Vector calculus MA3VC 2016–17: Assignment 2

MA3VC: Part 3 students only.
Handed out: Thursday 17th November. Due: **Thursday 24th November, 12 noon**.
You can use formulas and identities from the lecture notes. Do not use red pen nor pencil.
Marking will be anonymous, so please write your name only on the "assessed work coversheet" and not on your work. Write your student number both on the back of the coversheet and each page of your work.
Total marks: 25. (10% of the total marks for the module.)

(Exercise 1 — 18 marks) Consider the following (planar) curves:

$$\begin{aligned}
\vec{\mathbf{a}}(t) &= -\mathrm{e}^t\hat{\boldsymbol{\imath}} + \mathrm{e}^t\hat{\boldsymbol{\jmath}} & -\infty < t \le 0,\\
\vec{\mathbf{b}}(t) &= \sqrt{3}t^2\hat{\boldsymbol{\imath}} + (t^3 - t)\hat{\boldsymbol{\jmath}} & -1 \le t \le 1,\\
\vec{\mathbf{c}}(t) &= \cos t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}} & 0 \le t \le \pi,\\
\vec{\mathbf{d}}(t) &= -t\hat{\boldsymbol{\imath}} + \Big(1 - \frac{(t+1)^{3/2}}{\sqrt{2}}\Big)\hat{\boldsymbol{\jmath}} & -1 \le t \le 1.
\end{aligned}$$

Denote by $\Gamma_a, \Gamma_b, \Gamma_c$ and Γ_d the paths of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}, \vec{\mathbf{d}}$, respectively.

(i) Which of these curves are loops? **(2 marks)**

(ii) Two of these curves share the same endpoints, which ones? **(2 marks)**

(iii) Which of the four curves has the shortest path? Which the longest? **(5 marks)**

(iv) Compute the line integral of the scalar field $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$ along the path Γ_a. **(2 marks)**

(v) Compute the line integral of the vector field $x^2\hat{\boldsymbol{\imath}} + xy\hat{\boldsymbol{\jmath}}$ along the path Γ_c. **(2 marks)**

(vi) Without computing any integral, show that

$$\int_{\Gamma_b}(x^{10}\hat{\boldsymbol{\imath}} + y^{10}\hat{\boldsymbol{\jmath}})\cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_d}\vec{\mathbf{r}}\cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_c}(\cosh x \tan y\hat{\boldsymbol{\imath}} - \cosh x \tan y\hat{\boldsymbol{\jmath}})\cdot \mathrm{d}\vec{\mathbf{r}} = 0.$$

(3 marks)

(vii) Find a conservative vector field $\vec{\mathbf{F}}$ such that its line integral over each of the four paths is 1, or prove that no such field exists. **(2 marks)**

Justify all your answers.
Hint: Sketch the paths of the curves and use definition and results available in the lecture notes.

(Exercise 2 — 7 marks) Compute the double integral $\iint_R f\,\mathrm{d}V$, where $f(\vec{\mathbf{r}}) = (x^2+y^2)^2x^{-2}$ and R is the region

$$R = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2, \quad 1 < x^2+y^2 < 4, \quad 0 < \frac{y}{x} < 1, \quad x > 0\Big\}.$$

Hint: Use a suitable change of variables, as suggested by the definition of the region of integration, so that $\iint_R f\,\mathrm{d}V$ can be computed as a double integral over a rectangle.

Please check carefully the list of common errors on page 110 of the notes and try not to commit them!