Vector calculus MA3VC 2015–16: Assignment 1 SOLUTIONS

(Exercise 1) *Consider the vector field* $\vec{\mathbf{F}} = -x^3y^4\hat{\boldsymbol{\imath}} + 3x^2y^4z\hat{\boldsymbol{k}}$.
Compute the divergence and the curl of $\vec{\mathbf{F}}$.
Is $\vec{\mathbf{F}}$ *solenoidal, irrotational?*
Is $\vec{\mathbf{F}}$ *conservative? If the answer is positive compute a scalar potential.*
Does $\vec{\mathbf{F}}$ *admit a vector potential* $\vec{\mathbf{A}}$*? If the answer is positive compute a potential. (In this case, look for the simplest one!)*

We apply the definitions of divergence and curl:

$$\begin{aligned} &\vec{\mathbf{F}} = -x^3y^4\hat{\boldsymbol{\imath}} + 3x^2y^4z\hat{\boldsymbol{k}}, \\ \text{divergence:}\quad &\vec{\nabla}\cdot\vec{\mathbf{F}} = -3x^2y^4 + 3x^2y^4 = 0 && \Rightarrow \quad \vec{\mathbf{F}} \text{ is solenoidal,} \\ \text{curl:}\quad &\vec{\nabla}\times\vec{\mathbf{F}} = 12x^2y^3z\hat{\boldsymbol{\imath}} - 6xy^4z\hat{\boldsymbol{\jmath}} - 4x^3y^3\hat{\boldsymbol{k}} && \Rightarrow \quad \vec{\mathbf{F}} \text{ is not irrotational.} \end{aligned}$$

Since the field is not irrotational, it is not conservative and there exists no scalar potential (recall the box on page 26).

On the other hand, the field is solenoidal, so it may admit a vector potential $\vec{\mathbf{A}}$. (Actually, $\vec{\mathbf{F}}$ is solenoidal and defined on the whole of $\mathbb{R}^3$, so it admits a vector potential by Remark 1.68.) The vector potential $\vec{\mathbf{A}}$ has to satisfy $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{F}}$, which, by definition (23) of curl, is equivalent to the following three conditions:

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = F_1 = -x^3y^4, \qquad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = F_2 = 0, \qquad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = F_3 = 3x^2y^4z.$$

There are many possible vector potentials, we look for the simplest possible. In particular we look for $\vec{\mathbf{A}}$ with only one non-zero component. From the conditions we have just written, we see that the non-zero component must necessarily be A_2, as it is the only one entering the two equations with non-zero right-hand sides. So, setting $A_1 = A_3 = 0$, we have

$$-\frac{\partial A_2}{\partial z} = -x^3y^4, \qquad \frac{\partial A_2}{\partial x} = 3x^2y^4z.$$

Integrating the first equation with respect to z we have $A_2 = x^3y^4z + g(x,y)$ for some g independent of z. We see immediately that $A_2 = x^3y^4z$ with $g = 0$ satisfies the second condition $\frac{\partial A_2}{\partial x} = 3x^2y^4z$, so $\boxed{\vec{\mathbf{A}} = x^3y^4z\hat{\boldsymbol{\jmath}}}$. Recall that there are plenty of other correct vector potentials, your solution might differ from this one.

(Exercise 2) *Let f be a smooth scalar field. Prove the following identity:*

$$\vec{\nabla}\cdot\big(\vec{\nabla}f \times (\vec{\mathbf{r}}f)\big) = 0.$$

Hint: use the identities of Section 1.4 and the values of the curl and the divergence of the position vector $\vec{\mathbf{r}}$. *Recall also Exercise 1.15.*

