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Vector calculus
Solutions to exercises in appendix C of the notes

Exercise C.1. (Exercise 4 in assignment 1 of MA2VC 2013–14.) Show that, for any unit vector $\hat{\boldsymbol{n}}$, and any vector $\vec{\mathbf{w}}$ perpendicular to $\hat{\boldsymbol{n}}$, the identity

$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{w}}$$

holds true. You can make use of the identities in Section 1.1. Demonstrate this identity for the vectors

$$\hat{\boldsymbol{n}} = \frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{w}} = 3\hat{\boldsymbol{\jmath}}.$$

From identity (4) in Exercise 1.12 of the notes, the definition of perpendicularity ($\hat{\boldsymbol{n}} \cdot \vec{\mathbf{w}} = 0$), and the unit length of $\hat{\boldsymbol{n}}$ ($\sqrt{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}} = |\hat{\boldsymbol{n}}| = 1$), we obtain the required identity:

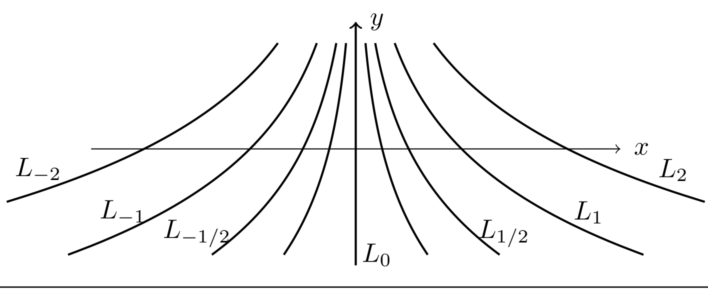
$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\mathbf{w}}) \overset{(4)}{=} \hat{\boldsymbol{n}}(\underbrace{\vec{\mathbf{w}} \cdot \hat{\boldsymbol{n}}}_{=0}) - \vec{\mathbf{w}}(\underbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}}_{=|\hat{\boldsymbol{n}}|^2=1}) = 0\hat{\boldsymbol{n}} - 1\vec{\mathbf{w}} = \vec{\mathbf{0}} - 1\vec{\mathbf{w}} = -\vec{\mathbf{w}}.$$

We demonstrate the identity for the two vectors given by using twice the vector product definition (2):

$$\begin{aligned}
\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\mathbf{w}}) &= \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right) \times \left(\left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right) \times 3\hat{\boldsymbol{\jmath}}\right) \\
&\overset{(2)}{=} \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right) \times \left(\left(0 - \frac{4}{5}3\right)\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + \left(\frac{3}{5}3 - 0\right)\hat{\boldsymbol{k}}\right) \\
&= \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right) \times \left(-\frac{12}{5}\hat{\boldsymbol{\imath}} + \frac{9}{5}\hat{\boldsymbol{k}}\right) \overset{(2)}{=} 0\hat{\boldsymbol{\imath}} + \left(\frac{4}{5}\left(\frac{-12}{5}\right) - \frac{3}{5}\frac{9}{5}\right)\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}} = -\frac{75}{25}\hat{\boldsymbol{\jmath}} = -3\hat{\boldsymbol{\jmath}} = -\vec{\mathbf{w}}.
\end{aligned}$$

Exercise C.2. Compute and draw the level sets of the scalar field $f(\vec{\mathbf{r}}) = x\mathrm{e}^y$.

The field f depends only on the variables x and y, so we can draw the level sets as level curves in the xy-plane. The level curves are defined as the sets $L_\lambda = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ f(\vec{\mathbf{r}}) = \lambda\}$ for some $\lambda \in \mathbb{R}$. For our specific field they read $\boxed{L_\lambda = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ x\mathrm{e}^y = \lambda\}}$, which can be rewritten as $L_\lambda = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ x = \lambda \mathrm{e}^{-y}\}$. For $\lambda = 0$ we have $L_0 = \{x = 0\}$, i.e. the vertical line corresponding to the y-axis. For any other value of λ, L_λ is the graph of the real function $x = \lambda\mathrm{e}^{-y}$, where x and y are swapped with respect to the usual convention, thus the graph is tilted 45 degrees across the line $\{x = y\}$. Some level sets are shown in figure.

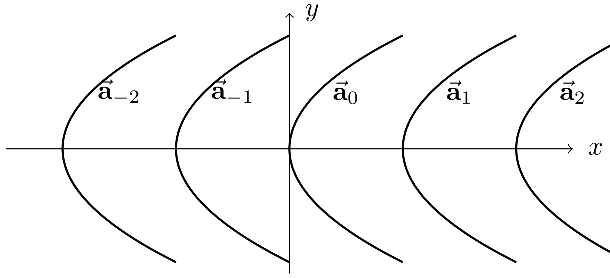


Exercise C.3. Draw some paths of the curves

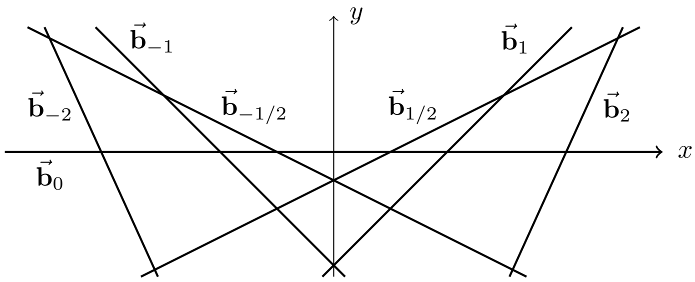
$$\vec{\mathbf{a}}(t) = (\lambda + t^2)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{b}}(t) = t\hat{\boldsymbol{\imath}} + (\lambda t - \lambda^2)\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{c}}(t) = t\cos(t+\lambda)\hat{\boldsymbol{\imath}} + t\sin(t+\lambda)\hat{\boldsymbol{\jmath}}$$

for different values of $\lambda \in \mathbb{R}$. Are they level sets of any scalar field?

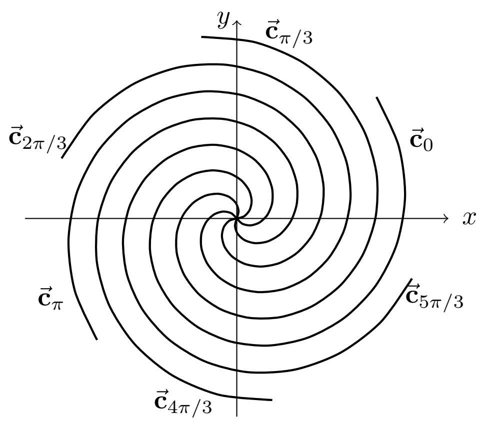
We first note that all curves are planar, as the component in the z direction is zero. All points in the paths of the curves $\vec{\mathbf{a}}(t) = (\lambda + t^2)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}$ satisfy $x = \lambda + t^2 = \lambda + y^2$ (recall that the coordinates x and y of the points correspond to the coefficients of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the curve expression). As in the previous exercise, the sets $x = \lambda + y^2$ are tilted graphs, in this case of parabolas. This graphs are translated copies of the same path and they do not intersect each other, so they might be level sets for some scalar field. We can write λ as a function of x and y as $\lambda = x - t^2 = x - y^2$, thus the scalar field $f = x - y^2$ takes value λ exactly on the points of the path of $\vec{\mathbf{a}}_\lambda$. This is the same as saying that the curves are level sets for f.



Also the paths of the curves $\vec{\mathbf{b}}$ can be written as graphs, indeed $y = \lambda x - \lambda^2 = \lambda(x - \lambda)$. So the paths of $\vec{\mathbf{b}}$ are straight lines with slope λ, intersecting the x-axis in the point $\lambda\hat{\boldsymbol{\imath}}$. If we draw some of them, as you can see in the figure, we note that each two of these lines intersect each other (of course having different slopes they are not parallel, so they must intersect). Since level lines of a field never intersect each other (because a field can have only one value in a given point), these lines are not level set of any field. (Try to prove the following: there exists a parabola Γ, such that all points below Γ belong to the paths of two different $\vec{\mathbf{b}}$'s, the points on Γ belong to only one of these lines, the points above to none.)



The curves $\vec{\mathbf{c}}$ are a bit harder to draw. We first note that the curves for a given λ and for $\lambda + 2\pi n$, with $n \in \mathbb{Z}$, coincide, so we can consider the case $0 \leq \lambda < 2\pi$. We assume $t > 0$. We start from simpler curves: we recall from Remark 1.24 that the path of $\vec{\mathbf{d}}(t) = \cos(t + \lambda)\hat{\boldsymbol{\imath}} + \sin(t + \lambda)\hat{\boldsymbol{\jmath}}$ is the circle of radius one centred at the origin, run starting from the point $\cos(\lambda)\hat{\boldsymbol{\imath}} + \sin(\lambda)\hat{\boldsymbol{\jmath}}$ in anti-clockwise direction. The curve $\vec{\mathbf{c}}$ satisfies $\vec{\mathbf{c}}(t) = t\vec{\mathbf{d}}(t)$, thus its points have same direction of those of $\vec{\mathbf{d}}$ but magnitude multiplied by t, i.e. magnitude increasing from 0 to ∞ for $t > 0$. The paths we obtain are the Archimedean spirals in the figure. This spirals intersect each other only at the origin, so they can be level set of a field not defined at the origin[1].



Exercise C.4. Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ be two unit vectors orthogonal to each other. Show that the curve $\vec{\mathbf{a}}(t) = \hat{\mathbf{u}}\cos t + \hat{\mathbf{w}}\sin t$ lies on the unit sphere for all $t \in \mathbb{R}$. Which curve is this?

The unit sphere is the set of points with magnitude equal to one. The magnitude of a point in the path of $\vec{\mathbf{a}}$ is

$$|\vec{\mathbf{a}}(t)| = \sqrt{\vec{\mathbf{a}}(t) \cdot \vec{\mathbf{a}}(t)} = \sqrt{\cos^2 t \underbrace{\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}}_{=1} + 2\cos t \sin t \underbrace{\hat{\mathbf{u}} \cdot \hat{\mathbf{w}}}_{=0,\text{ because orthogonal}} + \sin^2 t \underbrace{\hat{\mathbf{w}} \cdot \hat{\mathbf{w}}}_{=1}} = \sqrt{\cos^2 t + \sin^2 t} = 1,$$

[1]After we learn about polar coordinates in Section 2.3.1, we will see that these paths are level sets for the scalar field $f = \theta - r$, θ and r being the polar coordinates. The field f is discontinuous not only at the origin but also on the half line $\{y = 0, x \leq 0\}$.

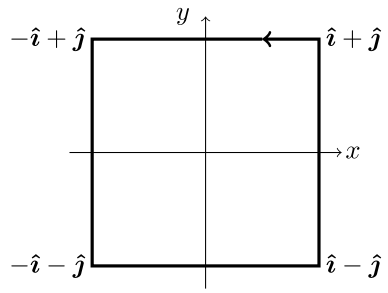
thus all its points belong to the unit sphere. We can also see that all points described by $\vec{\mathbf{a}}$ lie in the plane spanned by $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{w}}$. The intersection between the sphere and a plane through its centre is a circle: the path of $\vec{\mathbf{a}}$ is the circle of unit radius, centred at the origin in the plane spanned by $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{w}}$.

Exercise C.5. Draw the path of the curve

$$\vec{\mathbf{a}}(t) = \frac{(\cos t - \sin t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath}}{|\cos t| + |\sin t|}, \qquad 0 \le t \le 2\pi.$$

This is nothing else than the parametrisation of the boundary of the square with vertices $\hat{\imath}+\hat{\jmath}$ (at $t=0$), $-\hat{\imath}+\hat{\jmath}$ (at $t=\pi/2$), $-\hat{\imath}-\hat{\jmath}$ (at $t=\pi$), $\hat{\imath}-\hat{\jmath}$ (at $t=3\pi/2$). This can be verified as follows. For $0 \le t \le \pi/2$ we have $\cos t \ge 0$ and $\sin t \ge 0$, so $\cos t + \sin t = |\cos t| + |\sin t|$ and $\vec{\mathbf{a}}(t) = \frac{\cos t - \sin t}{|\cos t|+|\sin t|}\hat{\imath} + 1\hat{\jmath}$; the second component of the curve is identically one, which means that this part of the path is part of the straight line $\{y=1\}$. For $\pi/2 \le t \le \pi$ we have $a_1(t) = -1$ so this part of the path lies in the line $\{x=-1\}$ and similarly for the remaining two sides. You can check it out in Matlab with `VCplotter`, by calling:

```
VCplotter(1,1, @(t) (cos(t)-sin(t))/(abs(cos(t))+abs(sin(t))), ...
          @(t) (cos(t)+sin(t))/(abs(cos(t))+abs(sin(t))), 0, 2*pi);
```



Exercise C.6. Find a quotient rule for the gradient, i.e. give a formula for $\vec{\nabla}(f/g)$ where f and $g \neq 0$ are scalar fields. Hint: use parts 2 and 3 of Proposition 1.33.

