jmath}}$ for $t \in [0, 4\pi]$ and the scalar field $f(\vec{\mathbf{r}}) = 1/\sqrt{1 + |\vec{\mathbf{r}}|^2}$. Draw the path Γ of the curve, compute the line integral $\int_\Gamma f\,\mathrm{d}s$ and the total derivative of f evaluated along $\vec{\mathbf{a}}$.

Exercise C.17. Compute the line integral of the vector field $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = y^2\hat{\boldsymbol{\imath}} + 2xy\hat{\boldsymbol{\jmath}}$ along the five curves of Example 2.10.

Exercise C.18. Compute the integral of the scalar field $f = x + y$ on the square Q with vertices $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, -\hat{\boldsymbol{\imath}}, -\hat{\boldsymbol{\jmath}}$.

Exercise C.19. Compute the area of the "eye" domain $\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, 1 - y^2 < |x| < 2(1 - y^2)\}$.

Hint: draw a sketch of the domain. This domain is neither y-simple nor x-simple, to compute the double integral you need to use two of the fundamental properties of integrals in list (40) in the notes.

Exercise C.20. Compute the area of $\vec{\mathbf{T}}(R)$, where $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, 0 < x < 1, 0 < y < 1\}$ and $\vec{\mathbf{T}}(x, y) = (x + y)\hat{\boldsymbol{\xi}} + y^3\hat{\boldsymbol{\eta}}$.

Exercise C.21. Find a change of variables from the trapezoid R with vertices $-2\hat{\boldsymbol{\imath}}, 2\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ to the square Q with vertices $\vec{\mathbf{0}}, \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\eta}}$.

Use the change of variables you have found to compute $\iint_R \mathrm{e}^{(2-y)^2}\,\mathrm{d}x\,\mathrm{d}y$.

Exercise C.22. Let S be the surface parametrised by the chart $\vec{\mathbf{X}}(u, w) = u\hat{\boldsymbol{\imath}} + \frac{w}{u}\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ defined over the triangular region $R = \{0 < w < u < 1\}$. Compute the flux through S of the field $\vec{\mathbf{F}} = \hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}}$.

Rewrite the surface as the graph of a scalar field $g(x, y)$ over a suitable region $\widetilde{R}$ and compute again the same flux using formula (74).

Exercise C.23. Compute the triple integral of $f = x^2 + y^2$ over the domain D lying inside the ball of radius $a > 0$ centred at the origin and above the cone $z = \sqrt{x^2 + y^2}$.

Exercise C.24. Compute the area of the region bounded by $\vec{\mathbf{a}}(t) = t(2\pi - t)(\cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}})$, $0 \le t \le 2\pi$.

Exercise C.25. London and Astana (Kazakhstan) approximately lie on the parallel $51°N$, and have longitude $0°$ and $71°E$, respectively. Describe the location of the two cities in a suitable special coordinates system. Compute the distance between the two cities, if the distance is measured:

(i) along a straight (underground) segment,

(ii) on the Earth surface along the parallel $51°N$,

(iii) along the shortest surface path.

Assume that Earth is a sphere of radius 6371 km.

Exercises for Section 3.

Exercise C.26. Compute the area of the region bounded by the curve $\vec{\mathbf{a}}(t) = \sin t\hat{\boldsymbol{\imath}} + \sin 2t\hat{\boldsymbol{\jmath}}$, $0 \le t \le 2\pi$.

Exercise C.27. Compute the flux of $\vec{\mathbf{F}} = xy\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}} + zx\hat{\boldsymbol{k}}$ through the boundary of the cube $C = (0, 2)^3 = \{0 < x, y, z < 2\}$.

Exercise C.28. Demonstrate the divergence theorem for $\vec{\mathbf{F}} = \frac{1}{2}z^2\hat{\boldsymbol{k}}$ and the upper half unit ball $B = \{z > 0, |\vec{\mathbf{r}}| < 1\}$.

Exercise C.29. Let $\vec{\mathbf{F}}$ be a vector field, and a and ϵ be two positive numbers such that $\vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot \vec{\mathbf{r}} \geq \epsilon$ for all $\vec{\mathbf{r}} \in \mathbb{R}^3$ with $|\vec{\mathbf{r}}| = a$. Prove that $\vec{\mathbf{F}}$ is not solenoidal in the ball B of radius a centred at the origin.

