Vector calculus MA2VC 2013–14 — Assignment 4

Handed out: Wednesday 27th November. Due: **Wednesday 11th December, 12 noon**.
Late assignments will not be accepted. Do not use red pen or pencil.
You can use formulas and identities from the lecture notes. Drawing a sketch of the domains may help you.
Total marks: 25. (5% of the total marks for the module.)

(1) (10 marks) Demonstrate Green's theorem for the field $\vec{\mathbf{G}} = x^3\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}}$ and the region R, defined in polar coordinates as

$$R = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} = r\hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}, \text{ such that } 0 < r < 2,\ 0 < \theta < \frac{\pi}{4}\Big\}.$$

Hints: draw a sketch of the domain; use polar coordinates to compute the double integral; find simple parametrisations of the sides of R; exploit some orthogonalities to compute the line integral along a part of ∂R.

(2) (6 marks) Consider the domain $D = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ 1 < |\vec{\mathbf{r}}| < 2\}$ and the field $\vec{\mathbf{H}}(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2\vec{\mathbf{r}}$. Use the divergence theorem and spherical coordinates to compute the flux of $\vec{\mathbf{H}}$ through ∂D.

(3) (9 marks) Let $\vec{\mathbf{F}}$ be a vector field that satisfies the following properties:

(i) $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth and irrotational;

(ii) $\vec{\mathbf{F}}$ is parallel to the x axis, i.e. $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = F_1(\vec{\mathbf{r}})\hat{\boldsymbol{\imath}}$ for all $\vec{\mathbf{r}} \in \mathbb{R}^3$;

(iii) on the x axis $\vec{\mathbf{F}}$ has expression $\vec{\mathbf{F}}(x, 0, 0) = x^2\hat{\boldsymbol{\imath}}$.

Compute the line integral $\int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where Γ is the horizontal straight segment going from $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ to $\hat{\boldsymbol{\jmath}}$.

Hint: the tools you need are Green's theorem and a square.