We first note that f/g can be written as the product of the scalar fields f and $1/g$, so we can apply product rule (13) to it. This involves the gradient of $1/g$: to compute this we write $1/g = G(g)$, where $G : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the real function $G(t) = 1/t$. So we can compute the gradient $\vec{\nabla}(1/g) = \vec{\nabla}G(g)$ using chain rule (14). Putting everything together, we have:

$$\boxed{\vec{\nabla}\left(\frac{f}{g}\right)} = \vec{\nabla}\left(f\frac{1}{g}\right) \overset{(13)}{=} \frac{1}{g}\vec{\nabla}f + f\vec{\nabla}\left(\frac{1}{g}\right) = \frac{1}{g}\vec{\nabla}f + f\vec{\nabla}G(g) \overset{(14)}{=} \frac{1}{g}\vec{\nabla}f + fG'(g)\vec{\nabla}g = \frac{1}{g}\vec{\nabla}f + f\frac{(-1)}{g^2}\vec{\nabla}g$$
$$\boxed{= \frac{g\vec{\nabla}f - f\vec{\nabla}g}{g^2}}.$$

Exercise C.7.

- Compute the vector field $\hat{\boldsymbol{n}}$ of unit length defined on the sphere of radius $R > 0$, which is orthogonal to the sphere itself and points outwards. (This is called "outward-pointing unit normal vector field".) Hint: use part 4 of Proposition 1.33.
- Compute the outward-pointing unit normal vector field on the boundary of the parabolic "cup" $\{x^2 + y^2 < z < 1\}$.

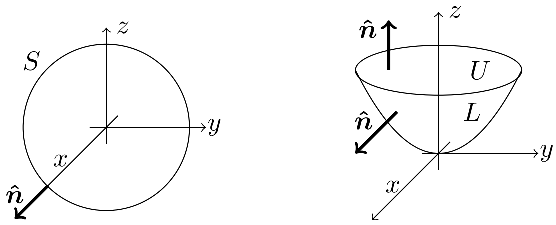
For both sets we have to find a vector field $\hat{\boldsymbol{n}}$ defined on the given surface that satisfies three conditions: (i) is orthogonal to the surface, (ii) has length one, (iii) points "outwards". The general strategy is the following: we first compute a non-zero field that satisfies (i) using part 4 of Proposition 1.33; then we divide it by its magnitude so that condition (ii) is satisfied; and finally we check whether the field obtained points outwards, if so we are done, otherwise we multiply it by -1.

The sphere S of radius R is the set of points lying at distance R from the origin, thus it can be written with the equation $S = \{|\vec{\mathbf{r}}| = R\}$ or, equivalently, $S = \{|\vec{\mathbf{r}}|^2 = R^2\}$. To proceed as in Example 1.38 we want to find a scalar field f such that S is a level set for f. The simplest choice is $f = |\vec{\mathbf{r}}|^2$, so that

$S = \{f = R^2\}$ is the level set for f corresponding to the value R^2 (the choice $\widetilde{f} = |\vec{\mathbf{r}}|$ might seem even simpler, but the gradient of f is slightly simpler to compute than that of $\widetilde{f}$). The gradient of f and its magnitude are

$$\vec{\nabla} f = \vec{\nabla}(|\vec{\mathbf{r}}|^2) = \vec{\nabla}(x^2 + y^2 + z^2) = 2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2y\hat{\boldsymbol{k}} = 2\vec{\mathbf{r}}, \qquad |\vec{\nabla} f| = 2|\vec{\mathbf{r}}| = 2R \text{ on } S.$$

In the last equality we used that all points of S have magnitude equal to R. By part 4 of Proposition 1.33, $\vec{\nabla} f$ is orthogonal to S in each point, and $\vec{\nabla} f/|\vec{\nabla} f| = \vec{\mathbf{r}}/R$ is orthogonal to S and has unit length. We only have to verify whether it points outwards: we choose a point on the sphere, e.g. $R\hat{\boldsymbol{\imath}}$ and we compute $\vec{\nabla} f(R\hat{\boldsymbol{\imath}})/|\vec{\nabla} f(\hat{\boldsymbol{\imath}})| = R\hat{\boldsymbol{\imath}}/R = \hat{\boldsymbol{\imath}}$, which indeed points outwards. Thus $\boxed{\hat{\boldsymbol{n}} = \vec{\nabla} f/(2R) = \vec{\mathbf{r}}/R}$ is the outward-pointing unit normal vector field on S.



The parabolic cup $C = \{x^2 + y^2 < z < 1\}$ is defined by two inequalities: $x^2 + y^2 < z$ and $z < 1$. We reach its boundary when one of the inequalities is turned into an equality, thus the boundary is made of two parts: a lower part $L = \{z = x^2 + y^2, z < 1\}$ (the cup) and an upper one $U = \{z = 1, x^2 + y^2 < z\}$ (the lid), this should be intuitive from the figure. We have to compute separately the unit normal on each of the two parts. The upper part is the easiest: it is a subset of the plane $\{z = 1\}$, whose unit normal is $\hat{\boldsymbol{k}}$, which points upwards, thus out of C, thus it is the desired $\hat{\boldsymbol{n}}$ (we can also see that is the gradient of the field $f_U = z$, and U is a subset of the level set $f_U = 1$). For the lower part L we need to find a field f_L such that L is a subset of one of its level sets. From the definition $L = \{z = x^2 + y^2, z < 1\}$ we see that we can take $f_L = z - x^2 - y^2$ because f_L has constant value 0 on L. The gradient of f_L and its magnitude are

$$\vec{\nabla} f_L = -2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}, \qquad |\vec{\nabla} f_L| = \sqrt{4x^2 + 4y^2 + 1},$$

thus $\vec{\nabla} f_L/|\vec{\nabla} f_L| = (-2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})/\sqrt{4x^2 + 4y^2 + 1}$ is orthogonal to S and has unit length. However this has positive z component, so from the picture of C we see that it points inside C. To find the outward-pointing unit normal we just reverse its direction by changing the sign of all its components. Thus we conclude:

$$\boxed{\hat{\boldsymbol{n}} = \begin{cases} \hat{\boldsymbol{k}} & \text{on } U, \\ -\frac{\vec{\nabla} f_L}{|\vec{\nabla} f_L|} = \frac{2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}}{\sqrt{4x^2 + 4y^2 + 1}} & \text{on } L. \end{cases}}$$

Exercise C.8.

- Fix $\vec{\mathbf{F}} = e^{xyz}\hat{\boldsymbol{\imath}}$. Show that $\vec{\mathbf{F}}$ and $\vec{\nabla} \times \vec{\mathbf{F}}$ are orthogonal to each other at each point of $\mathbb{R}^3$.
- Prove that any vector field with only one non-zero component is orthogonal to its own curl.
- Find a vector field that is not orthogonal to its own curl.

(i) We just compute $\vec{\nabla} \times \vec{\mathbf{F}}$ and its scalar product with $\vec{\mathbf{F}}$:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{\partial F_1}{\partial z}\hat{\boldsymbol{\jmath}} - \frac{\partial F_2}{\partial y}\hat{\boldsymbol{k}} = xye^{xyz}\hat{\boldsymbol{\jmath}} - xze^{xyz}\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}} \cdot \vec{\nabla} \times \vec{\mathbf{F}} = e^{xyz}\hat{\boldsymbol{\imath}} \cdot (xye^{xyz}\hat{\boldsymbol{\jmath}} - xze^{xyz}\hat{\boldsymbol{k}}) = 0.$$

(ii) The solution of the first point already shows that the precise expression of a vector field is not relevant to be orthogonal to its curl, provided only the first component of the field is non-zero.

Let $\vec{\mathbf{G}}$ be a vector field with only one non-zero component. Without loss of generality, we can assume that the non-zero component is the first one: $\vec{\mathbf{G}} = G_1\hat{\boldsymbol{\imath}}$. The definition (23) of the curl shows that the first component of the curl of $\vec{\mathbf{G}}$ is zero, thus the scalar product with $\vec{\mathbf{G}}$ itself is zero:

$$\vec{\mathbf{G}} = G_1\hat{\boldsymbol{\imath}} \qquad \Rightarrow \qquad \vec{\nabla} \times \vec{\mathbf{G}} = \frac{\partial G_1}{\partial z}\hat{\boldsymbol{\jmath}} - \frac{\partial G_2}{\partial y}\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{G}} \cdot \vec{\nabla} \times \vec{\mathbf{G}} = G_1\, 0 + 0\frac{\partial G_1}{\partial z} - 0\frac{\partial G_2}{\partial y} = 0.$$

(iii) We have seen in the second point of the exercise that a field can be non-orthogonal to its curl only if it has at least two non-zero components. How to construct a field as required? We look for the simplest example. One of the simplest vector fields with non-zero curl is $z\hat{\boldsymbol{\imath}}$, which has constant curl equal to $\hat{\boldsymbol{\jmath}}$. If we add a constant term to $z\hat{\boldsymbol{\imath}}$ we do not change its curl, so we choose to add a term that gives a non-zero scalar product with the curl of $z\hat{\boldsymbol{\imath}}$, i.e. $\hat{\boldsymbol{\jmath}}$. The simplest choice is thus

$$\boxed{\vec{\mathbf{F}} = z\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}} \qquad \Rightarrow \qquad \nabla \times \vec{\mathbf{F}} = \hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{F}} \cdot \nabla \times \vec{\mathbf{F}} = (z\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \cdot \hat{\boldsymbol{\jmath}} = 1 \neq 0.$$

Exercise C.9. Find a scalar field f defined on the complement of the x axis, whose direction of maximal increase is $(\vec{\mathbf{r}} - x\hat{\boldsymbol{\imath}})/(y^2+z^2)^{1/2}$.

The "x axis" is the set $\{\vec{\mathbf{r}} \in \mathbb{R}^3, y = z = 0\}$, thus the "complement of the x axis" is the set

$$D = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3, \text{ such that either } y \neq 0 \text{ or } z \neq 0\big\} = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3, y^2 + z^2 \neq 0\big\}.$$

By part 5 of Proposition 1.33, the direction of maximal increase of a scalar field f is the direction of the gradient: $\vec{\nabla} f/|\vec{\nabla} f|$. So the exercise just asks to find a scalar field f defined on D such that

$$\frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{\vec{\mathbf{r}} - x\hat{\boldsymbol{\imath}}}{(y^2+z^2)^{1/2}} = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}} - x\hat{\boldsymbol{\imath}}}{(y^2+z^2)^{1/2}} = \frac{y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}}{(y^2+z^2)^{1/2}}.$$

Since the denominator of this fraction is nothing else than the magnitude of the numerator, it is enough to find f such that $\vec{\nabla} f = y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$. The field $\boxed{f = \frac{1}{2}(y^2+z^2)}$ satisfies this requirement. (Actually the coefficient $\frac{1}{2}$ is not necessary, as every positive multiple of f has the same direction of maximal increase.)

Exercise C.10. (Exercises 1–2 in assignment 1 of MA2VC 2013–14.) Prove the following identity:

$$\vec{\nabla} \cdot (\vec{\mathbf{r}} fg) = 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g)f,$$

where $\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$ is the position vector, and $f(\vec{\mathbf{r}})$ and $g(\vec{\mathbf{r}})$ are two scalar fields.
Hint: you can either use the vector differential identities in the boxes of Propositions 1.52 and 1.55, or the definitions of gradient and divergence.

Demonstrate the above identity for the scalar fields $f = e^{xy}$ and $g = y^4 z$.

At least three different ways of proving the identity are possible.
(Version i) The easiest proof is to compute $\vec{\nabla} \cdot \vec{\mathbf{r}} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ and use the vector identities, first separating $\vec{\mathbf{r}}$ from the two fields in the parenthesis:

$$\vec{\nabla} \cdot (\vec{\mathbf{r}} fg) \overset{(29)}{=} (\vec{\nabla} \cdot \vec{\mathbf{r}}) fg + \vec{\mathbf{r}} \cdot \vec{\nabla}(fg) \overset{(28)}{=} 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g)f.$$

(Version ii) One can use the same vector identity twice, first separating one of the fields (e.g. f) from the second field and $\vec{\mathbf{r}}$, and then separating the latter two objects:

$$\vec{\nabla} \cdot (\vec{\mathbf{r}} fg) \overset{(29)}{=} (\vec{\nabla} \cdot (\vec{\mathbf{r}} g))f + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g \overset{(29)}{=} \big((\vec{\nabla} \cdot \vec{\mathbf{r}})g + \vec{\mathbf{r}} \cdot \vec{\nabla} g\big)f + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g = 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g)f.$$

(Version iii) One can directly use the definitions of gradient and divergence, and the formula for the partial derivative of a product:

$$\begin{aligned}
\vec{\nabla} \cdot (\vec{\mathbf{r}} fg) &= \vec{\nabla} \cdot \big(xfg\hat{\boldsymbol{\imath}} + yfg\hat{\boldsymbol{\jmath}} + zfg\hat{\mathbf{k}}\big) \\
&= \frac{\partial(xfg)}{\partial x} + \frac{\partial(yfg)}{\partial y} + \frac{\partial(zfg)}{\partial z} \\
&= \Big(fg + xg\frac{\partial f}{\partial x} + xf\frac{\partial g}{\partial x}\Big) + \Big(fg + gy\frac{\partial f}{\partial y} + fy\frac{\partial g}{\partial y}\Big) + \Big(fg + zg\frac{\partial f}{\partial z} + zf\frac{\partial g}{\partial z}\Big) \\
&= 3fg + \Big(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\Big)g + \Big(x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + z\frac{\partial g}{\partial z}\Big)f \\
&= 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f)g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g)f.
\end{aligned}$$

All versions are correct, but if you learn how to properly use the vector identities as in the first two versions above, you will avoid mistakes and save a lot of time and effort.