(Version 1.) We use the product rules for divergence and curl (31) and (32), the curl-of-gradient identity (26), the fact that $\vec{\nabla}\times\vec{\mathbf{r}} = \vec{\mathbf{0}}$ (recall e.g. Exercise 1.60) and the properties of the triple and vector products ($\vec{\mathbf{u}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{w}}) = \vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{u}}) = \vec{\mathbf{w}}\cdot\vec{\mathbf{0}} = 0$, by Exercise 1.15)[1]:

$$\vec{\nabla}\cdot\Big((\vec{\nabla}f)\times(\vec{\mathbf{r}}f)\Big) \overset{(31)}{=} \underbrace{(\vec{\nabla}\times\vec{\nabla}f)}_{=\vec{\mathbf{0}},\text{ by (26)}}\cdot\vec{\mathbf{r}}f - \vec{\nabla}f\cdot\big(\vec{\nabla}\times(\vec{\mathbf{r}}f)\big) \overset{(32)}{=} -\vec{\nabla}f\cdot\big(\vec{\nabla}f\times\vec{\mathbf{r}} + f\underbrace{\nabla\times\vec{\mathbf{r}}}_{=\vec{\mathbf{0}}}\big) \overset{1.15}{=} -\vec{\mathbf{r}}\cdot\underbrace{(\vec{\nabla}f)\times(\vec{\nabla}f)}_{=\vec{\mathbf{0}}} = 0.$$

(Version 2.) We can also prove the identity by expanding in coordinates, however this solution is more complicated and more prone to errors. Using the definition of the vector product (2), of the divergence (22), the product rule for partial derivatives (8) and Clairault's theorem (17):

$$\vec{\nabla}\cdot\Big((\vec{\nabla}f)\times(\vec{\mathbf{r}}f)\Big) = \vec{\nabla}\cdot\bigg(\Big(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}\Big)\times(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})f\bigg)$$

[1] Alternatively, one can see that the argument of the divergence can be bracketed in equivalent way: $\vec{\nabla}f\times(\vec{\mathbf{r}}f) = (\vec{\nabla}f\times\vec{\mathbf{r}})f$. This leads to a slightly different proof:

$$\begin{aligned} \vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}f) &= \vec{\nabla}\cdot((\vec{\nabla}f\times\vec{\mathbf{r}})f) \overset{(30)}{=} (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) + f\vec{\nabla}\cdot((\vec{\nabla}f)\times\vec{\mathbf{r}}) \\ &\overset{(31)}{=} (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) + f\big((\underbrace{\vec{\nabla}\times\vec{\nabla}f}_{=\vec{\mathbf{0}},\text{ by (26)}})\cdot\vec{\mathbf{r}} - (\vec{\nabla}f)\cdot(\underbrace{\nabla\times\vec{\mathbf{r}}}_{=\vec{\mathbf{0}}})\big) = (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) \overset{\text{Ex. 1.15}}{=} \vec{\mathbf{r}}\cdot\underbrace{(\vec{\nabla}f)\times(\vec{\nabla}f)}_{=\vec{\mathbf{0}}} = 0. \end{aligned}$$

$$
\begin{aligned}
&\overset{(2)}{=} \vec\nabla\cdot\Big(\Big(\frac{\partial f}{\partial y}zf-\frac{\partial f}{\partial z}yf\Big)\hat{\boldsymbol\imath}+\Big(\frac{\partial f}{\partial z}xf-\frac{\partial f}{\partial x}zf\Big)\hat{\boldsymbol\imath}+\Big(\frac{\partial f}{\partial x}yf-\frac{\partial f}{\partial y}xf\Big)\hat{\boldsymbol\imath}\Big)\\
&\overset{(22)}{=} \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}zf-\frac{\partial f}{\partial z}yf\Big)+\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial z}xf-\frac{\partial f}{\partial x}zf\Big)+\frac{\partial}{\partial z}\Big(\frac{\partial f}{\partial x}yf-\frac{\partial f}{\partial y}xf\Big)\\
&\overset{(8)}{=} \frac{\partial^2 f}{\partial x\partial y}zf+\frac{\partial f}{\partial y}z\frac{\partial f}{\partial x}-\frac{\partial^2 f}{\partial x\partial z}yf-\frac{\partial f}{\partial z}y\frac{\partial f}{\partial x}\\
&\quad+\frac{\partial^2 f}{\partial y\partial z}xf+\frac{\partial f}{\partial z}x\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial y\partial x}zf-\frac{\partial f}{\partial x}z\frac{\partial f}{\partial y}\\
&\quad+\frac{\partial^2 f}{\partial z\partial x}yf+\frac{\partial f}{\partial x}y\frac{\partial f}{\partial z}-\frac{\partial^2 f}{\partial z\partial y}xf-\frac{\partial f}{\partial y}x\frac{\partial f}{\partial z}\\
&=\Big(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\Big)xf+\Big(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\Big)yf+\Big(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\Big)zf\overset{(17)}{=}0.
\end{aligned}
$$

