# MA240 Vector Analysis Sheet 1 PKS

1. Let $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$; $\mathbf{c} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. Evaluate
   (a) $\mathbf{a}.\mathbf{b}$;
   (b) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$;
   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

2. Let **a**, **b**, **c** be defined as in question 1. Verify that:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}.$$

3. Let $\mathbf{a}(t) = (3t^2 + 2t)\mathbf{i} + \tan t\mathbf{j} - te^t\mathbf{k}$. Determine $\frac{d\mathbf{a}}{dt}$.

4. Let $\mathbf{F}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$. Find the value of

$$\frac{d\mathbf{F}}{dt} \quad \text{and} \quad \frac{d^2\mathbf{F}}{dt^2}$$

   when $t = 1$.

5. Let $\mathbf{a}(u, v) = uv^2\mathbf{i} + u^2v\mathbf{j} + (u+v)^2\mathbf{k}$. Determine the partial derivatives:

$$\frac{\partial \mathbf{a}}{\partial u}; \quad \frac{\partial \mathbf{a}}{\partial v}; \quad \frac{\partial^2 \mathbf{a}}{\partial u^2}; \quad \frac{\partial^2 \mathbf{a}}{\partial u \partial v}; \quad \frac{\partial^2 \mathbf{a}}{\partial v \partial u}; \quad \frac{\partial^2 \mathbf{a}}{\partial v^2}.$$

6. Let $\mathbf{F}(u) = (u^2 + u)\mathbf{i} + (u^3 - 2u^2)\mathbf{j} + ue^u\mathbf{k}$. Determine

$$\int_1^2 \mathbf{F}(u)du.$$

MA240 Vector Analysis Solutions 1 PKS

1.(a) $\underset{\sim}{a}\cdot\underset{\sim}{b} = (3\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k})\cdot(2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}) = 6-1-2 = 3$

(b) $$\underset{\sim}{b}\times\underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = (2-3)\underset{\sim}{i} - (-4-1)\underset{\sim}{j} + (6-(-1))\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$$

$$\therefore\ \underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = (3\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k})\cdot(-\underset{\sim}{i}+5\underset{\sim}{j}+7\underset{\sim}{k}) = -3+5-14 = -12$$

(c) $$\underline{a}\times(\underline{b}\times\underline{c}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & -2 \\ -1 & 5 & 7 \end{vmatrix} = (7+10)\underset{\sim}{i} - (21-2)\underset{\sim}{j} + (15+1)\underset{\sim}{k}$$
$$= 17\underset{\sim}{i} - 19\underset{\sim}{j} + 16\underset{\sim}{k}$$

2. $\underset{\sim}{a}\cdot\underset{\sim}{c} = (3\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k})\cdot(\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}) = 3+3+4 = 10$

$$\therefore\ (\underset{\sim}{a}\cdot\underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{c} = 10(2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}) - 3(\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k})$$
$$= 17\underset{\sim}{i} - 19\underset{\sim}{j} + 16\underset{\sim}{k}$$
$$= \underline{a}\times(\underline{b}\times\underline{c})$$

3. $$\frac{d\underset{\sim}{a}}{dt} = \frac{d}{dt}(3t^2+2t)\underset{\sim}{i} + \frac{d}{dt}(\tan t)\underset{\sim}{j} - \frac{d}{dt}(te^t)\underset{\sim}{k}$$
$$= (6t+2)\underset{\sim}{i} + \sec^2 t\,\underset{\sim}{j} - (e^t+te^t)\underset{\sim}{k}$$

4. $$\frac{d\underset{\sim}{F}}{dt} = \frac{d}{dt}t^3\underset{\sim}{i} + \frac{d}{dt}t^2\underset{\sim}{j} + \frac{d}{dt}t\underset{\sim}{k} = 3t^2\underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{k}$$
$$= 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} \quad \text{at } t=1$$

$$\frac{d^2\underset{\sim}{F}}{dt} = \frac{d}{dt}\left(\frac{d\underset{\sim}{F}}{dt}\right) = \frac{d(3t^2)}{dt}\underset{\sim}{i} + \frac{d(2t)}{dt}\underset{\sim}{j} + \frac{d(1)}{dt}\underset{\sim}{k}$$
$$= 6t\underset{\sim}{i} + 2\underset{\sim}{j} = 6\underset{\sim}{i} + 2\underset{\sim}{j} \quad \text{at } t=1$$

5. $\underset{\sim}{a}(u,v) = uv^2\,\underset{\sim}{i} + u^2v\,\underset{\sim}{j} + (u+v)^2\,\underset{\sim}{k}$

$$\frac{\partial \underset{\sim}{a}}{\partial u} = \frac{\partial}{\partial u}(uv^2)\underset{\sim}{i} + \frac{\partial}{\partial u}(u^2v)\underset{\sim}{j} + \frac{\partial}{\partial u}(u+v)^2\underset{\sim}{k} = \underline{v^2\underset{\sim}{i} + 2uv\underset{\sim}{j} + 2(u+v)\underset{\sim}{k}}$$

$$\frac{\partial \underset{\sim}{a}}{\partial v} = \frac{\partial}{\partial v}(uv^2)\underset{\sim}{i} + \frac{\partial}{\partial v}(u^2v)\underset{\sim}{j} + \frac{\partial}{\partial v}(u+v)^2\underset{\sim}{k} = \underline{2uv\underset{\sim}{i} + u^2\underset{\sim}{j} + 2(u+v)\underset{\sim}{k}}$$

$$\frac{\partial^2 \underset{\sim}{a}}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \underset{\sim}{a}}{\partial u}\right) = \frac{\partial(v^2)}{\partial u}\underset{\sim}{i} + \frac{\partial}{\partial u}(2uv)\underset{\sim}{j} + \frac{\partial}{\partial u}(2(u+v))\underset{\sim}{k} = \underline{2v\underset{\sim}{j} + 2\underset{\sim}{k}}$$

$$\frac{\partial^2 a}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial \underset{\sim}{a}}{\partial v}\right) = \frac{\partial}{\partial u}(2uv)\underset{\sim}{i} + \frac{\partial}{\partial u}(u^2)\underset{\sim}{j} + \frac{\partial}{\partial u}(2(u+v))\underset{\sim}{k} = \underline{2v\underset{\sim}{i} + 2u\underset{\sim}{j} + 2\underset{\sim}{k}}$$

$$\frac{\partial^2 a}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial \underset{\sim}{a}}{\partial u}\right) = \frac{\partial}{\partial v}(v^2)\underset{\sim}{i} + \frac{\partial}{\partial v}(2uv)\underset{\sim}{j} + \frac{\partial}{\partial v}(2(u+v))\underset{\sim}{k} = \underline{2v\underset{\sim}{i} + 2u\underset{\sim}{j} + 2\underset{\sim}{k}}$$

$$\frac{\partial^2 a}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial \underset{\sim}{a}}{\partial v}\right) = \frac{\partial}{\partial v}(2uv)\underset{\sim}{i} + \frac{\partial}{\partial v}(u^2)\underset{\sim}{j} + \frac{\partial}{\partial v}(2(u+v))\underset{\sim}{k} = \underline{2u\underset{\sim}{i} + 2\underset{\sim}{k}}$$