Demonstration. We have to evaluate both the left-hand side and the right-hand side of the identity using the given fields f and g. We should not use the vector identities here, otherwise we just repeat the first part of the exercise. From the definition of divergence, the left-hand side of the identity reads:

$$\begin{aligned}\vec{\nabla}\cdot(\vec{\mathbf{r}}fg) &= \vec{\nabla}\cdot\big(xy^4ze^{xy}\hat{\boldsymbol{\imath}}+y^5ze^{xy}\hat{\boldsymbol{\jmath}}+y^4z^2e^{xy}\hat{\boldsymbol{k}}\big)\\ &=\big(y^4ze^xy+xy^5ze^{xy}\big)+\big(5y^4ze^{xy}+xy^5ze^{xy}\big)+2y^4ze^{xy}=8y^4ze^xy+2xy^5ze^{xy}.\end{aligned}$$

The right-hand side reads:

$$\begin{aligned}3fg+(\vec{\mathbf{r}}\cdot\vec{\nabla}f)g+(\vec{\mathbf{r}}\cdot\vec{\nabla}g)f &= 3y^4ze^{xy}+\big(\vec{\mathbf{r}}\cdot(ye^{xy}\hat{\boldsymbol{\imath}}+xe^{xy}\hat{\boldsymbol{\jmath}})\big)y^4z+\big(\vec{\mathbf{r}}\cdot(4y^3z\hat{\boldsymbol{\jmath}}+y^4\hat{\boldsymbol{k}})\big)e^{xy}\\ &=3y^4ze^{xy}+2xy^5ze^{xy}+5y^4ze^{xy}\\ &=8y^4ze^xy+2xy^5ze^{xy},\end{aligned}$$

thus the two expressions are equal to each other and the desired identity is demonstrated.

Exercise C.11. Prove identity (34) in the notes (curl of a vector product).

We prove the identity for the first component only, for the others it follows similarly. To expand the two sides of the identity we use the definitions of curl, vector product and advection operator. To prove that the expressions obtained coincide, we use the usual product rule for partial derivatives and we have to play around a bit with the terms, grouping them according to which field is derived and to the sign, and adding and subtracting the missing terms.

$$\begin{aligned}&\big(\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}})\big)_1\\
&\overset{(23)}{=}\frac{\partial}{\partial y}(\vec{\mathbf{F}}\times\vec{\mathbf{G}})_3-\frac{\partial}{\partial z}(\vec{\mathbf{F}}\times\vec{\mathbf{G}})_2 && \text{definition of curl}\\
&\overset{(2)}{=}\frac{\partial}{\partial y}(F_1G_2-F_2G_1)-\frac{\partial}{\partial z}(F_3G_1-F_1G_3) && \text{definition of vector product}\\
&\overset{(8)}{=}\frac{\partial F_1}{\partial y}G_2+\frac{\partial G_2}{\partial y}F_1-\frac{\partial F_2}{\partial y}G_1-\frac{\partial G_1}{\partial y}F_2-\frac{\partial F_3}{\partial z}G_1-\frac{\partial G_1}{\partial z}F_3+\frac{\partial F_1}{\partial z}G_3+\frac{\partial G_3}{\partial z}F_1\\
&\qquad\text{product rule for partial derivatives}\\
&=\Big(\frac{\partial G_2}{\partial y}F_1+\frac{\partial G_3}{\partial z}F_1\Big)-\Big(\frac{\partial F_2}{\partial y}G_1+\frac{\partial F_3}{\partial z}G_1\Big)+\Big(\frac{\partial F_1}{\partial y}G_2+\frac{\partial F_1}{\partial z}G_3\Big)-\Big(\frac{\partial G_1}{\partial y}F_2+\frac{\partial G_1}{\partial z}F_3\Big)\\
&\qquad\text{reorder and collect terms}\\
&=\Big(\frac{\partial G_1}{\partial x}F_1+\frac{\partial G_2}{\partial y}F_1+\frac{\partial G_3}{\partial z}F_1\Big)-\Big(\frac{\partial F_1}{\partial x}G_1+\frac{\partial F_2}{\partial y}G_1+\frac{\partial F_3}{\partial z}G_1\Big)\\
&\quad+\Big(\frac{\partial F_1}{\partial x}G_1+\frac{\partial F_1}{\partial y}G_2+\frac{\partial F_1}{\partial z}G_3\Big)-\Big(\frac{\partial G_1}{\partial x}F_1+\frac{\partial G_1}{\partial y}F_2+\frac{\partial G_1}{\partial z}F_3\Big)\\
&\qquad\text{add and subtract terms }\frac{\partial G_1}{\partial x}F_1\text{ and }\frac{\partial F_1}{\partial x}G_1\\
&\overset{(22)}{=}(\vec{\nabla}\cdot\vec{\mathbf{G}})F_1-(\vec{\nabla}\cdot\vec{\mathbf{F}})G_1+(\vec{\mathbf{G}}\cdot\nabla)F_1-(\vec{\mathbf{F}}\cdot\nabla)G_1\\
&\qquad\text{definition of divergence and advection operator (footnote 7)}\\
&\overset{(35)}{=}\Big((\vec{\nabla}\cdot\vec{\mathbf{G}})\vec{\mathbf{F}}-(\vec{\nabla}\cdot\vec{\mathbf{F}})\vec{\mathbf{G}}+(\vec{\mathbf{G}}\cdot\vec{\nabla})\vec{\mathbf{F}}-(\vec{\mathbf{F}}\cdot\vec{\nabla})\vec{\mathbf{G}}\Big)_1.\end{aligned}$$

It might be slightly simpler to first expand both left- and right-hand side, and then to verify that they lead to the same expression, so they are equal.

Exercise C.12.

- Find a vector field $\vec{\mathbf{F}}$ such that: $\vec{\mathbf{F}}$ is irrotational, $\vec{\mathbf{F}}$ is not solenoidal, all its streamlines (recall caption to Figure 7) are straight lines passing through the origin.
  This last condition means that $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ is parallel to $\vec{\mathbf{r}}$ in all points $\vec{\mathbf{r}}\neq\vec{\mathbf{0}}$.
- [Harder!] Find a vector field $\vec{\mathbf{G}}$ (defined on a suitable domain) such that: $\vec{\mathbf{G}}$ is solenoidal, $\vec{\mathbf{G}}$ is not irrotational, all its streamlines are straight lines passing through the origin.

For each $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$, the value of the field $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ has to be parallel to $\vec{\mathbf{r}}$ itself, so $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \alpha\vec{\mathbf{r}}$. The constant of proportionality α may depend on $\vec{\mathbf{r}}$, so it is a scalar field and we can write the condition on the streamlines as $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}\alpha(\vec{\mathbf{r}})$ for all $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$ and for some scalar field α. We need to find α such that $\vec{\mathbf{F}} = \vec{\mathbf{r}}\alpha$ is irrotational but not solenoidal. The divergence and the curl of $\vec{\mathbf{F}}$ can be computed by using the vector identities of Section 1.4:

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = \vec{\nabla} \cdot (\vec{\mathbf{r}}\alpha) \overset{(29)}{=} (\vec{\nabla} \cdot \vec{\mathbf{r}})\alpha + \vec{\mathbf{r}} \cdot \vec{\nabla}\alpha = 3\alpha + \vec{\mathbf{r}} \cdot \vec{\nabla}\alpha,$$
$$\vec{\nabla}\times\vec{\mathbf{F}} = \vec{\nabla} \times (\vec{\mathbf{r}}\alpha) \overset{31)}{=} (\vec{\nabla} \times \vec{\mathbf{r}})\alpha + \vec{\mathbf{r}} \times \vec{\nabla}\alpha = \vec{\mathbf{r}} \cdot \vec{\nabla}\alpha,$$

where we also used that $\vec{\nabla} \cdot \vec{\mathbf{r}} = 3$ and $\vec{\nabla}\times\vec{\mathbf{r}} = \vec{\mathbf{0}}$, as we learned in Exercise 1.62. We need to find a scalar field α such that the first expression above is non-zero and the second one is zero. The simplest choice is the constant field $\alpha = 1$, so that its gradient is zero and $\boxed{\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}}$ is simply the position vector field.

For the second part of the exercise, we need to find a scalar field β such that the vector field $\vec{\mathbf{G}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}\beta(\vec{\mathbf{r}})$ is solenoidal but not irrotational, i.e.

$$\vec{\nabla} \cdot \vec{\mathbf{G}} = 3\beta + \vec{\mathbf{r}} \cdot \vec{\nabla}\beta = 3\beta + x\frac{\partial\beta}{\partial x} + y\frac{\partial\beta}{\partial y} + z\frac{\partial\beta}{\partial z} = 0, \qquad \vec{\nabla}\times\vec{\mathbf{G}} = \vec{\mathbf{r}} \cdot \vec{\nabla}\beta \neq \vec{\mathbf{0}}.$$

We look for the simplest possible β, namely a field depending only on the variable x, so that $\frac{\partial\beta}{\partial y} = \frac{\partial\beta}{\partial z} = 0$. With this choice, the identity above becomes $\frac{\partial\beta}{\partial x} = -3x^{-1}\beta$, whose solution is $\beta = x^{-3}$ (or a constant multiple of it). Thus a field satisfying the requests of the exercise is $\boxed{\vec{\mathbf{G}} = \vec{\mathbf{r}}/x^3 = x^{-2}\hat{\boldsymbol{\imath}} + yx^{-3}\hat{\boldsymbol{\jmath}} + zx^{-3}\hat{\boldsymbol{k}}}$, which is defined on $\{x \neq 0\}$, the complement of the yz-plane.

Exercise C.13. Compute the total derivative of the field $f(\vec{\mathbf{r}}) = (x^2 + y^2)^\beta$, for $\beta \in \mathbb{R}$, evaluated along the curve $\vec{\mathbf{a}}(t) = \cos 2\pi t\hat{\boldsymbol{\imath}} + \sin 2\pi t\hat{\boldsymbol{\jmath}} + e^{-t^2}\hat{\boldsymbol{k}}$.
Interpret the result geometrically.

Instead of using chain rule (37), we compute the composition $f \circ \vec{\mathbf{a}}$ and see that its derivative is zero:

$$f\big(\vec{\mathbf{a}}(t)\big) = (\cos^2 2\pi t + \sin^2 2\pi t)^\beta = 1^\beta = 1, \qquad \frac{d\big(f(\vec{\mathbf{a}})\big)}{dt} = \frac{d1}{dt} = \boxed{0}.$$

The geometric reason of this is simple: the value of the field $\vec{\mathbf{f}}(\vec{\mathbf{r}})$ depends only on the distance $(x^2+y^2)^{1/2}$ between $\vec{\mathbf{r}}$ and the z-axis, so it is constant on each cylinder centred around this axis. In other words, the cylinders $\{x^2 + y^2 = R^2\}$ are level sets for f. The first two components of the curve $\vec{\mathbf{a}}$ describe the unit circle in the xy-plane, so the path of $\vec{\mathbf{a}}$ lies on the vertical cylinder above it. So the value of the field is constant along the whole path of $\vec{\mathbf{a}}$ and its total derivative is zero.

Exercise C.14. Let $F : (0, \infty) \to \mathbb{R}$ be a smooth function. Let $f : \mathbb{R}^3 \setminus \{\vec{\mathbf{0}}\} \to \mathbb{R}$ be the scalar field defined by $f(\vec{\mathbf{r}}) = F(|\vec{\mathbf{r}}|)$.

- Compute $\vec{\nabla} f$.
- Prove that $\Delta f = F''(|\vec{\mathbf{r}}|) + \frac{2F'(|\vec{\mathbf{r}}|)}{|\vec{\mathbf{r}}|}$. Hint: use one of the identities of Section 1.4.