Exercise C.30. Prove the following "integration by parts formula" for the curl: given a domain $D \subset \mathbb{R}^3$ and two vector fields $\vec{\mathbf{F}}, \vec{\mathbf{G}}$,

$$\iiint_D (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{G}} \,\mathrm{d}V = \iiint_D \vec{\mathbf{F}} \cdot (\vec{\nabla} \times \vec{\mathbf{G}}) \,\mathrm{d}V + \oiint_{\partial D} (\vec{\mathbf{G}} \times \hat{\boldsymbol{n}}) \cdot \vec{\mathbf{F}} \,\mathrm{d}S.$$

Exercise C.31. Demonstrate Stokes' theorem for the field $\vec{\mathbf{F}} = zy\hat{\boldsymbol{\imath}}$ and the quadrilateral S with vertices $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}, \hat{\boldsymbol{\jmath}}, 2\hat{\boldsymbol{k}}$.

Exercise C.32. Consider the field $\vec{\mathbf{G}} = 2x^2y^2z\hat{\boldsymbol{\jmath}} - 2x^2yz^2\hat{\boldsymbol{k}}$ and a triangle T with the three vertices on the x-, y- and z-axis, respectively. Show that the flux of $\vec{\mathbf{G}}$ through T is zero.

Exercise C.33. Use Stokes' theorem to compute the flux of $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} - 2z\hat{\boldsymbol{k}}$ through the cylindrical surface $S = \{x^2 + y^2 = 1, 0 < z < H\}$, where $H > 0$.

D Alphabetical index

E References

[1] R.A. Adams, C. Essex, *Calculus, a complete course*, Pearson Canada, Toronto, 7th ed., 2010. UoR library call number: FOLIO–515-ADA.

[2] E. Carlen, *Multivariable Calculus, Linear Algebra and Differential Equation*, lecture notes, 2013, available on: http://www.math.rutgers.edu/~carlen/291F14/index.html.

[3] M. Corral, *Vector Calculus*, 2013, available on: http://www.mecmath.net/.

[4] O. Knill, http://www.math.harvard.edu/~knill/teach/index.html. Here you can find plenty of materials: lecture notes, exercises and instructive illustrations. Click on one of the Math(s)21a links.

[5] S. Lang, *Calculus of Several Variables*, Springer Undergraduate Texts in Mathematics, 3rd ed., 1996.

[6] C.J. Smith, *Vector Calculus Primer*, 2011, Reading, available on Blackboard and on course web page.

[7] M. Spiegel, S. Lipschutz, D. Spellman, *Vector Analysis*, McGraw-Hill, 2nd ed., 2009.

F Common errors in vector calculus

This is a list of errors I found dozens of times in previous vector calculus assignments and exams. Many of them are trivial and can easily be avoided, nonetheless they appear too often, watch out!

- The most common source of mistakes is treating scalar quantities as vector ones and vice versa. E.g. summing or equating scalars to vectors, writing the divergence of a scalar field, evaluating fields at a number and curves at a vector. One of the main aims of this module is to learn how to manipulate these objects, so take the greatest care to avoid these mistakes! Remember: **never confuse scalars with vectors**.
- An unbelievably large number of errors happen because **brackets** are not used or are badly placed. Typical example: instead of $\frac{\partial}{\partial x}(fg)$ you write $\frac{\partial}{\partial x}fg$, in the next line you have forgotten that also g was inside the derivation and you go on with $\frac{\partial f}{\partial x}g$, which is wrong. When in doubt, always write the brackets.
- You should know very well that the signs "$=$", "$\iff$" and "$\Rightarrow$" cannot be used in place of each other. The equal sign can only stay between two numbers, vectors, sets, matrices, or any two other comparable mathematical objects. The implication $\Rightarrow$ and the double implication $\iff$ ("if and only if") stay between two propositions, statements or equations (in which case they contain an equal sign).
- Whenever you use a mathematical object denoted by a certain letter for the first time in an exercise, you must introduce it, unless it is known from a standard definition (or from the statement of the exercise). E.g. the number π and the position vector $\vec{\mathbf{r}}$ do not need to be defined every time and you can use them directly; fields, curves, domains $f, g, \vec{\mathbf{F}}, \vec{\mathbf{a}}, D\dots$ are *not* standard and you need to define them. Examples of typical correct sentences: "given a scalar field $f\dots$", "let $D \subset \mathbb{R}^3$ be a domain…", "fix $\vec{\mathbf{a}}(t) := t^3\hat{\boldsymbol{\imath}}$ for $t \in (-1,1)\dots$"
- Every time you write an (either single or multiple) integral sign, you must also write *(1)* the correct infinitesimal element (e.g. $\mathrm{d}t$, $\mathrm{d}S, \cdot\,\mathrm{d}\vec{\mathbf{r}}$, $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\dots$) and *(2)* either a domain of integration (placed under it, e.g. D in $\iiint_D \cdots \mathrm{d}V$) or the extremes of integration (e.g. $\int_0^{1-y} \cdots \mathrm{d}x$, only for single integrals or line integrals of conservative fields). In this module we never consider indefinite integrals.
- Read carefully the main verb of the exercise request: "prove" (or "show"), "compute" (or "find", "calculate"), "demonstrate", "use", "state"… are not all the same! E.g. if you are asked to *demonstrate* a certain identity $A = B$, you have to compute both A and B and verify they are equal; if you are asked to *use* the same identity to compute B, you only compute A and state the identity.
- Do not mix up the different theorems. You need to know and remember well the statements of all the main ones, it is not so hard! (E.g. the divergence theorem is that one containing the divergence, not surprisingly, and not the curl or the partial derivative in z.) You do not need to remember by hearth the identities in Proposition 1.55.
- When you prove a vector identity (either differential as in Proposition 1.55 or integral like (103)) and you consider only the first component *of the identity*, you cannot ignore the second and third components *of the fields* involved, nor the derivatives with respect to the second and third variables.
- Whenever you can use the differential identities of Proposition 1.52 and 1.55, use them! Once you become comfortable with them, their use is much easier than the expansion in coordinates, see the example in Remark 1.58.
- The symbols $\vec{\nabla}, \vec{\nabla}\cdot, \vec{\nabla}\times, \Delta$ are different objects; every time you use one of them in place of another one you do a mistake.
- In scalar products and in the divergence, the dot symbol "$\cdot$" is *not* optional, it must be written clearly.
- If at the beginning of an exercise you report the request, distinguish *clearly* which are the data you have and which is the assertion you want to prove or demonstrate or the entity you want to compute. If you simply list many formulas it is not clear what you are doing.
- Often common sense helps: a negative length, area or volume is surely wrong.
- You must always write what you are computing. E.g. if an exercise requires to compute the potential φ, your solution should include either a sentence such as "the desired potential is equal to $xy\mathrm{e}^z$", or a formula like "$\varphi = xy\mathrm{e}^z$", and not simply the result "$xy\mathrm{e}^z$".
- In the notes there are several "warning" remarks describing frequent errors, read them carefully. See 1.4, 1.16, 1.27, 1.45, 1.49.