6. $$\int_1^2 \underset{\sim}{F}(u)\,du = \int_1^2 (u^2+u)\,du\ \underset{\sim}{i} + \int_1^2 (u^3-2u^2)\,du\ \underset{\sim}{j} + \int_1^2 ue^u\,du\ \underset{\sim}{k}$$

$$= \left[\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2\right]_1^2\underset{\sim}{i} + \left[\tfrac{1}{4}u^4 - \tfrac{2}{3}u^3\right]_1^2\underset{\sim}{j} + \left\{\left[ue^u\right]_1^2 - \int_1^2 e^u\,du\right\}\underset{\sim}{k}$$

$$= \left(\tfrac{8}{3} + 2 - \tfrac{1}{3} - \tfrac{1}{2}\right)\underset{\sim}{i} + \left(4 - \tfrac{16}{3} - \tfrac{1}{4} + \tfrac{2}{3}\right)\underset{\sim}{j} + \left\{2e^2 - e - \left[e^u\right]_1^2\right\}\underset{\sim}{k}$$

$$= \left(\frac{7}{3} + \frac{3}{2}\right)\underset{\sim}{i} + \left(\frac{15}{4} - \frac{14}{3}\right)\underset{\sim}{j} + \left\{2e^2 - e - (e^2 - e)\right\}\underset{\sim}{k}$$

$$= \underline{\frac{23}{6}\underset{\sim}{i} - \frac{11}{12}\underset{\sim}{j} + e^2\underset{\sim}{k}}$$

**MA240** **Vector Analysis Sheet 2** **PKS**

1. Which of the following are vector fields and which are scalar fields (and which, if any, are neither): Temperature, gravity, velocity, density, speed, time, force, height above sea level, acceleration.

2. Sketch the level surfaces of the following scalar fields:

   (a) $\phi(x, y, z) = x + y$
   (b) $\phi(x, y, z) = x^2 + y^2$
   (c) $\phi(x, y, z) = 4x^2 + y^2$

3. Determine the equations of the field lines for

$$\mathbf{F}(x, y, z) = x\mathbf{i} + x^2\mathbf{j}$$

4. Calculate $\nabla\phi$ and find the unit normal to the level surfaces of $\phi$ where

   (a) $\phi(x, y, z) = xyz$
   (b) $\phi(x, y, z) = x^2y + 3z$
   (c) $\phi(x, y, z) = \cos x + \sin y$

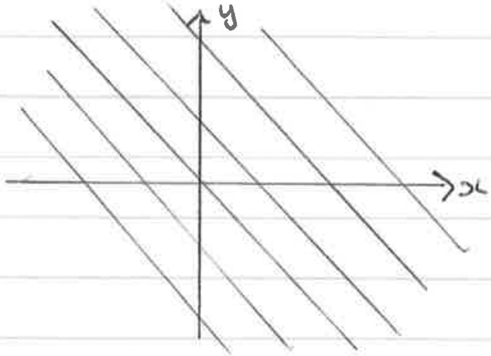
MA240 Vector Analysis Solutions 2 PKS

1.

| scalar field | vector field | neither |
|---|---|---|
| temperature | gravity | time |
| density | velocity | |
| speed | force | |
| height above sealevel | acceleration | |

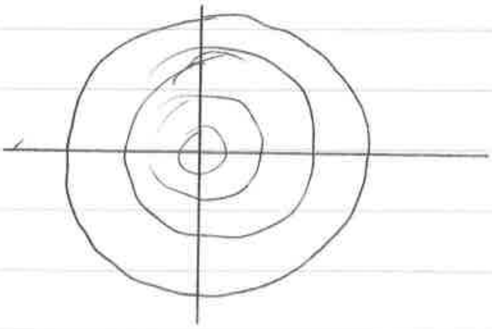
2. Since none of scalar fields involve z the sketches below represent the intersection of level surfaces with plane z = constant.

a) $x+y = \text{const}$, i.e. $y = \text{const} - x$



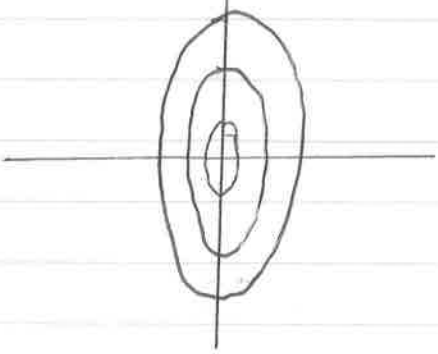
planes perpendicular to page

b) $x^2+y^2 = \text{const}$ – circles about origin



cylinders perpendicular to page

c) $4x^2+y^2 = \text{const}$ – ellipses about origin – ~~longer axis y~~



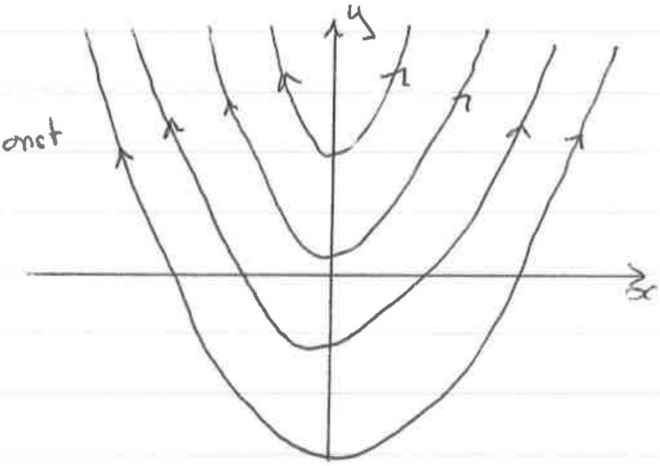
elliptical cylinders perpendicular to page

3. $\underset{\sim}{F}(x,y,z) = x\underset{\sim}{i} + x^2\underset{\sim}{j}$

Field lines given by $x^2\frac{dx}{ds} = x\frac{dy}{ds}$ ; $0 = x\frac{dz}{ds}$ , $0 = x^2\frac{dz}{ds}$

$\therefore \int x\frac{dx}{ds}\,ds = \int \frac{dy}{ds}\,ds$ (cancelling through $x$)

i.e. $\int x\,dx = \int dy \Rightarrow \underline{\tfrac{1}{2}x^2 = y + c}$ - parabolas



<u>and</u> $\frac{dz}{ds} = 0 \Rightarrow z = d$

i.e. in planes $z=\text{const}$

At $x = a$ $\underset{\sim}{F} = (a, a^2, 0)$

$\therefore$ if $a < 0$ $\underset{\sim}{F} \equiv \nwarrow$

$a > 0$ $\underset{\sim}{F} \equiv \nearrow$

hence directions shown

4.