The gradient of f can be computed using the chain rule (14), where the inner scalar field is the magnitude field $|\vec{\mathbf{r}}|$, and the formula $\vec{\nabla}|\vec{\mathbf{r}}| = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}$ which was computed in Exercise 1.35:

$$\vec{\nabla} f = \vec{\nabla}\big(F(|\vec{\mathbf{r}}|)\big) \overset{(14)}{=} F'(|\vec{\mathbf{r}}|)\vec{\nabla}(|\vec{\mathbf{r}}|) = \boxed{F'(|\vec{\mathbf{r}}|)\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}}. \qquad (\spadesuit)$$

The identity at the second item can be proved using twice the formula just obtained, once to $F(|\vec{\mathbf{r}}|)$ and once to $F'(|\vec{\mathbf{r}}|)$:

$$\Delta f \overset{(24)}{=} \vec{\nabla} \cdot (\vec{\nabla} f)$$
$$\overset{(\spadesuit)}{=} \vec{\nabla} \cdot \left(F'(|\vec{\mathbf{r}}|)\frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}\right) \qquad \text{apply } (\spadesuit) \text{ to } F(|\vec{\mathbf{r}}|)$$

$$\overset{(29)}{=} \big(\vec\nabla F'(|\vec{\mathbf r}|)\big)\cdot\frac{\vec{\mathbf r}}{|\vec{\mathbf r}|} + F'(|\vec{\mathbf r}|)\vec\nabla\cdot\frac{\vec{\mathbf r}}{|\vec{\mathbf r}|} \qquad \text{product rule for divergence}$$

$$\overset{(\spadesuit)}{=} F''(|\vec{\mathbf r}|)\frac{\vec{\mathbf r}}{|\vec{\mathbf r}|}\cdot\frac{\vec{\mathbf r}}{|\vec{\mathbf r}|} + F'(|\vec{\mathbf r}|)2\frac{1}{|\vec{\mathbf r}|} \qquad \text{apply } (\spadesuit) \text{ to } F'(|\vec{\mathbf r}|),\text{ divergence of } |\vec{\mathbf r}|^{-1}\vec{\mathbf r} \text{ from Ex. 1.62}$$

$$= F''(|\vec{\mathbf r}|) + \frac{2}{|\vec{\mathbf r}|}F'(|\vec{\mathbf r}|).$$

Exercise C.15. Let $\vec{\mathbf F}:\mathbb R^3\to\mathbb R^3$ be a *conservative force* field, i.e. a conservative field with $\vec{\mathbf F} = -\vec\nabla\psi$, where the scalar field ψ is the *potential energy*. Let a particle of *mass* m move with trajectory $\vec{\mathbf a}(t)$ according to Newton's law $\vec{\mathbf F}(\vec{\mathbf a}(t)) = m\frac{d^2\vec{\mathbf a}}{dt^2}(t)$ (force equals mass times acceleration). Define the *kinetic energy* of the particle $T(t) = \frac12 m\left|\frac{d\vec{\mathbf a}}{dt}(t)\right|^2$. Prove that the *total energy* $E(t) = \psi(\vec{\mathbf a}(t)) + T(t)$ of the particle is constant in time.

Hint: you only need to use chain and product rule for the total derivative.

The product rule (36) for the total derivatives gives: $\frac{d}{dt}(|\vec{\mathbf b}|^2) = \frac{d(\vec{\mathbf b}\cdot\vec{\mathbf b})}{dt} = \vec{\mathbf b}\cdot\frac{d\vec{\mathbf b}}{dt} + \frac{d\vec{\mathbf b}}{dt}\cdot\vec{\mathbf b} = 2\vec{\mathbf b}\cdot\frac{d\vec{\mathbf b}}{dt}$. Applying this formula to to $\vec{\mathbf b} = \frac{d\vec{\mathbf a}}{dt}$, using the definitions and the chain rule we have:

$$\frac{dE}{dt} = \frac{d\big(\psi(\vec{\mathbf a}) + T\big)}{dt} \qquad \text{definition of } E = \psi(\vec{\mathbf a}) + T$$

$$= \frac{d\big(\psi(\vec{\mathbf a})\big)}{dt} + \frac12 m\frac{d}{dt}\left(\left|\frac{d\vec{\mathbf a}}{dt}\right|^2\right) \qquad \text{definition of } T = \frac12 m\left|\frac{d\vec{\mathbf a}}{dt}\right|^2$$

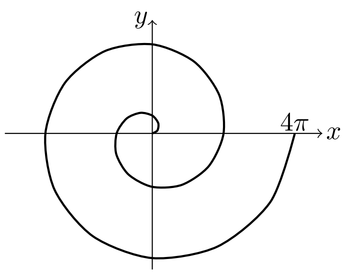
$$= \vec\nabla\psi(\vec{\mathbf a})\cdot\frac{d\vec{\mathbf a}}{dt} + \frac12 m\left(2\frac{d\vec{\mathbf a}}{dt}\cdot\frac{d^2\vec{\mathbf a}}{dt^2}\right) \qquad \text{chain rule (37)}$$

$$= -F\big(\vec{\mathbf a}(t)\big)\cdot\frac{d\vec{\mathbf a}}{dt} + \frac{d\vec{\mathbf a}}{dt}\cdot F\big(\vec{\mathbf a}(t)\big) = 0. \qquad \text{conservative field } \vec{\mathbf F} = -\vec\nabla\psi \text{ and Newton's law.}$$

The total energy E is constant as its derivative is zero.

Exercise C.16. Consider the curve $\vec{\mathbf a}(t) = t\cos t\,\hat{\boldsymbol\imath} + t\sin t\,\hat{\boldsymbol\jmath}$ for $t\in[0,4\pi]$ and the scalar field $f(\vec{\mathbf r}) = 1/\sqrt{1+|\vec{\mathbf r}|^2}$. Draw the path Γ of the curve, compute the line integral $\int_\Gamma f\,\mathrm ds$ and the total derivative of f evaluated along $\vec{\mathbf a}$.

We have already drawn the path of $\vec{\mathbf a}$ in Exercise C.3 (it was called $\vec{\mathbf c}$, with $\lambda = 0$), see the figure.



To compute the required line integral and the total derivative, we first calculate the value of f along $\vec{\mathbf a}$, the total derivative of $\vec{\mathbf a}$ and its magnitude:

$$|\vec{\mathbf a}(t)|^2 = t^2\cos^2 t + t^2\sin^2 t = t^2, \qquad f\big(\vec{\mathbf a}(t)\big) = \frac{1}{\sqrt{1+|\vec{\mathbf a}(t)|^2}} = \frac{1}{\sqrt{1+t^2}},$$

$$\frac{d\vec{\mathbf a}}{dt} = (\cos t - t\sin t)\hat{\boldsymbol\imath} + (\sin t + t\cos t)\hat{\boldsymbol\jmath},$$

$$\left|\frac{d\vec{\mathbf a}}{dt}\right|^2 = \cos^2 t + t^2\sin^2 t - 2t\sin t\cos t + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t = 1 + t^2,$$

$$\int_\Gamma f\,\mathrm ds \overset{(41)}{=} \int_0^{4\pi} f\big(\vec{\mathbf a}(t)\big)\left|\frac{d\vec{\mathbf a}}{dt}\right|\mathrm dt = \int_0^{4\pi}\frac{1}{\sqrt{1+t^2}}\sqrt{1+t^2}\,\mathrm dt = \int_0^{4\pi} 1\,\mathrm dt = \boxed{4\pi},$$

$$\frac{d\big(f(\vec{\mathbf a})\big)}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right) = \boxed{\frac{-t}{(1+t^2)^{3/2}}}.$$

Alternatively, one could write $f(\vec{\mathbf{r}}) = G(|\vec{\mathbf{r}}|^2)$ for $G(s) = \frac{1}{\sqrt{1+s}}$, whose derivative is $G'(s) = \frac{-1}{2(1+s)^{3/2}}$, and compute the total derivative of f along $\vec{\mathbf{a}}$ with the chain rules (37) and (14):

$$
\begin{aligned}
\frac{d\big(f(\vec{\mathbf{a}})\big)}{dt} &\overset{(37)}{=} \vec{\nabla} f(\vec{\mathbf{a}}) \cdot \frac{d\vec{\mathbf{a}}}{dt} \\
&\overset{(14)}{=} G'(|\vec{\mathbf{a}}|^2)\, \vec{\nabla}(|\vec{\mathbf{a}}|^2) \cdot \frac{d\vec{\mathbf{a}}}{dt} \\
&= \frac{-1}{2(1+|\vec{\mathbf{a}}|^2)^{3/2}} 2\vec{\mathbf{a}} \cdot \Big((\cos t - t \sin t)\hat{\boldsymbol{\imath}} + (\sin t + t\cos t)\hat{\boldsymbol{\jmath}} \Big) && \text{using } \vec{\nabla}(|\vec{\mathbf{r}}|^2) = 2\vec{\mathbf{r}} \\
&= \frac{-1}{2(1+t^2)^{3/2}} 2\big(t\cos^2 t - t^2 \cos t \sin t + t \sin^2 t + t^2 \sin t \cos t \big) && \text{from } \vec{\mathbf{a}}(t) = t\cos t\, \hat{\boldsymbol{\imath}} + t \sin t\, \hat{\boldsymbol{\jmath}} \\
&= \frac{-t}{(1+t^2)^{3/2}}.
\end{aligned}
$$

This solution, however, is definitely more complicated than the direct derivation of $f(\vec{\mathbf{a}})$ as done above.

Exercise C.17. Compute the line integral of the vector field $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = y^2\hat{\boldsymbol{\imath}} + 2xy\hat{\boldsymbol{\jmath}}$ along the five curves of Example 2.10.

We recall the five curves and their total derivatives:

$$
\begin{aligned}
&\vec{\mathbf{a}}_A(t) = t\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{a}}_A}{dt} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, && 0 \le t \le 1, \\
&\vec{\mathbf{a}}_B(t) = \sin t\, \hat{\boldsymbol{\imath}} + \sin t\, \hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{a}}_B}{dt} = \cos t\, \hat{\boldsymbol{\imath}} + \cos t\, \hat{\boldsymbol{\jmath}}, && 0 \le t \le \frac{\pi}{2}, \\
&\vec{\mathbf{a}}_C(t) = \mathrm{e}^{-t}\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{a}}_C}{dt} = -\mathrm{e}^{-t}\hat{\boldsymbol{\imath}} - \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}}, && 0 \le t < \infty, \\
&\vec{\mathbf{a}}_D(t) = t^2\hat{\boldsymbol{\imath}} + t^4\hat{\boldsymbol{\jmath}}, && \frac{d\vec{\mathbf{a}}_D}{dt} = 2t\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}}, && 0 \le t \le 1, \\
&\vec{\mathbf{a}}_E(t) = \begin{cases} t\hat{\boldsymbol{\imath}}, \\ \hat{\boldsymbol{\imath}} + (t-1)\hat{\boldsymbol{\jmath}}, \end{cases} && \frac{d\vec{\mathbf{a}}_E}{dt} = \begin{cases} \hat{\boldsymbol{\imath}}, & 0 \le t \le 1, \\ \hat{\boldsymbol{\jmath}}, & 1 < t \le 2. \end{cases}
\end{aligned}
$$

We directly compute the integrals using formula (44):

$$
\begin{aligned}
&\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_A = \int_0^1 \big(y^2(t)\hat{\boldsymbol{\imath}} + 2x(t)y(t)\hat{\boldsymbol{\jmath}} \big) \cdot (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 \big(t^2\hat{\boldsymbol{\imath}} + 2t^2\hat{\boldsymbol{\jmath}} \big) \cdot (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 3t^2\,\mathrm{d}t = t^3\Big|_0^1 = 1, \\
&\int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_B = \int_0^{\frac{\pi}{2}} \big(\sin^2 t\, \hat{\boldsymbol{\imath}} + 2\sin^2 t\, \hat{\boldsymbol{\jmath}} \big) \cdot (\cos t\, \hat{\boldsymbol{\imath}} + \cos t\, \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} 3\sin^2 t \cos t\,\mathrm{d}t = \sin^3 t\Big|_0^{\frac{\pi}{2}} = 1, \\
&\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_C = \int_0^{\infty} \big(\mathrm{e}^{-2t}\hat{\boldsymbol{\imath}} + 2\mathrm{e}^{-2t}\hat{\boldsymbol{\jmath}} \big) \cdot (-\mathrm{e}^{-t}\hat{\boldsymbol{\imath}} - \mathrm{e}^{-t}\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^{\infty} (-3\mathrm{e}^{-3t})\,\mathrm{d}t = \mathrm{e}^{-3t}\Big|_0^{\infty} = 0 - 1 = -1, \\
&\int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_0^1 (t^8\hat{\boldsymbol{\imath}} + 2t^6\hat{\boldsymbol{\jmath}}) \cdot (2t\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 (2t^9 + 8t^9)\,\mathrm{d}t = t^{10}\Big|_0^1 = 1, \\
&\int_{\Gamma_E} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = \int_0^1 (0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}) \cdot \hat{\boldsymbol{\imath}}\,\mathrm{d}t + \int_1^2 \big((t-1)^2\hat{\boldsymbol{\imath}} + 2(t-1)\hat{\boldsymbol{\jmath}} \big) \cdot \hat{\boldsymbol{\jmath}}\,\mathrm{d}t = 0 + \int_1^2 2(t-1)\,\mathrm{d}t = (t^2 - 2t)\Big|_1^2 \\
&\qquad = (4-4) - (1-2) = 1.
\end{aligned}
$$

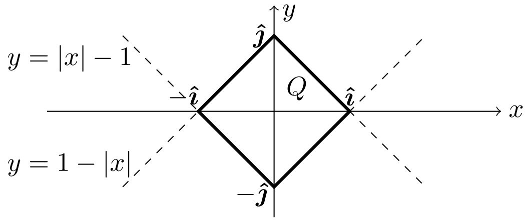
We have $\boxed{\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_A = \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_B = -\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_C = \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_{\Gamma_E} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = 1.}$

All integrals on the paths from $\vec{\mathbf{0}}$ to $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ have value one, while the only one on the path Γ_C from $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ to $\vec{\mathbf{0}}$ has opposite value.

