Vector calculus MA2VC 2013–14 — Assignment 1 SOLUTIONS

(1) At least three different ways of proving the identity are possible.
(Version i) The easiest proof is to compute $\vec{\nabla} \cdot \vec{\mathbf{r}} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ and use the vector identities, first separating $\vec{\mathbf{r}}$ from the two fields in the parenthesis:

$$\begin{aligned}\vec{\nabla} \cdot \big(\vec{\mathbf{r}} f g\big) &\overset{(27)}{=} (\vec{\nabla} \cdot \vec{\mathbf{r}}) f g + \vec{\mathbf{r}} \cdot \vec{\nabla}(fg) \\ &\overset{(25)}{=} 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g) f.\end{aligned}$$

(Version ii) One can use the same vector identity twice, first separating one of the fields (e.g. f) from the second field and $\vec{\mathbf{r}}$, and then separating the latter two objects:

$$\begin{aligned}\vec{\nabla} \cdot \big(\vec{\mathbf{r}} f g\big) &\overset{(27)}{=} \big(\vec{\nabla} \cdot (\vec{\mathbf{r}} g)\big) f + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g \\ &\overset{(27)}{=} \big((\vec{\nabla} \cdot \vec{\mathbf{r}}) g + \vec{\mathbf{r}} \cdot \vec{\nabla} g\big) f + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g \\ &= 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g) f.\end{aligned}$$

(Version iii) One can directly use the definitions of gradient and divergence, and the formula for the partial derivative of a product:

$$\begin{aligned}\vec{\nabla} \cdot \big(\vec{\mathbf{r}} f g\big) &= \vec{\nabla} \cdot \big(x f g \hat{\boldsymbol{\imath}} + y f g \hat{\boldsymbol{\jmath}} + z f g \hat{\boldsymbol{k}}\big) \\ &= \frac{\partial (xfg)}{\partial x} + \frac{\partial (yfg)}{\partial y} + \frac{\partial (zfg)}{\partial z} \\ &= \Big(fg + xg\frac{\partial f}{\partial x} + xf\frac{\partial g}{\partial x}\Big) + \Big(fg + gy\frac{\partial f}{\partial y} + fy\frac{\partial g}{\partial y}\Big) + \Big(fg + zg\frac{\partial f}{\partial z} + zf\frac{\partial g}{\partial z}\Big) \\ &= 3fg + \Big(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\Big) g + \Big(x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + z\frac{\partial g}{\partial z}\Big) f \\ &= 3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g) f.\end{aligned}$$

All versions are correct, but if you learn how to properly use the vector identities as in the first two versions above, you will avoid mistakes and save a lot of time and effort.

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(2) We have to evaluate both the left-hand side and the right-hand side of the identity using the given fields f and g. We should not use the vector identities here, otherwise we just repeat exercise 1.
From the definition of divergence, the left-hand side of the identity reads:

$$\begin{aligned}\vec{\nabla} \cdot \big(\vec{\mathbf{r}} f g\big) &= \vec{\nabla} \cdot \big(xy^4 z e^{xy} \hat{\boldsymbol{\imath}} + y^5 z e^{xy} \hat{\boldsymbol{\jmath}} + y^4 z^2 e^{xy} \hat{\boldsymbol{k}}\big) \\ &= \big(y^4 z e^x y + xy^5 z e^{xy}\big) + \big(5y^4 z e^{xy} + xy^5 z e^{xy}\big) + 2y^4 z e^{xy} \\ &= 8y^4 z e^x y + 2xy^5 z e^{xy}.\end{aligned}$$

The right-hand side reads:

$$\begin{aligned}3fg + (\vec{\mathbf{r}} \cdot \vec{\nabla} f) g + (\vec{\mathbf{r}} \cdot \vec{\nabla} g) f &= 3y^4 z e^{xy} + \big(\vec{\mathbf{r}} \cdot (y e^{xy} \hat{\boldsymbol{\imath}} + x e^{xy} \hat{\boldsymbol{\jmath}})\big) y^4 z + \big(\vec{\mathbf{r}} \cdot (4y^3 z \hat{\boldsymbol{\jmath}} + y^4 \hat{\boldsymbol{k}})\big) e^{xy} \\ &= 3y^4 z e^{xy} + 2xy^5 z e^{xy} + 5y^4 z e^{xy} \\ &= 8y^4 z e^x y + 2xy^5 z e^{xy},\end{aligned}$$

thus the two expressions are equal to each other and the desired identity is demonstrated.

(3) We immediately see that $\vec{\nabla}\cdot\vec{\mathbf{F}} = \frac{\partial(y^2)}{\partial x} + \frac{\partial(z^3)}{\partial y} + \frac{\partial 0}{\partial z} = 0+0+0 = 0$, so $\vec{\mathbf{F}}$ is solenoidal.