a) $\phi = xyz$ $\quad \underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k} = yz\underset{\sim}{i} + xz\underset{\sim}{j} + xy\underset{\sim}{k}$

$$\underset{\sim}{\hat{n}} = \frac{\underset{\sim}{\nabla}\phi}{|\underset{\sim}{\nabla}\phi|} = \frac{yz\underset{\sim}{i} + xz\underset{\sim}{j} + xy\underset{\sim}{k}}{\sqrt{y^2z^2 + x^2z^2 + x^2y^2}}$$

b) $\phi = x^2y + 3z$ $\quad \underset{\sim}{\nabla}\phi = 2xy\underset{\sim}{i} + x^2\underset{\sim}{j} + 3\underset{\sim}{k}$

$$\underset{\sim}{\hat{n}} = \frac{2xy\underset{\sim}{i} + x^2\underset{\sim}{j} + 3\underset{\sim}{k}}{\sqrt{4x^2y^2 + x^4 + 9}}$$

c) $\phi = \cos x + \sin y$ $\quad \underset{\sim}{\nabla}\phi = -\sin x\,\underset{\sim}{i} + \cos y\,\underset{\sim}{j}$

$$\underset{\sim}{\hat{n}} = \frac{-\sin x\,\underset{\sim}{i} + \cos y\,\underset{\sim}{j}}{\sqrt{\sin^2 x + \cos^2 y}}$$

# MA240 Vector Analysis Sheet 3 PKS

1. Let
$$\phi = x^2y + xz + y^2z^3.$$
   (a) Calculate $\nabla\phi$

   (b) Find $\nabla . \nabla\phi$

   (c) Find $\nabla^2\phi$

   (d) Verify that in this case $\nabla \times \nabla\phi = 0$.

   (e) Show (by substituting in the definition of $\nabla$) that for a general $\phi$
$$\nabla \times \nabla\phi = 0$$
   (You may assume that $\phi$ is such that $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ etc.)

2. Let
$$\mathbf{F} = (x^2y + yz)\mathbf{i} + (xy^2 + z)\mathbf{j} + xyz\mathbf{k}.$$
   (a) Find $\nabla . \mathbf{F}$

   (b) Find $\nabla \times \mathbf{F}$

   (c) Verify that
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla . \mathbf{F}) - \nabla^2\mathbf{F}$$

3. (a) Let
$$r = |\mathbf{r}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}.$$
   By using te definition of $\nabla$ show that for any integer $n$:
$$\nabla r^n = nr^{n-2}\mathbf{r}.$$
   (b) Again using the definition of $\nabla$, show that for any constant vector $\mathbf{a}$
$$\nabla(\mathbf{r}.\mathbf{a}) = \mathbf{a}.$$

MA240 Vector Analysis Solutions 3 PKS

1. $\phi = x^2y + xz + y^2z^3$

(a) $\underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k} = (2xy+z)\underset{\sim}{i} + (x^2+2yz^3)\underset{\sim}{j} + (x+3y^2z^2)$

(b) $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\phi = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\Big((2xy+z)\underset{\sim}{i} + (x^2+2yz^3)\underset{\sim}{j} + (x+3y^2z^2)\underset{\sim}{k}$

$= \frac{\partial}{\partial x}(2xy+z) + \frac{\partial}{\partial y}(x^2+2yz^3) + \frac{\partial}{\partial z}(x+3y^2z^2)$

$= 2y + 2z^3 + 6y^2z$

(c) $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2y + 2z^3 + 6y^2z$

(d) $$\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\phi = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy+z) & (x^2+2yz^3) & (x+3y^2z^2) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x+3y^2z^2) - \frac{\partial}{\partial z}(x^2+2yz^3)\right)\underset{\sim}{i}$$
$$- \left(\frac{\partial}{\partial x}(x+3y^2z^2) - \frac{\partial}{\partial z}(2xy+z)\right)\underset{\sim}{j}$$
$$+ \left(\frac{\partial}{\partial x}(x^2+2yz^3) - \frac{\partial}{\partial y}(2xy+z\right)\underset{\sim}{k}$$

$$= (6yz^2 - 6yz^2)\underset{\sim}{i} - (1-1)\underset{\sim}{j} + (2x-2x)\underset{\sim}{k}$$

$$= \underset{\sim}{0}$$

(e) $$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla}\phi = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\underset{\sim}{i} - \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\underset{\sim}{k}$$

$$\underline{= 0 \quad \text{assuming} \quad \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \quad \text{etc.}}$$

2. $\underset{\sim}{F} = (x^2y + yz)\underset{\sim}{i} + (xy^2 + z)\underset{\sim}{j} + xyz\,\underset{\sim}{k}$

(a) $$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) \cdot \left((x^2y+yz)\underset{\sim}{i} + (xy^2+z)\underset{\sim}{j} + xyz\underset{\sim}{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y+yz) + \frac{\partial}{\partial y}(xy^2+z) + \frac{\partial}{\partial z}(xyz)$$

$$= 2xy + 2xy + xy$$

$$= 5xy$$

(b) $$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y+yz & xy^2+z & xyz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(xy^2+z)\right)\underset{\sim}{i} - \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(x^2y+yz)\right)\underset{\sim}{j}$$
$$+ \left(\frac{\partial}{\partial x}(xy^2+z) - \frac{\partial}{\partial y}(x^2y+yz)\right)\underset{\sim}{k}$$

$$= (xz-1)\underset{\sim}{i} - (yz-y)\underset{\sim}{j} + (y^2-x^2-z)\underset{\sim}{k}$$

(c) $$\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{F}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz-1 & y-yz & y^2-x^2-z \end{vmatrix}$$

$$= (2y+y)\underset{\sim}{i} - (-2x-x)\underset{\sim}{j} + (0-0)\underset{\sim}{k}$$

$$= 3y\underset{\sim}{i} + 3x\underset{\sim}{j}$$

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{F}) = \frac{\partial}{\partial x}(5xy)\underset{\sim}{i} + \frac{\partial}{\partial y}(5xy)\underset{\sim}{j} + \frac{\partial}{\partial z}(5xy)\underset{\sim}{k}$$

$$= 5y\underset{\sim}{i} + 5x\underset{\sim}{j}$$

$$\nabla^2\underset{\sim}{F} = \nabla^2(x^2y+yz)\underset{\sim}{i} + \nabla^2(xy^2+z)\underset{\sim}{j} + \nabla^2(xyz)\underset{\sim}{k}$$

$$= 2y\underset{\sim}{i} + 2x\underset{\sim}{j} + 0\underset{\sim}{k}$$

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{F}) - \nabla^2\underset{\sim}{F} = (5y\underset{\sim}{i} + 5x\underset{\sim}{j}) - (2y\underset{\sim}{i} + 2x\underset{\sim}{j})$$

$$= 3y\underset{\sim}{i} + 3x\underset{\sim}{j} = \underline{\nabla}\times(\underline{\nabla}\times\underline{F})$$