After learning about the fundamental theorem of vector calculus in Section 2.1.3, we can solve the exercise in a much simpler way. We first note that $\varphi = xy^2$ is a scalar potential for $\vec{\mathbf{F}}$, so by the fundamental theorem of vector calculus (49)

$$
\begin{aligned}
&\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_A = \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_B = \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_D = \int_{\Gamma_E} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_E = \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) - \varphi(\vec{\mathbf{0}}) = 1 - 0 = 1, \\
&\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{a}}_C = \varphi(\vec{\mathbf{0}}) - \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = -1.
\end{aligned}
$$

Exercise C.18. Compute the integral of the scalar field $f = x + y$ on the square Q with vertices $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, -\hat{\boldsymbol{\imath}}, -\hat{\boldsymbol{\jmath}}$.

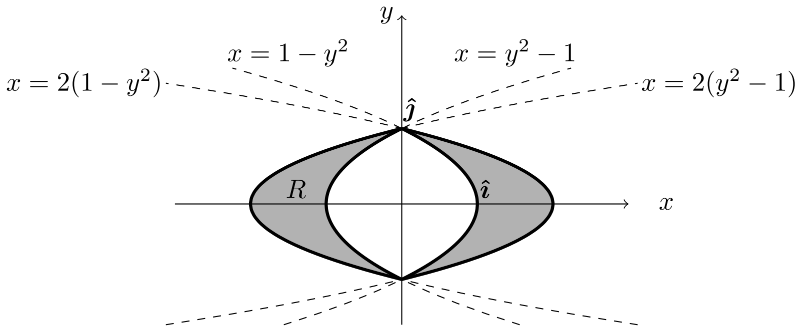


The square Q (in the figure) is the portion of the plane comprised between the graphs of $|x| - 1$ and $1 - |x|$. can be written as $Q = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ |x| - 1 < y < 1 - |x|\}$. We use the iterated integral formula (53) to compute the integral:[2]

$$\iint_Q (x+y)\,\mathrm{d}s = \int_{-1}^{1}\int_{|x|-1}^{1-|x|}(x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^{1}\Big(xy + \frac{y^2}{2}\Big)\Big|_{y=|x|-1}^{1-|x|}\mathrm{d}x = \int_{-1}^{1} 2x(1-|x|)\,\mathrm{d}x$$
$$= \int_{-1}^{0}(2x+2x^2)\,\mathrm{d}x + \int_{0}^{1}(2x-2x^2)\,\mathrm{d}x = \Big(x^2 + \frac{2x^3}{3}\Big)\Big|_{-1}^{0} + \Big(x^2 - \frac{2x^3}{3}\Big)\Big|_{0}^{1} = 0 - 1 + \frac{2}{3} + 1 - \frac{2}{3} - 0 = \boxed{0}$$

Exercise C.19. Compute the area of the "eye" domain $\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, 1 - y^2 < |x| < 2(1-y^2)\}$.

Hint: draw a sketch of the domain. This domain is neither y-simple nor x-simple, to compute the double integral you need to use two of the fundamental properties of integrals in list (40) in the notes.



The domain $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}, 1 - y^2 < |x| < 2(1-y^2)\}$ is the shaded region in the figure, and is made of two components. It is bounded by the paths $x = 1 - y^2$ and $x = 2(1-y^2)$ for positive x and by $x = y^2 - 1$ and $x = 2(y^2 - 1)$ for negative x. If we name R_+ and R_- the components of R in the positive- and in the negative-x half planes, respectively, we obviously have $\mathrm{Area}(R) = \mathrm{Area}(R_-) + \mathrm{Area}(R_+)$. This is a consequence of the additivity of integrals in (40), since $R = R_- \cup R_+$ and $R_- \cap R_+ = \emptyset$. Since the two subregions are symmetric across the y-axis, we have $\mathrm{Area}(R) = 2\mathrm{Area}(R_+)$. Moreover, R_+ is an x-simple domain, so we can compute its area using an iterated integral with x as inner variable:

$$\begin{aligned}\mathrm{Area}(R) &= 2\mathrm{Area}(R_+)\\ &= 2\iint_{R_+} 1\,\mathrm{d}A\\ &= 2\int_{-1}^{1}\int_{1-y^2}^{2(1-y^2)} 1\,\mathrm{d}x\,\mathrm{d}y = 2\int_{-1}^{1}(1-y^2)\,\mathrm{d}y = 2\Big(y - \frac{1}{3}y^3\Big)\Big|_{-1}^{1} = 2\Big(1 - \frac{1}{3} + 1 - \frac{1}{3}\Big) = \boxed{\frac{8}{3}}.\end{aligned}$$

---

[2] Could we guess that the integral vanishes without computing it? Yes, using the symmetry of the domain Q and of the integrand f. The square Q is symmetric with respect to the y axis, so the integral of x, which is odd in the variable x, is zero. The square is also symmetric with respect to the x axis, so the integral of y, which is odd in the variable y, is zero. Finally, by linearity, the integral of $f = x + y$ is the sum of two vanishing integrals so it must vanish as well. This is true for all terms containing odd powers of either x or y: for example in the integral over Q of $g = x^3 + xy^5 - x^2y + x^{37}y^{50} + x^4$, only the last term contributes to the integral as it is the only one with no odd powers:

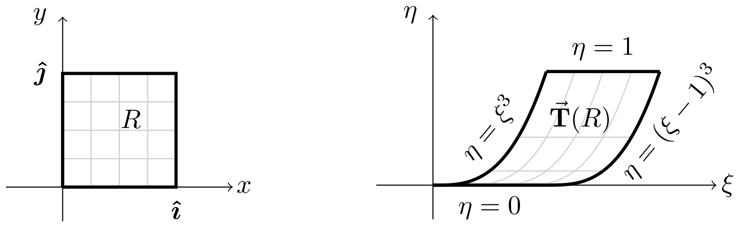
$$\iint_Q (x^3 + xy^5 - x^2y + x^{37}y^{50} + x^4)\,\mathrm{d}s = \iint_Q x^4\,\mathrm{d}s = \int_{-1}^{1}\int_{|x|-1}^{1-|x|} x^4\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^{1} 2x^4(1-|x|)\,\mathrm{d}x$$
$$= \int_{-1}^{0}(2x^4 + 2x^5)\,\mathrm{d}x + \int_{0}^{1}(2x^4 - 2x^5)\,\mathrm{d}x = \Big(\frac{2x^5}{5} + \frac{x^6}{3}\Big)\Big|_{-1}^{0} + \Big(\frac{2x^5}{5} - \frac{2x^6}{3}\Big)\Big|_{0}^{1} = \frac{2}{5} - \frac{1}{3} + \frac{2}{5} - \frac{1}{3} = \frac{2}{15}.$$

Exercise C.20. Compute the area of $\vec{\mathbf{T}}(R)$, where $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\, 0 < x < 1,\, 0 < y < 1\}$ and $\vec{\mathbf{T}}(x,y) = (x+y)\hat{\boldsymbol{\xi}} + y^3\hat{\boldsymbol{\eta}}$.

This is a straightforward application of the change of variables formula (57), where now the roles of (x,y) and of (ξ,η) are swapped:

$$\xi = x+y, \quad \eta = y^3, \qquad J\vec{\mathbf{T}} = \begin{pmatrix} \frac{\partial(x+y)}{\partial x} & \frac{\partial(x+y)}{\partial y} \\ \frac{\partial(y^3)}{\partial x} & \frac{\partial(y^3)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 3y^2 \end{pmatrix}, \qquad \frac{\partial(\xi,\eta)}{\partial(x,y)} = \det(J\vec{\mathbf{T}}) = 3y^2,$$

$$\mathrm{Area}\big(\vec{\mathbf{T}}(R)\big) = \iint_{\vec{\mathbf{T}}(R)} 1 \,\mathrm{d}\xi\,\mathrm{d}\eta = \iint_R \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 3y^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 1\,\mathrm{d}x \int_0^1 3y^2\,\mathrm{d}y = \boxed{1}.$$

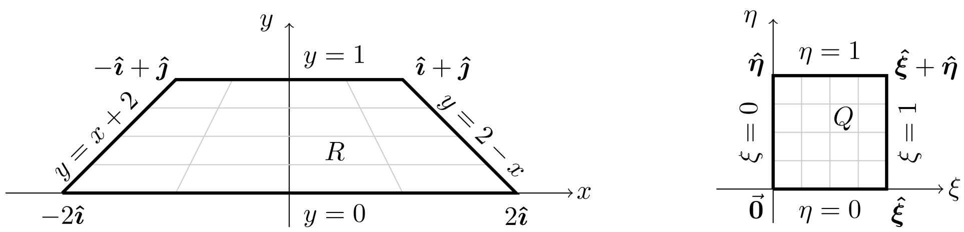


If we want to draw the picture of $\vec{\mathbf{T}}(R)$ (not required by the question), we observe where the four sides of the unit square R are mapped to, by substituting their four equations into $\xi = x+y$ and $\eta = y^3$:

$$\{x=0\} \mapsto \{\eta = \xi^3\}, \qquad \{x=1\} \mapsto \{\eta = (\xi-1)^3\}, \qquad \{y=0\} \mapsto \{\eta = 0\}, \qquad \{y=1\} \mapsto \{\eta = 1\}.$$

Exercise C.21. Find a change of variables from the trapezoid R with vertices $-2\hat{\boldsymbol{\imath}}, 2\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}, -\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}$ to the square Q with vertices $\vec{\mathbf{0}}, \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\eta}}$.

Use the change of variables you have found to compute $\iint_R \mathrm{e}^{(2-y)^2}\,\mathrm{d}x\,\mathrm{d}y$.



We first draw the regions R, in the xy-plane, and Q, in the $\xi\eta$-plane, as in figure. We write also the equations of the straight lines containing the four sides of the two quadrilaterals, as in figure. There exist many changes of variables from R to Q: guided by the knowledge of the shape of the regions we have to choose the simplest possible change of variables. Intuitively, we can imagine a deformation of R that preserves the vertical coordinate of each point and squeezes the horizontal coordinate to fit into Q. "Preserving the vertical coordinate" is the same as setting $\eta = y$. This ensures that the lower side $\{y=0\}$ of R is mapped to the lower side $\{\eta = 0\}$ of Q and the upper side $\{y=1\}$ of R is mapped to the upper side $\{\eta=1\}$ of Q. With this choice we only need to find the expression of either $\eta(x,y)$ or $x(\xi,\eta)$. These transformations have to ensure that

the line $\xi = 0$ is image of the line $x - y = -2$, and the line $\xi = 1$ is image of the line $x + y = 2$.

In terms of the fields $x(\xi,\eta)$ and $y(\xi,\eta)$, this is the same as requiring

$$\begin{cases} x(0,\eta) - y(0,\eta) = -2, \\ x(1,\eta) + y(1,\eta) = 2, \end{cases} \quad \overset{\text{using } y(\xi,\eta)=\eta}{\Longrightarrow} \quad \begin{cases} x(0,\eta) - \eta = -2, \\ x(1,\eta) + \eta = 2, \end{cases} \quad \Longrightarrow \quad \begin{cases} x(0,\eta) = \eta - 2, \\ x(1,\eta) = 2 - \eta. \end{cases}$$

We need to find $x(\xi,\eta)$ such that $x(0,\eta) = \eta - 2$ and $x(1,\eta) = 2 - \eta$ for all $0 < \eta < 1$. Since the value of x at $\xi = 0$ and $\xi = 1$ are opposite to each other, we can look for some $f(\xi)$ such that $x(\xi,\eta) = (2-\eta)f(\xi)$ satisfies the conditions. We only need f such that $f(1) = 1$ and $f(0) = -1$. What is the simplest f satisfying these two requirements? It is $f(\xi) = 2\xi - 1$. Substituting this into $x = (2-\eta)f(\xi)$ and computing the inverse transformation we obtain the change of variables we were looking for:

$$\boxed{\begin{cases} x = (2\xi - 1)(2-\eta), \\ y = \eta, \end{cases} \qquad \begin{cases} \xi = \frac{1}{2} + \frac{x}{2(2-y)}, \\ \eta = y. \end{cases}}$$

It is easy to verify that the transformation in the box is bijective and thus an acceptable change of variables. As you can see, finding a change of variables is not an easy task! See also Remark 2.35 and Exercise 2.36.

Alternative solution. We could look for the change of variables in a more analytic way, without relying on geometric intuition. Using the equations of the sides of R and Q as in figure, we first write the two regions in terms of inequalities:

$$R = \big\{y-2 < x < y+2,\ 0 < y < 1\big\}, \qquad Q = \big\{0 < \xi < 1,\ 0 < \eta < 1\big\}.$$

We now want to apply some transformations to the inequalities in the definition of R until we arrive at bounding some function of x and y between 0 and 1. At each step we need to verify that the expression obtained is equivalent to the previous one. As before, the choice $y = \eta$ is obvious. Since $2 - y > 0$, we have the following equivalences:

$$y-2 < x < 2-y \overset{\text{divide by } 2-y}{\iff} -1 < \frac{x}{2-y} < 1 \overset{\text{add } 1}{\iff} 0 < 1+\frac{x}{2-y} < 2 \overset{\text{divide by } 2}{\iff} 0 < \frac{1}{2}+\frac{x}{2(2-y)} < 1.$$

Here we are saying that $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ belongs to R if and only if the quantities $\frac{1}{2} + \frac{x}{2(2-y)}$ and y are both bounded between 0 and 1. Calling these quantities $\xi = \frac{1}{2} + \frac{x}{2(2-y)}$ and $\eta = y$, we are saying that $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ belongs to R if and only if $\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}}$ belongs to Q, so this is a change of variables from R to Q.

