Vector calculus **MA3VC** 2014–15 — Assignment 2

MA3VC: Part 3 students only.
Handed out: Tuesday 25th November. Due: **Thursday 4th December, 12 noon**.
You can use formulas and identities from the lecture notes. Do not use red pen or pencil.
Marking will be anonymous, so please write your name only on the "assessed work coversheet" and not on your work. Write your student number both on the back of the coversheet and each page of your work.
Total marks: 25 (10% of the total marks for the module.)

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(1) (5 marks) Consider the curve $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + t^3\hat{\boldsymbol{k}}$ for $-1 \le t \le 1$ and denote by Γ its path. Compute the line integral over Γ of the vector field

$$\vec{\mathbf{F}} = y^2\hat{\boldsymbol{\imath}} + 2xy\hat{\boldsymbol{\jmath}}.$$

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(2) (7 marks) Consider the following four curves:

$$\begin{aligned}
\vec{\mathbf{a}}_A(t) &= \sin t\,\hat{\boldsymbol{\imath}} + \cos 44t\,\hat{\boldsymbol{\jmath}} + \sin 5t\,\hat{\boldsymbol{k}} && -\frac{\pi}{2} \le t \le \frac{\pi}{2},\\
\vec{\mathbf{a}}_B(t) &= (t+1)\hat{\boldsymbol{\imath}} + (t+1)^2\hat{\boldsymbol{\jmath}} + (t+1)^3\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
\vec{\mathbf{a}}_C(t) &= t^4\hat{\boldsymbol{\imath}} - t^{12}\hat{\boldsymbol{\jmath}} - t^2\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
\vec{\mathbf{a}}_D(t) &= \Big(\frac{4}{1+t} - 3\Big)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \frac{1}{1+t(1-t)}\hat{\boldsymbol{\jmath}} && 0 \le t \le 1.
\end{aligned}$$

Denote by $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D the corresponding paths. The integrals of the field $\vec{\mathbf{F}}$ defined above over $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D give the following values, listed in ascending order:

$$-2, \qquad 0, \qquad 2, \qquad 32.$$

Associate to every curve the value of the corresponding line integral. Justify your answer.

Hint 1: Use the result obtained in the first exercise.
Hint 2: You do not need to compute any integral.

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(3) (6 marks) Compute the double integral of the field $f = \frac{x+y}{x}$ over the region

$$R = \Big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 2, 0 < \frac{1}{2}\Big(\frac{y}{x} + 1\Big) < 1\Big\}.$$

Hint: The definition of the domain should suggest you a change of variables mapping R to a rectangle.

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(4) (7 marks) Consider the graph surface

$$S = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ z = g(x,y),\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in R\big\},$$

where

$$R = \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ x^2 + y^2 < 1\big\} \quad \text{and} \quad g(x,y) = x^2 - y^2.$$

Define on S the standard upward-pointing orientation $\hat{\boldsymbol{n}}$ such that $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} > 0$.

The boundary ∂S of S is a path, find a curve $\vec{\mathbf{a}}(t)$ parametrising it.

Compute the path orientation $\hat{\boldsymbol{\tau}}$ (the unit tangent vector to Γ in the direction of $\vec{\mathbf{a}}$) of the parametrisation you found.

State whether $\hat{\boldsymbol{\tau}}$ and the orientation of ∂S induced by the surface orientation $\hat{\boldsymbol{n}}$ of S coincide or not.

Hint 1: There exists a simple parametrisation defined only by a few trigonometric functions, arising from the parametrisation of the boundary of R, look for this one.

Hint 2: Recall the definition of "induced (path) orientation" given in Section 2.2.5. A sketch of S can be very helpful.