3.(a) $$\underset{\sim}{\nabla} r^n = \underset{\sim}{\nabla}(x^2+y^2+z^2)^{n/2}$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}\underset{\sim}{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}\underset{\sim}{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\underline{k}$$

$$= 2x.\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\underset{\sim}{i} + 2y.\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\underset{\sim}{j} + 2z.\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\underline{k}$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$$

$$= n r^{n-2}\underset{\sim}{r}$$

(b) $\underset{\sim}{\nabla}(\underset{\sim}{r} \cdot \underset{\sim}{a}) = \underset{\sim}{\nabla}((x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}))$

where $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

$$= \underset{\sim}{\nabla}(a_1x + a_2y + a_3z)$$

$$= \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\underset{\sim}{i} + \frac{\partial}{\partial y}(a_1x + a_2y + a_3z)\underset{\sim}{j} + \frac{\partial}{\partial z}(a_1x + a_2y + a_3z)\underset{\sim}{k}$$

$$= a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$$

$$= \underset{\sim}{a}$$

**MA240** **Vector Analysis Sheet 4** **PKS**

1. A scalar field is defined by

$$\phi(x, y, z) = xyz.$$

Find the directional derivative of $\phi$ in the direction of the vector

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

at the point $(1, 1, 1)$.

Let the curve $C$ be described by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

as $t$ varies between 0 and $2\pi$.

Evaluate

$$\int_C \nabla\phi \mathrm{dt} \qquad \text{and} \qquad \int_C \nabla\phi.\mathbf{dr}$$

2. For each of

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

   (b) $\mathbf{F} = xy\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$

calculate

$$\int_C \mathbf{F.dr}$$

where

   (i) $C$ is the straight line joining $(0, 0, 0)$ to $(1, 1, 1)$,

   (ii) $C$ is the two straight lines joining $(0, 0, 0)$ to $(0, 1, 0)$ to $(1, 1, 1)$.

**Based on your answers**, is it possible for either of the given vector fields to be conservative? Give reasons for your answer. Confirm your deductions by taking the curl of the vector fields and find the associated scalar potentials of any of the vector fileds which are conservative.

1. Unit vector in direction of $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is $\hat{\underset{\sim}{u}} = \dfrac{x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}}{\sqrt{x^2+y^2+z^2}}$

$$\underset{\sim}{\nabla}\phi = yz\underset{\sim}{i} + xz\underset{\sim}{j} + xy\underset{\sim}{k}$$

$\therefore$ directional derivative is $\underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}} = \dfrac{xyz + xyz + xyz}{\sqrt{x^2+y^2+z^2}}$

$$= \frac{3xyz}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3} \quad \text{at } (1,1,1)$$

$$\int_C \underset{\sim}{\nabla}\phi \, dt = \int_C yz\underset{\sim}{i} + xz\underset{\sim}{j} + xy\underset{\sim}{k} \, dt$$

But on $C$ $x = \cos t$, $y = \sin t$, $z = t$ and so

$$= \int_0^{2\pi} t\sin t\underset{\sim}{i} + t\cos t\underset{\sim}{j} + \cos t \sin t\underset{\sim}{k} \, d\theta$$

using integration by parts $= \left(\left[t(-\cos t)\right]_0^{2\pi} - \int_0^{2\pi} -\cos t\, dt\right)\underset{\sim}{i} + \left(\left[t\sin t\right]_0^{2\pi} - \int_0^{2\pi}\sin t\, dt\right)\underset{\sim}{j} + \left[\tfrac{1}{2}\sin^2 t\right]_0^{2\pi}\underset{\sim}{k}$

$$= \left(-2\pi + [\sin t]_0^{2\pi}\right)\underset{\sim}{i} + \left(0 - [-\cos t]_0^{2\pi}\right)\underset{\sim}{j} + 0\underset{\sim}{k}$$

$$= -2\pi\underset{\sim}{i}$$

$$\int_{C} \underset{\wedge}{\nabla}\phi \cdot d\underset{\sim}{r} = \int_{C'} \underset{\sim}{\nabla}\phi \cdot \frac{d\underset{\sim}{r}}{dt}\,dt$$

$$= \int_{0}^{2\pi} (t\sin t\,\underset{\wedge}{i} + t\cos t\,\underset{\sim}{j} + \sin t\cos t\,\underset{\sim}{k}) \cdot (-\sin t\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j} + \underset{\wedge}{k})\,dt$$

$$= \int_{0}^{2\pi} -t\sin^2 t + t\cos^2 t + \sin t\cos t$$

$$= \int_{0}^{2\pi} t\cos 2t + \frac{1}{2}\sin 2t\,dt$$

$$= \left[\frac{1}{2}t\sin 2t\right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{2}\sin 2t\,dt + \left[-\frac{1}{4}\cos 2t\right]_0^{2\pi}$$

$$= 0 - \left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} + 0$$

$$= 0$$

2. (a) $\underset{\wedge}{F} = x\underset{\wedge}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

(i) $C$ is $\underset{\sim}{r} = t\underset{\sim}{i} + t\underset{\sim}{j} + t\underset{\sim}{k} \qquad 0 \leq t \leq 1$

$$\frac{d\underset{\wedge}{r}}{dt} = \underset{\wedge}{i} + \underset{\wedge}{j} + \underset{\wedge}{k}$$

$$\therefore \int_{C} \underset{\sim}{F} \cdot d\underset{\wedge}{r} = \int_{C'} \underset{\wedge}{F} \cdot \frac{d\underset{\wedge}{r}}{dt}\,dt = \int_{C} x + y + z\,dt$$

$$= \int_0^1 (t + t + t)\,dt$$

$$= \left[\frac{3}{2}t^2\right]_0^1$$

$$= \underline{\underline{\frac{3}{2}}}$$

a)(ii) Curves are $C_1: \underset{\sim}{r} = t\underset{\sim}{j} \quad 0 \leq t \leq 1$

$C_2: \underset{\sim}{r} = t\underset{\sim}{i} + \underset{\sim}{j} + t\underset{\sim}{k} \quad 0 \leq t \leq 1$

$$\therefore \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} + \int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= \int_{C_1} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} dt + \int_{C_2} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} dt$$

$$= \int_{C_1} \underset{\sim}{F} \cdot \underset{\sim}{j} \, dt + \int_{C_2} \underset{\sim}{F} \cdot (\underset{\sim}{i} + \underset{\sim}{k}) dt$$

$$= \int_{C_1} y \, dt + \int_{C_2} (x + z) \, dt$$

$$= \int_0^1 t \, dt + \int_0^1 2t \, dt = \left[\tfrac{1}{2}t^2\right]_0^1 + \left[t^2\right]_0^1 = \underline{\underline{3/2}}$$

(b) $\underset{\sim}{F} = xy\underset{\sim}{i} + xz\underset{\sim}{j} + x^2\underset{\sim}{k}$