Integral. The exercise asks to compute the integral of $\mathrm{e}^{(2-y)^2}$ over R. We first compute the Jacobian determinant and then use the change of variables formula (57):

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \det\begin{pmatrix} \frac{\partial((2\xi-1)(2-\eta))}{\partial\xi} & \frac{\partial((2\xi-1)(2-\eta))}{\partial\eta} \\ \frac{\partial\eta}{\partial\xi} & \frac{\partial\eta}{\partial\eta}\end{pmatrix} = \det\begin{pmatrix} 2(2-\eta) & 1-2\xi \\ 0 & 1\end{pmatrix} = 2(2-\eta),$$

$$\iint_R \mathrm{e}^{(2-y)^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_{\vec{\mathbf{T}}(R)} \mathrm{e}^{(2-y(\xi,\eta))^2}\left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta$$
$$= \int_0^1\int_0^1 \mathrm{e}^{(2-\eta)^2}2(2-\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi = -\mathrm{e}^{(2-\eta)^2}\Big|_0^1 = \boxed{\mathrm{e}^4-\mathrm{e}} \approx 51.9.$$

Exercise C.22. Let S be the surface parametrised by the chart $\vec{\mathbf{X}}(u,w) = u\hat{\boldsymbol{\imath}} + \frac{w}{u}\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ defined over the triangular region $R = \{0 < w < u < 1\}$. Compute the flux through S of the field $\vec{\mathbf{F}} = \hat{\boldsymbol{\jmath}} + 2x\hat{\boldsymbol{k}}$.

Rewrite the surface as the graph of a scalar field $g(x,y)$ over a suitable region $\widetilde{R}$ and compute again the same flux using formula (74).

We first compute the flux using (73):

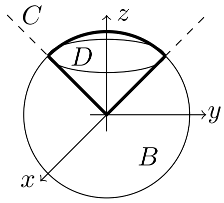
$$\vec{\mathbf{X}} = u\hat{\boldsymbol{\imath}} + \frac{w}{u}\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}, \qquad \frac{\partial\vec{\mathbf{X}}}{\partial u} = \hat{\boldsymbol{\imath}} - \frac{w}{u^2}\hat{\boldsymbol{\jmath}}, \qquad \frac{\partial\vec{\mathbf{X}}}{\partial w} = \frac{1}{u}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}, \qquad \frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w} = -\frac{w}{u^2}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{1}{u}\hat{\boldsymbol{k}},$$

$$\iint_S \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} = \iint_R \vec{\mathbf{F}}(\vec{\mathbf{X}})\cdot\frac{\partial\vec{\mathbf{X}}}{\partial u}\times\frac{\partial\vec{\mathbf{X}}}{\partial w}\,\mathrm{d}A$$
$$= \iint_R (\hat{\boldsymbol{\jmath}} + 2u\hat{\boldsymbol{k}})\cdot\Big(-\frac{w}{u^2}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{1}{u}\hat{\boldsymbol{k}}\Big)\,\mathrm{d}A = \int_0^1\int_0^u(-1+2)\,\mathrm{d}w\,\mathrm{d}u = \int_0^1 u\,\mathrm{d}u = \boxed{\frac{1}{2}}.$$

To write S as a graph, we have to find a region $\widetilde{R}\subset\mathbb{R}^2$ and a planar field $g:\widetilde{R}\to\mathbb{R}$ such that $S = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + g(x,y)\hat{\boldsymbol{k}},\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}\in\widetilde{R}\}$. Identifying the components of the chart $\vec{\mathbf{X}}(u,w)$ with $\vec{\mathbf{r}}$, we obtain $x = X_1 = u$ and $y = X_2 = w/u$, thus $\boxed{g(x,y)} = z = X_3 = w = uy = \boxed{xy}$. We also need to find the new region $\widetilde{R}$ corresponding to the projection on the xy-plane of S. From $x = u$, $y = w/u$ and the original region $\{0 < w < u < 1\}$ we have $0 < x < 1$ and $0 < w = uy = xy < x$ which, by dividing by x, gives $0 < y < 1$. So the region is the square $\boxed{\widetilde{R} = (0,1)^2 = \{0 < x < 1,\ 0 < y < 1\}}$. From $\vec{\nabla}g = \vec{\nabla}(xy) = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ we have

$$\iint_S \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} \overset{(74)}{=} \iint_{\widetilde{R}}\Big(-F_1\frac{\partial g}{\partial x} - F_2\frac{\partial g}{\partial y} + F_3\Big)\,\mathrm{d}A = \int_0^1\int_0^1(0 - y + 2x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\Big(-\frac{1}{2}+2x\Big)\,\mathrm{d}x = \frac{1}{2}.$$

Exercise C.23. Compute the triple integral of $f = x^2 + y^2$ over the domain D lying inside the ball of radius $a > 0$ centred at the origin and above the cone $z = \sqrt{x^2 + y^2}$.



The geometry of the problem suggests to use spherical coordinates (87). In this coordinate system, the scalar field reads

$$f = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi.$$

The ball B and the cone C, as described in the question, in spherical coordinates read

$$B = \big\{|\vec{\mathbf{r}}| < a\big\} = \big\{\rho < a\big\}, \qquad C = \big\{z > \sqrt{x^2+y^2}\big\} = \big\{\rho\cos\phi > \rho\sin\phi\big\} = \big\{0 < \phi < \pi/4\big\}$$

(recall that the maximal interval of definition for ϕ is $[0, \pi]$). Note that we can also easily derive the expression of the cone C from the geometric meaning of the colatitude ϕ, see Figure 39 in the notes. Thus the domain of integration is $D = B \cap C = \big\{\rho < a,\ 0 < \phi < \pi/4\big\}$, which we can imagine as a parallelepiped in the $\rho\phi\theta$-space. The desired integral is then:

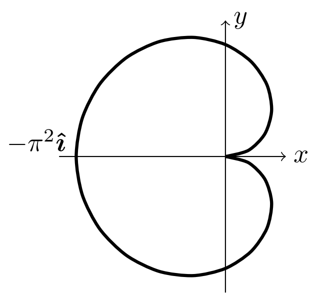
$$\begin{aligned}
\iiint_D f \,\mathrm{d}V &= \int_0^a \int_0^{\pi/4} \int_{-\pi}^{\pi} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho \\
&= \int_0^a \rho^4\,\mathrm{d}\rho \int_0^{\pi/4} \sin^3\phi\,\mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\theta \\
&= \frac{a^5}{5}\bigg(\int_0^{\pi/4} \sin\phi(1-\cos^2\phi)\,\mathrm{d}\phi\bigg) 2\pi \\
&= 2\pi \frac{a^5}{5}\bigg(-\cos\phi + \frac{1}{3}\cos^3\phi\bigg|_0^{\pi/4}\bigg) \\
&= 2\pi \frac{a^5}{5}\bigg(-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + 1 - \frac{1}{3}\bigg) = 2\pi \frac{a^5}{5}\frac{(-5+4\sqrt{2})}{6\sqrt{2}} = \boxed{\pi a^5 \frac{(8-5\sqrt{2})}{30}}.
\end{aligned}$$

Exercise C.24. Compute the area of the region bounded by $\vec{\mathbf{a}}(t) = t(2\pi - t)(\cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}})$, $0 \le t \le 2\pi$.

This is a polar graph as those in Example 2.62 in the notes, so its area is

$$\text{Area}(R) = \frac{1}{2}\int_0^{2\pi} \big(t(2\pi - t)\big)^2\,\mathrm{d}t = \int_0^{2\pi} \Big(2\pi^2 t^2 + \frac{1}{2}t^4 - 2\pi t^3\Big)\,\mathrm{d}t = \boxed{\frac{8\pi^5}{15}} \approx 163.$$

Note that since $g(t) = t(2\pi - t)$ is not periodic, we need to modify (79) to ensure we consider the correct interval of integration, i.e. $(0, 2\pi)$ instead of $(-\pi, \pi)$.



You can draw the region in Matlab using `VCplotter` calling one of the following commands

```
VCplotter(1, 1, @(t) (t*(2*pi-t))*cos(t), @(t) (t*(2*pi-t))*sin(t), 0, 2*pi);
VCplotter(1, 2, @(t) (t*(2*pi-t)), @(t) t, 0, 2*pi);
```

Exercise C.25. London and Astana (Kazakhstan) approximately lie on the parallel $51°N$, and have longitude $0°$ and $71°E$, respectively. Describe the location of the two cities in a suitable special coordinates system. Compute the distance between the two cities, if the distance is measured:

(i) along a straight (underground) segment,

(ii) on the Earth surface along the parallel $51°N$,

(iii) along the shortest surface path.

Assume that Earth is a sphere of radius 6371 km.

We first transform the coordinates of the cities in radians: 51° corresponds to $2\pi\frac{51}{360} \approx 0.8901$ and 71° to $2\pi\frac{71}{360} \approx 1.2392$. We represent London and Astana as two points $\vec{\mathbf{L}}$ and $\vec{\mathbf{A}}$ in $\mathbb{R}^3$, where the origin is fixed at centre of the Earth, the unit length equals 1 km, $\hat{\boldsymbol{k}}$ points to the North Pole and the $\{y = 0\}$ plane includes the Greenwich meridian. Then the two points have spherical coordinates

$$\rho_{\vec{\mathbf{L}}} = 6371, \quad \phi_{\vec{\mathbf{L}}} = \frac{\pi}{2} - 2\pi\frac{51}{360} \approx 0.6807, \quad \theta_{\vec{\mathbf{L}}} = 0, \qquad \rho_{\vec{\mathbf{A}}} = 6371, \quad \phi_{\vec{\mathbf{A}}} \approx 0.6807, \quad \theta_{\vec{\mathbf{A}}} = 1.2392.$$

(Recall the definition of spherical coordinates: ϕ is colatitude and θ is longitude.) In Cartesian coordinates:

$$\vec{\mathbf{L}} = \rho_{\vec{\mathbf{L}}} \sin\phi_{\vec{\mathbf{L}}} \cos\theta_{\vec{\mathbf{L}}}\hat{\boldsymbol{\imath}} + \rho_{\vec{\mathbf{L}}} \sin\phi_{\vec{\mathbf{L}}} \sin\theta_{\vec{\mathbf{L}}}\hat{\boldsymbol{\jmath}} + \rho_{\vec{\mathbf{L}}} \cos\phi_{\vec{\mathbf{L}}}\hat{\boldsymbol{k}} \approx 4009\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 4951\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{A}} \approx 1305\hat{\boldsymbol{\imath}} + 3791\hat{\boldsymbol{\jmath}} + 4951\hat{\boldsymbol{k}}.$$

(i) The distance along a straight underground segment is

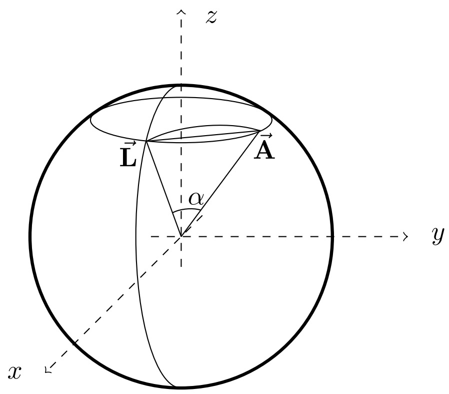
$$d_{(i)} = |\vec{\mathbf{A}} - \vec{\mathbf{L}}| = |-2704\hat{\boldsymbol{\imath}} + 3791\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}| = \boxed{4657\,\text{km}}.$$

(ii) The parallel $51°N$ is a circle of radius $R = \rho_{\vec{\mathbf{L}}} \sin\phi_{\vec{\mathbf{L}}} \approx 4009$ km. Thus the distance between the two cities along the parallel is $d_{(ii)} = R|\theta_{\vec{\mathbf{A}}} - \theta_{\vec{\mathbf{L}}}| = 4009 * 1.2392 = \boxed{4968 \text{ km}}$.

(iii) To compute the shortest surface distance, we first note that this is the length of the arc of a the circle centred at $\vec{\mathbf{0}}$ and endpoints $\vec{\mathbf{L}}$ and $\vec{\mathbf{A}}$; this is a great circle of Earth so it has radius 6371 km. Denote by α the angle at the origin separating $\vec{\mathbf{L}}$ and $\vec{\mathbf{A}}$ (see figure). Since $\vec{\mathbf{L}} \cdot \vec{\mathbf{A}} = |\vec{\mathbf{L}}||\vec{\mathbf{A}}| \cos\alpha$, we have

$$\alpha = \arccos\frac{\vec{\mathbf{L}} \cdot \vec{\mathbf{A}}}{|\vec{\mathbf{L}}||\vec{\mathbf{A}}|} = \arccos\frac{29744146}{6371^2} = 0.7484.$$

Thus the distance as the crow flies is $d_{(iii)} = r_{\vec{\mathbf{L}}}\alpha = \boxed{4768 \text{ km}}$. This result agrees with what you can compute on http://www.movable-type.co.uk/scripts/latlong.html, where you can also see the route on a map.