We seek a vector potential $\vec{\mathbf{A}} = A_1\hat{\boldsymbol{\imath}} + A_2\hat{\boldsymbol{\jmath}} + A_3\hat{\boldsymbol{k}}$ such that $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{F}}$, i.e., from the definition (4) of the curl,

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = y^2, \qquad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = z^3, \qquad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = 0.$$

If $A_3 = 0$, then A_1 and A_2 must both be non-zero, thus the only possibility for having a potential $\vec{\mathbf{A}}$ with only one non-zero component is to choose it parallel to the z axis, which means $\vec{\mathbf{A}} = A_3\hat{\boldsymbol{k}}$. From this choice we have

$$\frac{\partial A_3}{\partial y} = y^2, \qquad \frac{\partial A_3}{\partial x} = -z^3,$$

which immediately leads to the solution

$$\vec{\mathbf{A}} = \left(\frac{1}{3}y^3 - xz^3 + \lambda\right)\hat{\boldsymbol{k}},$$

where $\lambda \in \mathbb{R}$ may be any scalar.

If you don't see this last step, you can solve the corresponding differential equations:

$$\begin{aligned}
\frac{\partial A_3}{\partial y} = y^2 \qquad &\Rightarrow \qquad A_3(\vec{\mathbf{r}}) = \frac{1}{3}y^3 + b(x,z),\\
\frac{\partial A_3}{\partial x} = -z^3 \qquad &\Rightarrow \qquad \frac{\partial\big(\frac{1}{3}y^3 + b(x,z)\big)}{\partial x} = \frac{\partial b(x,z)}{\partial x} = -z^3 \qquad \Rightarrow \qquad b(x,z) = -xz^3 + \lambda,\\
&\Rightarrow \qquad A_3(\vec{\mathbf{r}}) = \frac{1}{3}y^3 - xz^3 + \lambda.
\end{aligned}$$

Other possible vector potentials found in the solutions handed in (not following the hint but perfectly correct) are:

$$\begin{aligned}
&\frac{1}{4}z^4\hat{\boldsymbol{\imath}} - y^2z\hat{\boldsymbol{\jmath}}, \qquad \frac{1}{4}z^4\hat{\boldsymbol{\imath}} + \frac{1}{3}y^3\hat{\boldsymbol{k}}, \qquad \frac{1}{4}z^4\hat{\boldsymbol{\imath}} + 2y^2z\hat{\boldsymbol{\jmath}} + y^3\hat{\boldsymbol{k}},\\
&-y^2z\hat{\boldsymbol{\jmath}} - xz^3\hat{\boldsymbol{k}}, \qquad x\hat{\boldsymbol{\imath}} - y^2z\hat{\boldsymbol{\jmath}} - xz^3\hat{\boldsymbol{k}}.
\end{aligned}$$

In particular, the first three of these are independent of the x variable.

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(4) From identity (5) in Exercise 1.11 of the notes, the definition of perpendicularity ($\hat{\boldsymbol{n}}\cdot\vec{\mathbf{w}} = 0$), and the unit length of $\hat{\boldsymbol{n}}$ ($\sqrt{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}} = |\hat{\boldsymbol{n}}| = 1$), we obtain the required identity:

$$\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\mathbf{w}}) \overset{(5)}{=} \hat{\boldsymbol{n}}(\underbrace{\vec{\mathbf{w}}\cdot\hat{\boldsymbol{n}}}_{=0}) - \vec{\mathbf{w}}(\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}}_{=|\hat{\boldsymbol{n}}|^2=1}) = 0\hat{\boldsymbol{n}} - 1\vec{\mathbf{w}} = \vec{\mathbf{0}} - `\vec{\mathbf{w}} = -\vec{\mathbf{w}}.$$

We demonstrate the identity for the two vectors given by using twice the definition (4) of the vector product:

$$\begin{aligned}
\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\mathbf{w}}) &= \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right)\times\left(\left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right)\times 3\hat{\boldsymbol{\jmath}}\right)\\
&\overset{(4)}{=} \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right)\times\left(\left(0 - \frac{4}{5}3\right)\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + \left(\frac{3}{5}3 - 0\right)\hat{\boldsymbol{k}}\right)\\
&= \left(\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{k}}\right)\times\left(-\frac{12}{5}\hat{\boldsymbol{\imath}} + \frac{9}{5}\hat{\boldsymbol{k}}\right)\\
&\overset{(4)}{=} 0\hat{\boldsymbol{\imath}} + \left(\frac{4}{5}\left(\frac{-12}{5}\right) - \frac{3}{5}\frac{9}{5}\right)\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\\
&= -\frac{75}{25}\hat{\boldsymbol{\jmath}} = -3\hat{\boldsymbol{\jmath}} = -\vec{\mathbf{w}}.
\end{aligned}$$