(i)

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} dt = \int_C xy + xz + x^2 \, dt = \int_0^1 t^2 + t^2 + t^2 dt$$

$$= \left[t^3\right]_0^1 = \underline{\underline{1}}$$

(ii)

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{C_1} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} dt + \int_{C_2} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} dt = \int_{C_1} xz \, dt + \int_{C_2} xy + x^2 dt$$

$$= \int_0^1 0 \, dt + \int_0^1 t + t^2 dt$$

$$= \left[\tfrac{1}{2}t^2 + \tfrac{1}{3}t^3\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3} = \underline{\underline{\frac{5}{6}}}$$

For (b) integral is different for the different curves so can not be conservative. For (a) is the integral is the same it is possible (but from this not certain) that the vector field is conservative

Taking curl

(a) $$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\underline{i} + 0\underline{j} + 0\underline{k} = 0$$

$\therefore$ conservative

(b) $$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & \cancel{xz} & x^2 \end{vmatrix} = -x\underline{i} - 2x\underline{j} + (z-x)\underline{k} \neq 0$$

$\therefore$ Not conservative

For (a) $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k} = \underline{\nabla}\phi$

$$\left.\begin{aligned} \therefore \frac{\partial \phi}{\partial x} = x &\Rightarrow \phi = \tfrac{1}{2}x^2 + f_1(y,z) \\ \frac{\partial \phi}{\partial y} = y &\Rightarrow \phi = \tfrac{1}{2}y^2 + f_2(x,z) \\ \frac{\partial \phi}{\partial z} = z &\Rightarrow \phi = \tfrac{1}{2}z^2 + f_3(x,y) \end{aligned}\right\} \Rightarrow \phi = \tfrac{1}{2}(x^2+y^2+z^2) + C$$

**MA240** **Vector Analysis Sheet 5** **PKS**

1. Evaluate
   (a) $\int_1^2 \int_2^4 (x+2y)\, dxdy$
   (b) $\int_1^2 \int_0^3 x^2y\, dxdy$
   (c) $\int_0^2 \int_0^{(2-y)} xy\, dxdy$ — sketch the region of integration

2. Calculate
$$\int_S \mathbf{F.dS}$$
   where
$$\mathbf{F} = 2x^2yz\mathbf{i} - xy^2z\mathbf{j} + 3xyz^2\mathbf{k}$$
   and $S$ is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

3. Calculate
$$\int_S \mathbf{F.dS}$$
   where
$$\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}$$
   and $S$ is the plane
$$2x + y + 2z = 2$$
   bounded by $x = 0$, $y = 0$ and $z = 0$, i.e. in the first octant.

MA240 Vector Analysis Solutions 5 PKS

1. (a) $\int_1^2 \int_2^4 (x+2y)\,dx\,dy = \int_1^2 \left[\tfrac{1}{2}x^2 + 2yx\right]_2^4 dy$

$= \int_1^2 (8+8y)-(2+4y)\,dy$

$= \int_1^2 6+4y\,dy = \left[6y+2y^2\right]_1^2 = 20-8$

$= \underline{\underline{12}}$

(b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 \left[\tfrac{1}{3}x^3\right]_0^3 y\,dy = \int_1^2 9y\,dy = \left[\tfrac{9}{2}y^2\right]_1^2 = 18 - \tfrac{9}{2}$

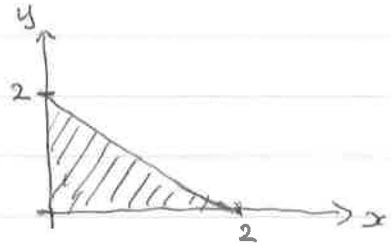
$= \underline{\underline{\tfrac{27}{2}}}$

(c) $\int_0^2 \int_0^{(2-y)} xy\,dx\,dy = \int_0^2 \left[\tfrac{1}{2}x^2\right]_0^{(2-y)} y\,dy$

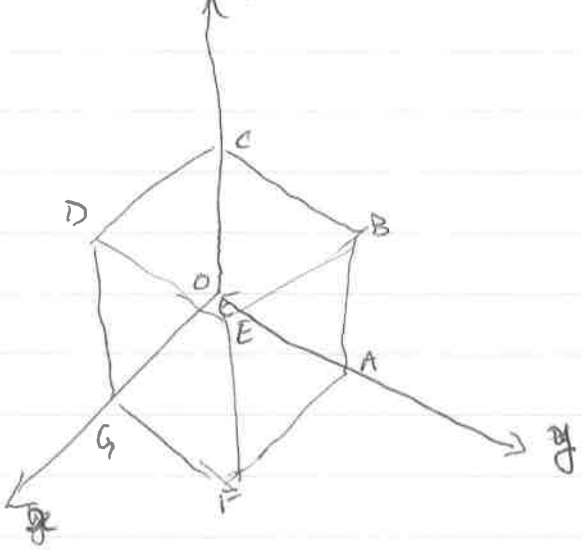
$= \int_0^2 \tfrac{1}{2}(2-y)^2 y\,dy = \int_0^2 \tfrac{1}{2}y^3 - 2y^2 + 2y\,dy$

$= \left[\tfrac{1}{8}y^4 - \tfrac{2}{3}y^3 + y^2\right]_0^2$

$= 2 - \tfrac{16}{3} + 4 = \underline{\underline{\tfrac{2}{3}}}$



2.



$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \sum_{\text{faces}} \int_{\text{face}} \underset{\sim}{F} \cdot d\underset{\sim}{S}$

$$\int_{OABC} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{OABC} \underset{\sim}{F} \cdot (-\underset{\sim}{i})\, dy\,dz = \int_{OABC} -2x^2yz\, dy\,dz = \underline{\underline{0}}$$ since $x = 0$ on this face

$$\int_{DEFG} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{DEFG} \underset{\sim}{F} \cdot \underset{\sim}{i}\, dy\,dz = \int_{DEFG} 2x^2yz\, dy\,dz = \int_0^1\int_0^1 2yz\, dy\,dz$$ since $x = 1$ on this face

$$= \int_0^1 \left[y^2\right]_0^1 z\, dz$$

$$= \int_0^1 z\, dz = \left[\tfrac{1}{2}z^2\right]_0^1 = \underline{\underline{\tfrac{1}{2}}}$$

$$\int_{OCDG} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{OCDG} \underset{\sim}{F} \cdot (-\underset{\sim}{j})\, dx\,dz = \int_{OCDG} xy^2z\, dx\,dz = \underline{\underline{0}}$$ since $y = 0$ on this face

$$\int_{ABEF} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{ABEF} \underset{\sim}{F} \cdot \underset{\sim}{j}\, dx\,dz = \int_{ABEF} -xy^2z\, dx\,dz = \int_0^1\int_0^1 -xz\, dx\,dz \quad (y=1)$$

$$= \int_0^1 -\left[\tfrac{1}{2}x^2\right]_0^1 z\, dz$$

$$= \int_0^1 -\tfrac{1}{2}z\, dz = \left[-\tfrac{1}{4}z^2\right]_0^1$$

$$= \underline{\underline{-\tfrac{1}{4}}}$$

$$\int_{OAFG} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{OAFG} \underset{\sim}{F} \cdot (-\underset{\sim}{k})\, dx\,dy = \int_{OAFG} -3xyz^2\, dx\,dy = \underline{\underline{0}}$$ since $z = 0$ on this face