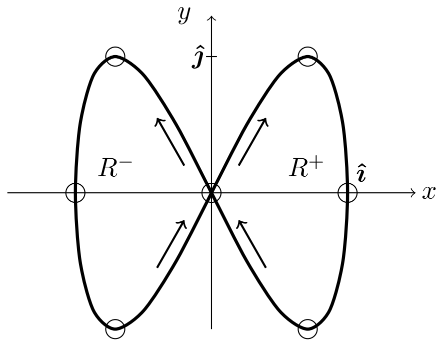
Exercise C.26. Compute the area of the region bounded by the curve $\vec{\mathbf{a}}(t) = \sin t\hat{\boldsymbol{\imath}} + \sin 2t\hat{\boldsymbol{\jmath}}$, $0 \le t \le 2\pi$.

We first note that this curve does not define a polar graph in the meaning of Example 2.62: we can not write $\vec{\mathbf{a}}(\tau) = g(\tau)(\cos\tau\hat{\boldsymbol{\imath}} + \sin\tau\hat{\boldsymbol{\jmath}})$ for a positive $g(\tau)$.

We can draw the curve by computing some values:

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\vec{\mathbf{a}}(t)$	$\vec{\mathbf{0}}$	$\frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$	$\hat{\boldsymbol{\imath}}$	$\frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$	$\vec{\mathbf{0}}$	$-\frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$	$-\hat{\boldsymbol{\imath}}$	$-\frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$	$\vec{\mathbf{0}}$

Connecting the dots, we have the following picture:



We can get this picture also using `VCplotter` and
`VCplotter(1, 1, @(t) sin(t), @(t) sin(2*t), 0, 2*pi);`

The region bounded by the path of $\vec{\mathbf{a}}$ is made of two identical subregions, which we call $R^{\pm}$, symmetric with respect to the origin: we can verify this noting that $\vec{\mathbf{a}}(-t) = -\vec{\mathbf{a}}(t)$ for all real t and using the periodicity of $\vec{\mathbf{a}}$. So we can compute the area of the right subregion and multiply by two. We can compute the area either as a line or as a double integral.

(Version 1) Since $\vec{\mathbf{a}}$, $0 \le t \le \pi$ parametrises ∂R^+ clockwise, i.e. in the opposite direction to the standard convention, we reverse the sign in the usual formula $\mathrm{Area}(Q) = -\int_{\partial Q} y \,\mathrm{d}x$ derived from Green's theorem in Example (3.9). Integrating we obtain:

$$\begin{aligned}\mathrm{Area}(R) = 2\mathrm{Area}(R^+) = 2\int_{\partial R^+} y \,\mathrm{d}x &= 2\int_{\partial R^+} a_2(t)\frac{da_1(t)}{dt}\,\mathrm{d}t \\ &= 2\int_0^\pi \sin 2t \cos t \,\mathrm{d}t = 4\int_0^\pi \sin t \cos^2 t \,\mathrm{d}t = -\frac{4}{3}\cos^3 t\Big|_0^\pi = \boxed{\frac{8}{3}}.\end{aligned}$$

(Note that if we blindly computed $\int_0^{2\pi} y \,\mathrm{d}x$ without drawing a picture of R, we would have obtained $-\frac{4}{3}\cos^3 t|_0^{2\pi} = 0$ which is clearly not the desired area: this is because $\vec{\mathbf{a}}$ parametrises ∂R^+ clockwise and ∂R^- anti-clockwise, so this integral is the difference between the two areas.)

(Version 2) Since R^+ is a y-simple region, one can compute directly the double integral. The first step is to write the y coordinate of $\vec{\mathbf{a}}$ as a function of the x coordinate:

$$y = a_2(t) = \sin 2t = 2\underbrace{\sin t}_{=a_2(t)=x}\cos t = 2x\sqrt{1-x^2}, \qquad 0 \le t \le \frac{\pi}{2}.$$

So R^+ can be written as $R^+ = \{0 < x < 1, -2x\sqrt{1-x^2} < y < 2x\sqrt{1-x^2}\}$ giving

$$\mathrm{Area}(R) = 2\mathrm{Area}(R^+) = 2\int_0^1\int_{-2x\sqrt{1-x^2}}^{2x\sqrt{1-x^2}} 1 \,\mathrm{d}y\,\mathrm{d}x = 8\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x = -\frac{8}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{8}{3}.$$

Exercise C.27. Compute the flux of $\vec{\mathbf{F}} = xy\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}} + zx\hat{\mathbf{k}}$ through the boundary of the cube $C = (0,2)^3 = \{0 < x, y, z < 2\}$.

The divergence theorem gives

$$\begin{aligned}\text{Flux of } \vec{\mathbf{F}} &= \oiint_{\partial C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} \\ &\overset{(101)}{=} \iiint_C \vec{\nabla}\cdot\vec{\mathbf{F}}\,\mathrm{d}V \\ &= \iiint_C (y+z+x)\,\mathrm{d}V \\ &= \int_0^2\int_0^2\int_0^2 (y+z+x)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^2\int_0^2 (2y+2+2x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 (4+4+4x)\,\mathrm{d}x = 8+8+8 = \boxed{24}.\end{aligned}$$

Alternatively, we can compute the flux directly by summing the contribution from each face. We name S_T, S_B, S_N, S_S, S_E, S_W the top, bottom, north, south, east and west faces of the cube, respectively, as in the proof of Lemma 3.13. For each of these faces we compute the outward-pointing unit normal, we take its scalar product with $\vec{\mathbf{F}}$ and integrate it on the face. You can see that this procedure is lengthier and much more prone to error than the use of the divergence theorem:

$$S_T = \{z=2,\ 0<x,y<2\}, \qquad \hat{\boldsymbol{n}}_T = \hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_T = zx = 2x, \qquad \iint_{S_T} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 2x\,\mathrm{d}y\,\mathrm{d}x = 8,$$

$$S_B = \{z=0,\ 0<x,y<2\}, \qquad \hat{\boldsymbol{n}}_B = -\hat{\boldsymbol{k}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_B = -zx = 0, \qquad \iint_{S_T} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 0\,\mathrm{d}y\,\mathrm{d}x = 0,$$

$$S_N = \{y=2,\ 0<x,z<2\}, \qquad \hat{\boldsymbol{n}}_N = \hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_N = yz = 2z, \qquad \iint_{S_N} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 2z\,\mathrm{d}z\,\mathrm{d}x = 8,$$

$$S_S = \{y=0,\ 0<x,z<2\}, \qquad \hat{\boldsymbol{n}}_S = -\hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_S = -yz = 0, \qquad \iint_{S_S} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 0\,\mathrm{d}z\,\mathrm{d}x = 0,$$

$$S_E = \{x=2,\ 0<y,z<2\}, \qquad \hat{\boldsymbol{n}}_E = \hat{\boldsymbol{\imath}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_E = xy = 2y, \qquad \iint_{S_E} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 2y\,\mathrm{d}z\,\mathrm{d}y = 8,$$

$$S_W = \{x=0,\ 0<y,z<2\}, \qquad \hat{\boldsymbol{n}}_W = -\hat{\boldsymbol{\imath}}, \qquad \vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}_W = -xy = 0, \qquad \iint_{S_W} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \int_0^2\int_0^2 0\,\mathrm{d}z\,\mathrm{d}y = 0,$$

$$\oiint_{\partial C} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \iint_{S_T} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_B} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_N} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_S} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_E} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} + \iint_{S_W} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = 24.$$

Exercise C.28. Demonstrate the divergence theorem for $\vec{\mathbf{F}} = \frac{1}{2}z^2\hat{\boldsymbol{k}}$ and the upper half unit ball $B = \{z>0, |\vec{\mathbf{r}}|<1\}$.

The volume integral in the divergence theorem is easily calculated using spherical coordinates. The domain is written in this system as $B = \{z>0, |\vec{\mathbf{r}}|<1\} = \{\phi<\pi/2,\ \rho<1\}$. We compute the divergence of $\vec{\mathbf{F}}$ in Cartesian coordinates and *then* we express it in spherical coordinates: $\vec{\nabla}\cdot\vec{\mathbf{F}} = z = \rho\cos\phi$. Thus the integral is:

$$\begin{aligned}
\iiint_B \vec{\nabla}\cdot\vec{\mathbf{F}}\,\mathrm{d}V &= \iiint_B \rho\cos\phi\,\mathrm{d}V = \int_0^1\int_0^{\pi/2}\int_{-\pi}^{\pi} (\rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho \\
&= \int_0^1 \rho^3\,\mathrm{d}\rho \int_0^{\pi/2}\cos\phi\sin\phi\,\mathrm{d}\phi \int_{-\pi}^{\pi}\mathrm{d}\theta \\
&= \frac{1}{4}\Big(\int_0^{\pi/2}\frac{1}{2}\sin 2\phi\,\mathrm{d}\phi\Big)2\pi = 2\pi\frac{1}{4}\Big(-\frac{1}{4}\cos 2\phi\Big)\Big|_0^{\pi/2} = \frac{\pi}{2}\frac{1+1}{4} = \boxed{\frac{\pi}{4}}.
\end{aligned}$$

We now have to compute the flux of $\vec{\mathbf{F}}$ through the boundary of B. The boundary is made of two parts: the disc $\{z=0,\ x^2+y^2<1\}$ and the cap $\{|\vec{\mathbf{r}}|^2=1,\ z>0\}$. The former lies in the horizontal plane $\{z=0\}$, thus the field $\vec{\mathbf{F}} = \frac{1}{2}z^2\hat{\boldsymbol{k}}$ vanishes there and the corresponding integral does not contribute to the total flux. The upper cap is the graph of the scalar field $g = \sqrt{1-x^2-y^2}$ (because its points satisfy $x^2+y^2+z^2=1$). We compute the flux using formula (74):

$$\begin{aligned}
\oiint_{\partial B} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} &= \iint_{\{z=0,\ x^2+y^2<1\}} \underbrace{\vec{\mathbf{F}}}_{=0}\cdot\mathrm{d}\vec{\mathbf{S}} + \iint_{\{|\vec{\mathbf{r}}|^2=1,\ z>0\}} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} \\
&= \iint_{\{|\vec{\mathbf{r}}|^2=1,\ z>0\}} \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{S}} \\
&\overset{(74)}{=} \iint_{\{x^2+y^2<1\}} \Big(-\underbrace{F_1}_{=0}\frac{\partial g}{\partial x} - \underbrace{F_2}_{=0}\frac{\partial g}{\partial y} + F_3\Big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\{x^2+y^2<1\}} \frac{1}{2}g^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\iint_{\{x^2+y^2<1\}} (1-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_0^1\int_{-\pi}^{\pi}(1-r^2)r\,\mathrm{d}\theta\,\mathrm{d}r = \frac{1}{2}2\pi\Big(\frac{r^2}{2}-\frac{r^4}{4}\Big)\Big|_0^1 = \frac{\pi}{4}.
\end{aligned}$$

Note that we computed the double integral in polar coordinates.

Exercise C.29. Let $\vec{\mathbf{F}}$ be a vector field, and a and ϵ be two positive numbers such that $\vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot \vec{\mathbf{r}} \geq \epsilon$ for all $\vec{\mathbf{r}} \in \mathbb{R}^3$ with $|\vec{\mathbf{r}}| = a$. Prove that $\vec{\mathbf{F}}$ is not solenoidal in the ball B of radius a centred at the origin.

The condition "$\vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot \vec{\mathbf{r}} \geq \epsilon$ for all $\vec{\mathbf{r}} \in \mathbb{R}^3$ with $|\vec{\mathbf{r}}| = a$" simply means that $\vec{\mathbf{F}}$ points outwards on the surface of the ball B of radius a centred at the origin. On the boundary $\partial B = \{|\vec{\mathbf{r}}| = a\}$ of this ball the outward-pointing unit vector field is $\hat{\boldsymbol{n}} = \vec{\mathbf{r}}/|\vec{\mathbf{r}}| = \vec{\mathbf{r}}/a$. From the divergence theorem and the definition of flux we have

$$\iiint_B \nabla \cdot \vec{\mathbf{F}} \,\mathrm{d}V = \oiint_{\partial B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \oiint_{\partial B} \vec{\mathbf{F}}(\hat{\boldsymbol{r}}) \cdot \frac{\vec{\mathbf{r}}}{a} \,\mathrm{d}S \geq \oiint_{\partial B} \frac{\epsilon}{a} \,\mathrm{d}S = 4\pi a \epsilon > 0.$$

The triple integral of the scalar field $\nabla \cdot \vec{\mathbf{F}}$ is positive, so $\nabla \cdot \vec{\mathbf{F}}$ can not be zero everywhere, which means that $\vec{\mathbf{F}}$ is not solenoidal.