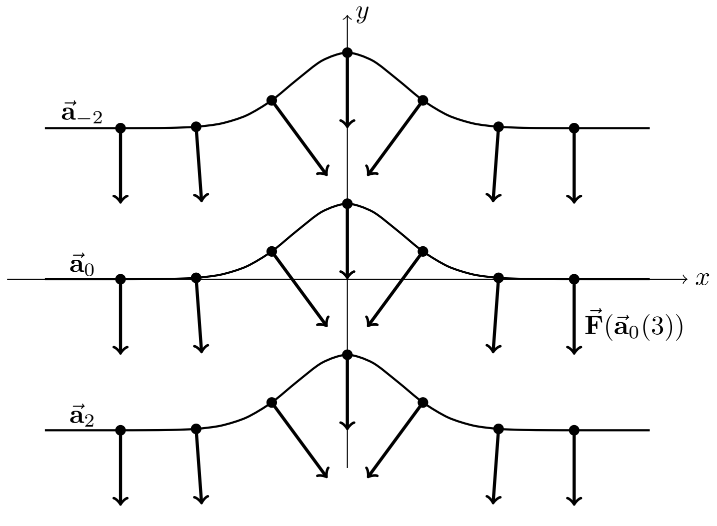
(Exercise 3) *Demonstrate the identity in Exercise 2 for the field $f=\sin(xy+z)$.*

We compute all the terms in the identity:

$$
\begin{aligned}
f&=\sin(xy+z), \qquad\qquad \vec\nabla f = y\cos(xy+z)\hat{\boldsymbol\imath}+x\cos(xy+z)\hat{\boldsymbol\jmath}+\cos(xy+z)\hat{\boldsymbol k},\\
\vec{\mathbf r}f&=x\sin(xy+z)\hat{\boldsymbol\imath}+y\sin(xy+z)\hat{\boldsymbol\jmath}+z\sin(xy+z)\hat{\boldsymbol k},\\
\vec\nabla f\times\vec{\mathbf r}f&\overset{(2)}{=}\sin(xy+z)\cos(xy+z)\Big((xz-y)\hat{\boldsymbol\imath}+(x-yz)\hat{\boldsymbol\jmath}+(y^2-x^2)\hat{\boldsymbol k}\Big),\\
\vec\nabla\cdot(\vec\nabla f\times\vec{\mathbf r}f)&\overset{(22)}{=}\Big(y(xz-y)\cos^2(xy+z)-y(xz-y)\sin^2(xy+z)+z\sin(xy+z)\cos(xy+z)\Big)\\
&\quad+\Big(x(x-yz)\cos^2(xy+z)-x(x-yz)\sin^2(xy+z)-z\sin(xy+z)\cos(xy+z)\Big)\\
&\quad+\Big((y^2-x^2)\cos^2(xy+z)-(y^2-x^2)\sin^2(xy+z)\Big)=0.
\end{aligned}
$$