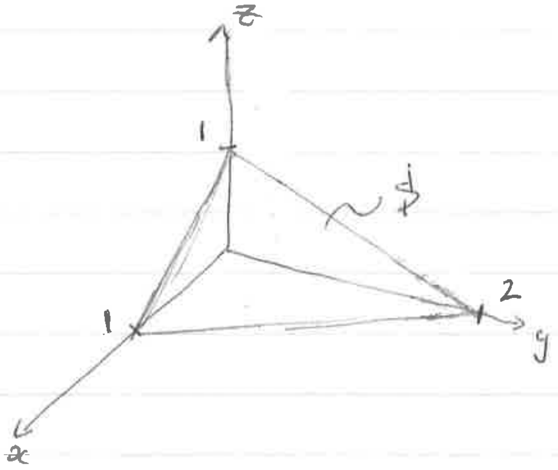
$$\int_{BCDE} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{BCDE} \underset{\sim}{F} \cdot \underset{\sim}{k}\, dx\,dy = \int_{BCDE} 3xyz^2\, dx\,dy = \int_0^1\int_0^1 3xy\, dx\,dy \quad (z=1)$$

$$= \int_0^1 \left[\tfrac{3}{2}x^2\right]_0^1 y\, dy$$

$$= \left[\tfrac{3}{4}y^2\right]_0^1 = \underline{\underline{3/4}}$$

$$\therefore \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + 3/4 = \underline{\underline{1}}$$

3.



$$\underset{\sim}{n} = \pm \frac{\underset{\sim}{\nabla}(2x+y+2z)}{|\underset{\sim}{\nabla}(2x+y+2z)|}$$

$$= \frac{2\underset{\sim}{i} + \underset{\sim}{j} + 2z}{\sqrt{4+1+4}}$$

$$= \frac{2}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$$

Project onto xy plane: $\underset{\sim}{n} \cdot \underset{\sim}{k} = \frac{2}{3}$ $\therefore\ dS' = \frac{dxdy}{|\underset{\sim}{n}\cdot\underset{\sim}{k}|}$

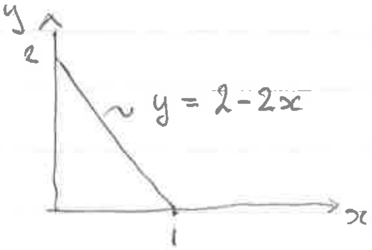
$$= \frac{3}{2}\,dxdy$$

$$\int_{S'} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_R \underset{\sim}{F}\cdot\underset{\sim}{n}\cdot\frac{3}{2}\,dxdy = \int_R (x^2\underset{\sim}{i} - y\underset{\sim}{j} + 2z\underset{\sim}{k})\cdot(\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \underset{\sim}{k})dxdy$$

$$= \int_R x^2 - \frac{1}{2}y + 2z\ dxdy$$

But $2z = 2 - 2x - y$ (from eqn of plane) $\therefore$

$$= \int_R x^2 - \frac{1}{2}y + 2 - 2x - y\ dxdy$$



$$= \int_R x^2 - 2x - \frac{3}{2}y + 2\ dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{2-2x} (x^2-2x+2) - \frac{3}{2}y\ dy\,dx$$

$$= \int_0^1 \Big[(x^2-2x+2)y - \frac{3}{4}y^2\Big]_0^{2-2x} dx$$

$$= \int_0^1 (x^2-2x+2)(2-2x) - \frac{3}{4}(2-2x)^2\ dx$$

$$= \int_0^1 -2x^3+6x^2-8x+4 - 3x^2+6x-3\ dx$$

$$= \int_0^1 -2x^3 + 3x^2 - 2x + 1 \, dx$$

$$= \left[ -\frac{1}{2}x^4 + x^3 - x^2 + x \right]_0^1$$

$$= -\frac{1}{2} + 1 - 1 + 1 \quad = \quad \underline{\underline{\frac{1}{2}}}$$

**MA240** **Vector Analysis Sheet 6** **PKS**

1. Verify Stoke's Theorem for
$$\mathbf{F} = zy\mathbf{i} + x\mathbf{j} + 4\mathbf{k}$$
over the unit square
$$(0 \le x \le 1) \times (0 \le y \le 1), \quad z = 0.$$

2. Verify Gauss' Divergence Theorem for the vector field
$$\mathbf{F} = xyz(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$
over the unit cube $(0 \le x \le 1) \times (0 \le y \le 1) \times (0 \le z \le 1)$.

3. By using Gauss' Divergence Theorem show that
$$\int_S \mathbf{F}.\mathbf{dS} = 4\pi$$
where
$$\mathbf{F} = 2x\mathbf{i} + x^2z^4\mathbf{j} + z\mathbf{k}$$
and $S$ is the surface of the sphere $x^2+y^2+z^2 = 1$. (Recall that $\int_V dV =$ volume of $V$.)

MA240 Vector Analysis Solutions 6 PKS

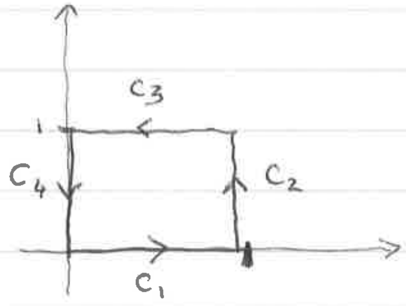
1. $$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot d\underset{\sim}{S}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zy & x & 4 \end{vmatrix} = 0\underset{\sim}{i} + y\underset{\sim}{j} + (1-z)\underset{\sim}{k}$$

Take +ve normal to be $\underset{\sim}{k}$ then $\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \int_S (y\underset{\sim}{j} + (1-z)\underset{\sim}{k}) \cdot \underset{\sim}{k}\, dx dy$

$$= \int_S (1-z)\, dx dy$$

But $z = 0$ on $S$ ∴ $= \int_S dx dy$ = area of $S$ = $\underline{\underline{1}}$



If $\underset{\sim}{n} = \underset{\sim}{k}$ then must integrate anticlockwise to be consistently orientated

On $C_1$ $d\underset{\sim}{r} = \underset{\sim}{i}\, dx$ since $y$ & $\overset{= 0}{z}$ both constant ∴ $\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 zy\, dx = \underline{\underline{0}}$

On $C_2$ $d\underset{\sim}{r} = \underset{\sim}{j}\, dy$ since $x (=1)$ & $z (=0)$ const ∴ $\int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 x dy = \int_0^1 dy = \underline{\underline{1}}$