G References to the relevant textbook sections and exercises

	Note sections	Book sections	Book exercises
Vectors, scalar product	1.1, 1.1.1	10.2	2–3, 16–31
Vector product	1.1.2	10.3	1–28
Scalar fields	1.2.1	12.1	11–42
Vector fields	1.2.2	15.1	1-8
Curves	1.2.3	11.1	
		11.3	1–12
Partial derivatives	1.3.1	12.3	1–12, 25–31
		12.5	
Gradient	1.3.2	12.7	1–6 (part a), 10–15
		16.1	
Jacobian matrix	1.3.3	12.6	
Second-order partial derivatives, Laplacian	1.3.4	12.4	1–14
Divergence and curl operator	1.3.5,1.3.6	16.1	1–8
Vector differential identities	1.4.1, 1.4.2	16.2	8, 11, 13, 14
Conservative fields, scalar potentials	1.5	15.2	1–10
Irrotational and solenoidal fields, potentials	1.5	16.2	9, 10, 12, 15, 16
Total derivative of curves	1.6	11.1	1–14, 27–32
Chain rule	1.7	12.5	1–12, 15–22
Line integral of scalar fields	2.1.1	8.4	1–9
		15.3	1–9
		11.3	13–21
Line integral of vector fields	2.1.2, 2.1.3	15.4	1–12, 15–22
Double integrals	2.2.1	14.1	13–22
		14.2	1–28
Change of variables	2.2.2	14.4	32–34
		14.6	
Triple integrals	2.2.3	14.5	1–20
Surface integrals	2.2.4	14.7	1–10
		15.5	3, 4, 7–10, 13–16
Flux integrals	2.2.5	15.6	1–13
Polar coordinates	2.3.1	8.5	1–28
		8.6	1-11
		14.4	1–30
Cylindrical and spherical coordinates	2.3.2, 2.3.3	10.6	1–14
		14.6	1–16
		15.2	19–22
		16.1	9–11
		16.7	1–4, 14
Green's theorem	3.1	16.3	1–7
Divergence theorem	3.2	16.4	1–14
Stokes' theorem	3.3	16.5	1–12

Table 4: *References to the relevant sections of the textbook [1] by Adams and Essex. The last column lists the exercises you should be able to attempt. Some of them are straightforward, while others are quite hard, or require some of the remarks marked with ★, or use a slightly different notation, or are presented in a different way from what you are used to: solving them is even more instructive. In many cases, drawing a sketch of the domain is crucial to solve the exercises.*

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