(Exercise 4 MA3VC) *Define the planar vector field $\vec{\mathbf F}=-2x\mathrm{e}^{-x^2}\hat{\boldsymbol\imath}-\hat{\boldsymbol\jmath}$. Compute at least one planar curve $\vec{\mathbf a}(t)=a_1(t)\hat{\boldsymbol\imath}+a_2(t)\hat{\boldsymbol\jmath}$, with $t\in\mathbb R$, that is perpendicular to $\vec{\mathbf F}$ at each point.*
Hint 1: Note that $\vec{\mathbf F}$ is conservative.
Hint 2: Recall that we have seen in Section 1.3.2 that some fields are orthogonal to some paths. Can you use this to compute the path of $\vec{\mathbf a}$?
Hint 3: Once you have the path of $\vec{\mathbf a}$, to find the parametrisation $\vec{\mathbf a}$ itself recall Remark 1.24 in the notes.

We easily see that $\varphi=(\mathrm{e}^{-x^2}-y)$ is a scalar potential for $\vec{\mathbf F}$, i.e. $\vec{\mathbf F}=\vec\nabla\varphi$. From Part 4 of Proposition 1.33 in the notes, we see that the level lines of φ are perpendicular to $\vec{\mathbf F}$. (Since the fields are planar, i.e. the component z does not play any role, the level sets are level lines, see e.g. Exercises C.1 and C.2 done in the tutorials.) The level lines of φ are the sets $L_\lambda=\{\vec{\mathbf r}\in\mathbb R^2,\mathrm{e}^{-x^2}-y=\lambda\}=\{\vec{\mathbf r}\in\mathbb R^2, y=\mathrm{e}^{-x^2}-\lambda\}$ for any $\lambda\in\mathbb R$. These are nothing else than the graphs of the real functions $G_\lambda(x)=\mathrm{e}^{-x^2}-\lambda$, i.e. all the vertical translates of the standard Gaussian. From the second item in Remark 1.24, we see that their simplest parametrisations are $\boxed{\vec{\mathbf a}_\lambda(t)=t\hat{\boldsymbol\imath}+(\mathrm{e}^{-t^2}-\lambda)\hat{\boldsymbol\jmath}}$. For each value of $\lambda\in\mathbb R$, the curve $\vec{\mathbf a}_\lambda$ is perpendicular to $\vec{\mathbf F}$ at each point. The figure shows three curves (for $\lambda=-2,0,2$) and the values of $\vec{\mathbf F}$ at a few points (the thick arrows).



A few students tried to solve Exercise 4 using differential equations. The field $\vec{\mathbf{F}}$ is orthogonal to a curve $\vec{\mathbf{a}}$ in the point $\vec{\mathbf{a}}(t)$ if it is orthogonal to the tangent vector to the curve. We know that the total derivative $\frac{d\vec{\mathbf{a}}}{dt}(t)$ is a scalar multiple of the tangent vector to Γ in the point $\vec{\mathbf{a}}(t)$. Thus the orthogonality amounts to the condition $\frac{d\vec{\mathbf{a}}(t)}{dt} \cdot \vec{\mathbf{F}}(\vec{\mathbf{a}}(t)) = 0$ for all t. For the given field $\vec{\mathbf{F}}$, this can be expanded as

$$0 = \frac{d\vec{\mathbf{a}}(t)}{dt} \cdot \vec{\mathbf{F}}(\vec{\mathbf{a}}(t)) = \frac{da_1(t)}{dt}(-2a_1(t)\mathrm{e}^{-a_1^2(t)}) - \frac{da_2(t)}{dt}.$$

Solving for a_2 as function of a_1 and using the chain rule, we have

$$\frac{da_2(t)}{dt} = -2a_1(t)\mathrm{e}^{-a_1^2(t)}\frac{da_1(t)}{dt} = \frac{d(\mathrm{e}^{-a_1^2(t)})}{dt}, \qquad \text{which gives} \qquad a_2(t) = \mathrm{e}^{-a_1^2(t)} + \lambda.$$

At this point, we can choose $a_1(t)$ arbitrarily, so we fix $a_1(t) = t$, and we obtain $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + (\mathrm{e}^{-t^2} + \lambda)\hat{\boldsymbol{\jmath}}$.