On $C_3$ $d\underset{\sim}{r} = \underset{\sim}{i}\, dx$ $(y = 1,\ z = 0)$ ∴ $\int_{C_3} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_1^0 zy\, dx = \underline{\underline{0}}$

↖ N.B limits ↙

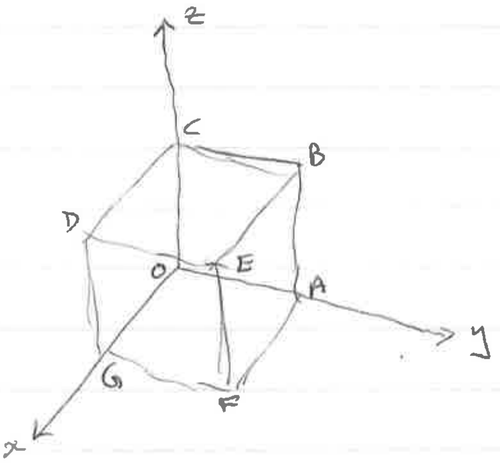
On $C_4$ $d\underset{\sim}{r} = \underset{\sim}{j}\, dy$ $(x = 0,\ z = 0)$ ∴ $\int_{C_4} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_1^0 x\, dy = 0$

Hence $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0 + 1 + 0 + 0 = \underline{\underline{1}}$ Hence Stoke's Theorem verified

2. $$\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\, dV = \int_S \underset{\sim}{F}\cdot d\underset{\sim}{S}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(xyz) = yz + 2xz + xy$$

$$\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\, dV = \int_0^1\int_0^1\int_0^1 yz + 2xz + xy \; dx\,dy\,dz = \int_0^1\int_0^1 \left[xyz + x^2 z + \tfrac{1}{2}x^2 y\right]_0^1 dy\,dz$$



$$= \int_0^1\int_0^1 yz + z + \tfrac{1}{2}y \; dy\,dz$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2 z + yz + \tfrac{1}{4}y^2\right]_0^1 dz$$

$$= \int_0^1 \tfrac{3}{2}z + \tfrac{1}{4}\, dz = \left[\tfrac{3}{4}z^2 + \tfrac{1}{4}z\right]_0^1$$

$$= \underline{\underline{1}}$$

$$\int_{OABC} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{OABC} \underset{\sim}{F}\cdot(-\underset{\sim}{i})\,dy\,dz = \int_{OABC} -xyz\, dy\,dz = 0 \text{ since } x = 0$$

$$\int_{DEFG} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{DEFG} \underset{\sim}{F}\cdot\underset{\sim}{i}\, dy\,dz = \int_{DEFG} yz\, dy\,dz = \int_0^1 z(\tfrac{1}{2}y^2]_0^1 dz = \left[\tfrac{1}{4}z^2\right]_0^1 = \tfrac{1}{4}$$

$$\int_{OCDG} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{OCDG} \underset{\sim}{F}\cdot(-\underset{\sim}{j})\, dx\,dz = \int_{OCDG} -2xyz\, dx\,dz = 0 \text{ since } y = 0$$

$$\int_{ABEF} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{ABEF} \underset{\sim}{F}\cdot\underset{\sim}{j}\, dx\,dz = \int_0^1\int_0^1 2xz\, dx\,dz = \int_0^1 [x^2]_0^1 z\, dz = \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{2}$$

$$\int_{OAFG} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{OAFG} \underset{\sim}{F}\cdot(-\underset{\sim}{k})\, dx\,dy = \int_{OAFG} -xyz\, dx\,dy = 0 \text{ since } z = 0$$

$$\int_{BCDE} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{BCDE} \underset{\sim}{F}\cdot\underset{\sim}{k}\, dx\,dy = \int_0^1\int_0^1 xy\, dx\,dy = \int_0^1 \left[\tfrac{1}{2}x^2\right]_0^1 y\,dy = \left[\tfrac{1}{4}y^2\right]_0^1 = \tfrac{1}{4}$$

$$\therefore \int_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = 0 + \tfrac{1}{4} + 0 + \tfrac{1}{2} + 0 + \tfrac{1}{4} = 1 = \int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\, dV$$

3.

$$\int_{S} \underset{\sim}{F} \circ d\underset{\sim}{S} = \int_{V} \underset{\sim}{\nabla} \circ \underset{\sim}{F} \; dV = \int_{V} \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(x^2 z^4) + \frac{\partial}{\partial z}(z) \; dV$$

$$= \int_{V} 2 + 0 + 1 \; dV$$

$$= 3 \int_{V} dV$$

$$= 3 \times \frac{4}{3} \pi \underset{\nwarrow_{=1}}{r^3} = \underline{\underline{4\pi}}$$

MA240 **Vector Analysis Assessment** PKS

**Due: 9am Friday 3rd December (at tutorial)**

1. (a) For what value of $\lambda$ is the vector field

$$\mathbf{F} = \lambda(\mathbf{a}.\mathbf{r})\mathbf{r} + (\mathbf{r}.\mathbf{r})\mathbf{a}$$

conservative, **a** being a constant vector.

(b) By substituting for **r** using question 3(a) from sheet 3 with $n = 2$ and for **a** using question 3(b) from sheet 3, give the associated scalar potential $\phi$ when $\lambda$ takes this value.

(c) Is $\phi$ harmonic - i.e. does it satisfy Laplace's equation $\nabla^2\phi = 0$?

2. If

$$\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$$

calculate

$$\int \mathbf{F}.d\mathbf{r} \qquad \text{from } (0,0,0) \text{ to } (1,1,1)$$

along

(a) the curve $x = t$, $y = t^2$, $z = t^3$,

(b) the straight line joining the two points,

(c) the three straight lines joing the two points via $(1, 0, 0)$ and $(1, 1, 0)$.

Is **F** a conservative vector field (give reasons)?

3. A solid consists of a hemisphere

$$x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.$$

Use Gauss' Divergence Theorem to show that

$$\int_{S_c} (e^z\mathbf{i} + (x^3 + 4y + z^4)\mathbf{j} + (1 - z)\mathbf{k}).d\mathbf{S} = 3\pi$$

where $S_c$ is the **curved** surface of the solid. (You may wish to remind yourselves of the formula for the volume of a sphere and that for the area of a disc.)