Exercise C.30. Prove the following "integration by parts formula" for the curl: given a domain $D \subset \mathbb{R}^3$ and two vector fields $\vec{\mathbf{F}}, \vec{\mathbf{G}}$,

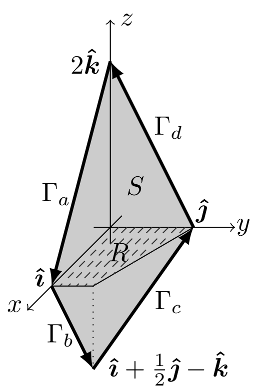
$$\iiint_D (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{G}} \,\mathrm{d}V = \iiint_D \vec{\mathbf{F}} \cdot (\vec{\nabla} \times \vec{\mathbf{G}}) \,\mathrm{d}V + \oiint_{\partial D} (\vec{\mathbf{G}} \times \hat{\boldsymbol{n}}) \cdot \vec{\mathbf{F}} \,\mathrm{d}S.$$

We first move the first term at the right-hand side to the left-hand side, collecting all volume integrals. As in the proof of any other integration by parts formula (see the comparison 3.26 in the notes), we use a product rule, in this case (30) for the divergence of a vector product. We are then able to apply the divergence theorem (101). We conclude recalling that a flux (72) is the surface integral of the normal component of a vector field and using the triple product identity proved in Exercise 1.15:

$$\begin{aligned}
\iiint_D (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{G}} \,\mathrm{d}V - \iiint_D \vec{\mathbf{F}} \cdot (\vec{\nabla} \times \vec{\mathbf{G}}) \,\mathrm{d}V \quad &= \quad \iiint_D \big((\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{G}} - \vec{\mathbf{F}} \cdot (\vec{\nabla} \times \vec{\mathbf{G}})\big) \,\mathrm{d}V \\
&\overset{(30)}{=} \quad \iiint_D \vec{\nabla} \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \,\mathrm{d}V \\
&\overset{(101)}{=} \quad \oiint_{\partial D} (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \cdot \mathrm{d}\vec{\mathbf{S}} \\
&\overset{(72)}{=} \quad \oiint_{\partial D} (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S \\
&\overset{\text{Ex. (1.15)}}{=} \quad \oiint_{\partial D} (\vec{\mathbf{G}} \times \hat{\boldsymbol{n}}) \cdot \vec{\mathbf{F}} \,\mathrm{d}S.
\end{aligned}$$

Exercise C.31. Demonstrate Stokes' theorem for the field $\vec{\mathbf{F}} = zy\hat{\boldsymbol{\imath}}$ and the quadrilateral S with vertices $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}}$, $2\hat{\boldsymbol{k}}$.

We first verify that the four vertices lie in the same plane, so that the quadrilateral is well-defined. The four points must belong to the graph of an affine scalar field in two variables $g(x, y) = g_0 + g_1 x + g_2 y$. For $\hat{\boldsymbol{\imath}}$ to belong to the graph of g we need $g(1, 0) = g_0 + g_1 = 0$, for $\hat{\boldsymbol{\jmath}}$ we have $g(0, 1) = g_0 + g_2 = 0$ and from $2\hat{\boldsymbol{k}}$ we have $g(0, 0) = g_0 = 2$. This gives $g = 2(1 - x - y)$. Since $g(1, \frac{1}{2}) = -1$, also the last vertex $\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$ belongs to the graph of S and all four of them are in the same plane.



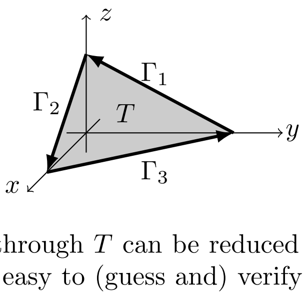
The surface S is the graph of g over the planar region R obtained projecting S on the horizontal xy-plane. This is the polygon with vertices $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{\jmath}}, \vec{\mathbf{0}}$ (the vertices of S with the z-component set to zero), i.e. $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ 0 < x < 1,\ 0 < y < 1 - \frac{1}{2}x\}$. This trapezium is the dashed area in the figure. We compute the flux term in the assertion of Stokes' theorem using the formula for the flux through a graph surface:

$$\begin{aligned}
\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} &= \iint_S (y\hat{\boldsymbol{\jmath}} - z\hat{\boldsymbol{k}}) \cdot \mathrm{d}\vec{\mathbf{S}} \\
&\overset{(74)}{=} \iint_R \Big(-0\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} - z \Big) \,\mathrm{d}A \\
&= \int_0^1 \int_0^{1-\frac{1}{2}x} \big(2y + 2(x+y-1)\big) \,\mathrm{d}y\,\mathrm{d}x \qquad \text{because } z = g(x,y) = 2(1-x-y) \text{ on } S \\
&= \int_0^1 \int_0^{1-\frac{1}{2}x} (2x + 4y - 2) \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \Big((2x-2)\Big(1 - \frac{1}{2}x\Big) + 2\Big(1 - \frac{1}{2}x\Big)^2 \Big) \,\mathrm{d}x = \int_0^1 \Big(-\frac{1}{2}x^2 + x \Big) \,\mathrm{d}x = -\frac{1}{6} + \frac{1}{2} = \boxed{\frac{1}{3}}.
\end{aligned}$$

To compute the boundary term in Stokes' theorem, we need to integrate $\vec{\mathbf{F}}$ along the four sides of S. However we will see that on three sides out of four the integral is zero. We call the four sides Γ_a, Γ_b, Γ_c and Γ_d as in figure. The field $\vec{\mathbf{F}} = zy\hat{\boldsymbol{\imath}}$ is zero on the plane $\{y = 0\}$, thus on the side Γ_a. Moreover, the segments Γ_b and Γ_d lie in the planes $\{x = 1\}$ and $\{x = 0\}$, respectively, so their tangential vectors ($(\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}})/\sqrt{5}$ and $(2\hat{\boldsymbol{k}} - \hat{\boldsymbol{\jmath}})/\sqrt{5}$, respectively) are orthogonal to $\hat{\boldsymbol{\imath}}$ and thus to $\vec{\mathbf{F}}$; since the path integral of $\vec{\mathbf{F}}$ is the integral of its tangential component, on these two sides it vanishes. The only side left is Γ_c, which is parametrised by $\vec{\mathbf{c}}(t) = (\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) + t(-\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) = (1-t)\hat{\boldsymbol{\imath}} + \frac{1}{2}(1+t)\hat{\boldsymbol{\jmath}} + (t-1)\hat{\boldsymbol{k}}$ for $0 \le t \le 1$ (recall Remark 1.24). The integral is calculated as follows:

$$\begin{aligned}
\oint_{\partial S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{\Gamma_a} \underbrace{\vec{\mathbf{F}}}_{=\vec{\mathbf{0}}} \cdot \mathrm{d}\vec{\mathbf{r}} + \underbrace{\int_{\Gamma_b} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}}_{=0} + \int_{\Gamma_c} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} + \underbrace{\int_{\Gamma_d} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}}_{=0} \\
&= \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{c}}) \cdot \frac{d\vec{\mathbf{c}}}{dt} \,\mathrm{d}t \\
&= \int_0^1 (zy\hat{\boldsymbol{\imath}}) \cdot \Big(-\hat{\boldsymbol{\imath}} - \frac{1}{2}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} \Big) \,\mathrm{d}t = \int_0^1 -(t-1)\frac{1}{2}(1+t) \,\mathrm{d}t = -\frac{1}{2}\int_0^1 (1-t^2) \,\mathrm{d}t = \frac{1}{3}.
\end{aligned}$$

Exercise C.32. Consider the field $\vec{\mathbf{G}} = 2x^2y^2z\hat{\boldsymbol{\jmath}} - 2x^2yz^2\hat{\boldsymbol{k}}$ and a triangle T with the three vertices on the x-, y- and z-axis, respectively. Show that the flux of $\vec{\mathbf{G}}$ through T is zero.

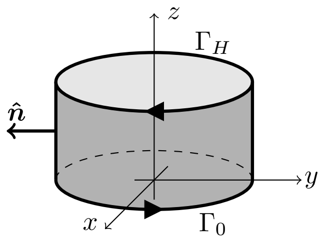


By Stokes' theorem, the flux of $\vec{\mathbf{G}}$ through T can be reduced to the circulation along ∂T of a vector potential of $\vec{\mathbf{G}}$, is there exist any. It is easy to (guess and) verify that $\vec{\mathbf{A}} = x^2y^2z^2\hat{\boldsymbol{\imath}}$ is a vector potential for $\vec{\mathbf{G}}$, i.e. $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{G}}$. The field $\vec{\mathbf{A}}$ is equal to zero on the three coordinate planes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. Each of the three sides Γ_1, Γ_2 and Γ_3 of the triangle lie in one of these planes (in the notation used in the figure, $\Gamma_1 \subset \{x = 0\}$, $\Gamma_2 \subset \{y = 0\}$ and $\Gamma_3 \subset \{z = 0\}$). Thus the value of $\vec{\mathbf{A}}$ on the boundary of T is zero and the line integral is zero:

$$\begin{aligned}
\iint_T \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{S}} &= \iint_T (\vec{\nabla}\times\vec{\mathbf{A}}) \cdot \mathrm{d}\vec{\mathbf{S}} \\
&= \oint_{\partial T} \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_1} \underbrace{x^2}_{=0} y^2z^2\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_2} x^2 \underbrace{y^2}_{=0} z^2\hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_3} x^2y^2 \underbrace{z^2}_{=0} \hat{\boldsymbol{\imath}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0.
\end{aligned}$$

Alternatively, one can use the divergence theorem in the bounded domain D delimited by T and the coordinate planes ($\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$), noting that $\vec{\nabla} \cdot \vec{\mathbf{G}} = 0$ and that $\vec{\mathbf{G}} = \vec{\mathbf{0}}$ on the three faces of the tetrahedron D other than T.

Exercise C.33. Use Stokes' theorem to compute the flux of $\vec{\mathbf{G}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} - 2z\hat{\boldsymbol{k}}$ through the cylindrical surface $S = \{x^2 + y^2 = 1, 0 < z < H\}$, where $H > 0$.



To use Stokes' theorem to compute the flux of $\vec{\mathbf{G}}$ we need to write $\vec{\mathbf{G}}$ as the curl of some vector potential $\vec{\mathbf{A}}$. Since all three components of $\vec{\mathbf{G}}$ are different from zero, the vector potential must have at least two non-zero components. Since the coordinates x and y play the same role in this problem, we look for $\vec{\mathbf{A}} = A_1\hat{\boldsymbol{\imath}} + A_2\hat{\boldsymbol{\jmath}}$. From the definition of the curl, for $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{G}}$ to hold we need $-\frac{\partial A_2}{\partial z} = x$, $\frac{\partial A_1}{\partial z} = y$, $\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = -2z$, which leads to $\vec{\mathbf{A}} = yz\hat{\boldsymbol{\imath}} - xz\hat{\boldsymbol{\jmath}}$.

The boundary of S is made of two components: the lower circle $\Gamma_0 = \{x^2 + y^2 = 1,\ z = 0\}$ and the the upper circle $\Gamma_H = \{x^2 + y^2 = 1,\ z = H\}$. On Γ_0 we have $z = 0$, so $\vec{\mathbf{A}} = \vec{\mathbf{0}}$ and this circle does not contribute to the line integral of $\vec{\mathbf{A}}$. We fix the unit normal vector field $\hat{\boldsymbol{n}}$ on S to be outward-pointing, as in the figure (this was not specified in the question). Then, the induced orientation on the upper circle Γ_H is that moving clockwise, recall the definition on page 59 of the notes. A parametrisation of this path is $\vec{\mathbf{a}}(t) = \cos t\hat{\boldsymbol{\imath}} - \sin t\hat{\boldsymbol{\jmath}} + H\hat{\boldsymbol{k}}$ (note the minus sign: since we move clockwise, this sign is opposite to the usual one). Putting everything together we have:

$$\begin{aligned}
\text{Flux of } \vec{\mathbf{G}} &= \iint_S \vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{S}}\\
&= \iint_S (\vec{\nabla}\times\vec{\mathbf{A}})\cdot \mathrm{d}\vec{\mathbf{S}}\\
&= \oint_{\partial S} \vec{\mathbf{A}}\cdot \mathrm{d}\vec{\mathbf{r}}\\
&= \int_{\Gamma_0} \underbrace{\vec{\mathbf{A}}}_{=\vec{\mathbf{0}}}\cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_H} \vec{\mathbf{A}}\cdot \mathrm{d}\vec{\mathbf{r}}\\
&= \int_0^{2\pi} \vec{\mathbf{A}}(\vec{\mathbf{a}})\cdot \frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}t\\
&= \int_0^{2\pi} (-H\sin t\hat{\boldsymbol{\imath}} - H\cos t\hat{\boldsymbol{\jmath}})\cdot(-\sin t\hat{\boldsymbol{\imath}} - \cos t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^{2\pi} H(\sin^2 + \cos^2 t)\,\mathrm{d}t = \boxed{2\pi H}.
\end{aligned}$$

In short, the strategy to tackle this problem is: (i) find the field $\vec{\mathbf{A}}$ to which we can apply Stokes' theorem, (ii) identify the boundary of S, (iii) compute the parametrisation of the (needed component of the) boundary, (iv) compute the line integral.

Note that in this example we could have directly computed the flux in a much simpler way, without using Stokes' theorem. Indeed, on S we have $\hat{\boldsymbol{n}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ and $x^2 + y^2 = 1$, so $\vec{\mathbf{G}}\cdot\hat{\boldsymbol{n}} = 1$ and $\iint_S \vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{S}} = \iint_S 1\,\mathrm{d}S = \text{Area}(S) = 2\pi H$.