MA240 Vector Analysis Assesment Solutions PKS

1. (a) Conservative if $\underset{\sim}{\nabla}\times\underset{\sim}{F} = 0$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{\nabla}\times\{\lambda(\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{r} + (\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{a}\}$$

$$= \lambda\,\underset{\sim}{\nabla}\times\{(\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{r}\} + \underset{\sim}{\nabla}\times\{(\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{a}\}$$

$$= \lambda\{\underset{\sim}{\nabla}(\underset{\sim}{a}\cdot\underset{\sim}{r})\times\underset{\sim}{r} + (\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{\nabla}\times\underset{\sim}{r}\} + \underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{r})\times\underset{\sim}{a} + (\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{\nabla}\times\underset{\sim}{a}$$

Using Sheet 3 Q3

$$= \lambda\{\underset{\sim}{a}\times\underset{\sim}{r} + (\underset{\sim}{a}\cdot\underset{\sim}{r})0\} + 2\underset{\sim}{r}\times\underset{\sim}{a} + (\underset{\sim}{r}\cdot\underset{\sim}{r})0$$

$$= (\lambda-2)\underset{\sim}{a}\times\underset{\sim}{r} \qquad (\text{since } \underset{\sim}{r}\times\underset{\sim}{a} = -\underset{\sim}{a}\times\underset{\sim}{r})$$

$$= 0 \text{ when } \underline{\underline{\lambda = 2}} \text{ i.e. conservative}$$

(b)

$$\underset{\sim}{F} = \underset{\sim}{\nabla}\phi = 2(\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{r} + (\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{a}$$

$$= 2(\underset{\sim}{a}\cdot\underset{\sim}{r})\tfrac{1}{2}\underset{\sim}{\nabla}r^2 + \underset{\overset{\shortparallel}{r^2}}{(\underset{\sim}{r}\cdot\underset{\sim}{r})}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{a}) \qquad \text{using Q3 Sheet3}$$

$$= \underset{\sim}{\nabla}((\underset{\sim}{a}\cdot\underset{\sim}{r})r^2)$$

$$= \underset{\sim}{\nabla}((\underset{\sim}{a}\cdot\underset{\sim}{r})r^2 + c) \qquad \text{for any const } c$$

$$\therefore\ \phi = (\underset{\sim}{a}\cdot\underset{\sim}{r})r^2 + c$$

(c)

$$\nabla^2\phi = \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\phi = \underset{\sim}{\nabla}\cdot\underset{\sim}{F} = \underset{\sim}{\nabla}\cdot\{2(\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{r} + (\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{a}\}$$

$$= \underset{\sim}{r}\cdot\underset{\sim}{\nabla}(2\underset{\sim}{a}\cdot\underset{\sim}{r}) + 2(\underset{\sim}{a}\cdot\underset{\sim}{r})\underset{\sim}{\nabla}\cdot\underset{\sim}{r} + \underset{\sim}{a}\cdot\underset{\sim}{\nabla}\overset{\overset{r^2}{\shortparallel}}{(\underset{\sim}{r}\cdot\underset{\sim}{r})} + r^2\underset{\sim}{\nabla}\cdot\underset{\sim}{a}$$

$$= \underset{\sim}{r}\cdot(2\underset{\sim}{a}) + 2(\underset{\sim}{a}\cdot\underset{\sim}{r})3 + \underset{\sim}{a}\cdot(2\underset{\sim}{r}) + 0$$

$$= 10\,\underset{\sim}{a}\cdot\underset{\sim}{r} \neq 0$$

2. (a) $\underset{\sim}{r} = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ on $C_1$ $\quad t \in [0,1]$

$$\int_{C_1} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{C} \underset{\sim}{F}\cdot\frac{d\underset{\sim}{r}}{dt}\,dt = \int_{C_1}(xy\underset{\sim}{i}+yz\underset{\sim}{j}+z^2\underset{\sim}{k})\cdot(\underset{\sim}{i}+2t\underset{\sim}{j}+3t^2\underset{\sim}{k})\,dt$$

$$= \int (t^3\underset{\sim}{i}+t^5\underset{\sim}{j}+t^6\underset{\sim}{k})\cdot(\underset{\sim}{i}+2t\underset{\sim}{j}+3t^2\underset{\sim}{k})\,dt$$

$$= \int_0^1 t^3+2t^6+3t^8\,dt$$

$$= \left[\tfrac{1}{4}t^4+\tfrac{2}{7}t^7+\tfrac{3}{9}t^9\right]_0^1 = \tfrac{1}{4}+\tfrac{2}{7}+\tfrac{1}{3} = \underline{\underline{\tfrac{73}{84}}}$$

(b) $\underset{\sim}{r} = t\underset{\sim}{i} + t\underset{\sim}{j} + t\underset{\sim}{k}$ on $C_2$ $\quad t \in [0,1]$

$$\int_{C_2} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{C_2} \underset{\sim}{F}\cdot\frac{d\underset{\sim}{r}}{dt}\,dt = \int_{C_2}(t^2\underset{\sim}{i}+t^2\underset{\sim}{j}+t^2\underset{\sim}{k})\cdot(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k})\,dt$$

$$= \int_0^1 3t^2\,dt = \left[t^3\right]_0^1 = \underline{\underline{1}}$$

(c) $C = C_1 + C_2 + C_3$

$C_1$: $\underset{\sim}{r} = t_1\underset{\sim}{i}$ $\quad t_1 \in [0,1]$ ; $\quad C_2$: $\underset{\sim}{r} = \underset{\sim}{i} + t_2\underset{\sim}{j}$ $\quad t_2 \in [0,1]$

$C_3$: $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + t_3\underset{\sim}{k}$ $\quad t_3 \in [0,1]$

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{C_1} \underset{\sim}{F}\cdot\underset{\sim}{i}\,dt_1 + \int_{C_2} \underset{\sim}{F}\cdot\underset{\sim}{j}\,dt_2 + \int_{C_3} \underset{\sim}{F}\cdot\underset{\sim}{k}\,dt_3$$

$$= \int_{C_1} x\underset{\overset{\uparrow}{0}}{y}\,dt_1 + \int_{C_2} y\underset{\overset{\uparrow}{0}}{z}\,dt_2 + \int_{C_3} z^2\,dt_3$$

$$= 0 + 0 + \int_0^1 t_3^2\,dt_3 = \left[\tfrac{1}{3}t_3^3\right]_0^1 = \underline{\underline{\tfrac{1}{3}}}$$

<u>Not</u> conservative since path dependent

3.

$$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV$$

ie. $$\int_{S_c} \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_{S_F} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV$$

$$\therefore \int_{S_c} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV - \int_{S_F} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$\underset{\sim}{F} = e^z \underset{\sim}{i} + (x^3 + 4y + z^4) \underset{\sim}{j} + (1 - z) \underset{\sim}{k}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0 + 4 + (-1) = 3 \qquad \therefore \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = 3 \int_V dV = 3 \times \frac{1}{2} \times \frac{4}{3} \pi$$

$$= \underline{\underline{2\pi}}$$

On $S_F$ $\underset{\sim}{n} = -\underset{\sim}{k}$ (pointing outwards)

$$\int_{S_F} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{S_F} -(1 - z)\, dx\,dy = -\int_{S_F} dx\,dy \quad \text{since } z = 0 \text{ on } S_F$$

$$= -\pi$$

$$\therefore \int_{S_c} \underset{\sim}{F} \cdot d\underset{\sim}{S} = 2\pi - (-\pi) = \underline{\underline{3\pi}}$$