Vector calculus MA3VC 2014–15 — Assignment 1 SOLUTIONS

(Exercise 1) *Compute a scalar potential φ for the vector field $\vec{\mathbf{F}} = yz(z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}) + \cos 2\pi x\hat{\boldsymbol{\imath}}$. Is $\vec{\mathbf{F}}$ solenoidal, irrotational? Does $\vec{\mathbf{F}}$ allow a vector potential?*

It is easy to find that the scalar potentials of $\vec{\mathbf{F}}$ are the scalar fields $\boxed{\varphi = \frac{1}{2\pi}\sin 2\pi x + \frac{1}{2}y^2z^2 + \lambda}$, where λ is a real constant:

$$\begin{aligned}
\frac{\partial\varphi}{\partial x} = \cos 2\pi x \quad &\Rightarrow \quad \varphi(x,y,z) = \frac{1}{2\pi}\sin 2\pi x + f(y,z) \qquad \text{for some two-dimensional scalar field } f,\\
\frac{\partial\varphi}{\partial y} = yz^2 \quad &\Rightarrow \quad \frac{\partial\big(\frac{1}{2\pi}\sin 2\pi x + f(y,z)\big)}{\partial y} = \frac{\partial f(y,z)}{\partial y} = yz^2 \quad \Rightarrow \quad f = \frac{1}{2}y^2z^2 + g(z)\\
&\Rightarrow \quad \varphi = \frac{1}{2\pi}\sin 2\pi x + \frac{1}{2}y^2z^2 + g(z) \qquad \text{for some real function } g,\\
\frac{\partial\varphi}{\partial z} = y^2z \quad &\Rightarrow \quad \frac{\partial\big(\frac{1}{2\pi}\sin 2\pi x + \frac{1}{2}y^2z^2 + g(z)\big)}{\partial z} = y^2z + \frac{\partial g(z)}{\partial z} = y^2z \quad \Rightarrow \quad \frac{\partial g(z)}{\partial z} = 0\\
&\Rightarrow \quad \varphi = \frac{1}{2\pi}\sin 2\pi x + \frac{1}{2}y^2z^2 + \lambda.
\end{aligned}$$

To verify that the scalar potential is correct, it is sufficient to check that $\vec{\nabla}\varphi = \vec{\mathbf{F}}$.

Since $\vec{\mathbf{F}}$ is conservative, by the box in Section 1.5 or by the identity $\vec{\nabla} \times (\vec{\nabla}\varphi) = \vec{\mathbf{0}}$, $\vec{\mathbf{F}}$ is irrotational.

The divergence of $\vec{\mathbf{F}}$ is not zero: $\vec{\nabla} \cdot \vec{\mathbf{F}} = -2\pi \sin 2\pi x + z^2 + y^2 \neq 0$, so $\vec{\mathbf{F}}$ is not solenoidal. This implies that $\vec{\mathbf{F}}$ does not admit a vector potential, again by the box in Section 1.5 or by the identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{A}}) = 0$.

(Exercise 2) *Let $\vec{\mathbf{F}}$ be a vector field with scalar potential φ, and let $\vec{\mathbf{G}}$ be a vector field with scalar potential ψ. Prove the following identity: $2\vec{\mathbf{F}} \cdot \vec{\mathbf{G}} = \Delta(\varphi\psi) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}}$.*

The identity to be proved is nothing else than the product rule (32) for the Laplacian in disguise. We can either use the known vector identities (simpler, *i*) or expand in partial derivatives (more complicated, *ii*).[1]

(Version i) We use three tools:

- The definition of scalar potential, namely $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ and $\vec{\mathbf{G}} = \vec{\nabla}\psi$;
- Identity (22) in the notes, which gives $\Delta\varphi = \vec{\nabla} \cdot (\vec{\nabla}\varphi) = \vec{\nabla} \cdot \vec{\mathbf{F}}$ and $\Delta\psi = \vec{\nabla} \cdot (\vec{\nabla}\psi) = \vec{\nabla} \cdot \vec{\mathbf{G}}$;
- The product rule (32) for the Laplacian.

These identities together lead to

$$\Delta(\varphi\psi) \overset{(32)}{=} (\Delta\varphi)\psi + 2\vec{\nabla}\varphi \cdot \vec{\nabla}\psi + (\Delta\psi)\varphi \overset{(22)}{=} \vec{\nabla} \cdot (\vec{\nabla}\varphi)\psi + 2\vec{\nabla}\varphi \cdot \vec{\nabla}\psi + \vec{\nabla} \cdot (\vec{\nabla}\psi)\varphi = (\vec{\nabla} \cdot \vec{\mathbf{F}})\psi + 2\vec{\mathbf{F}} \cdot \vec{\mathbf{G}} + (\vec{\nabla} \cdot \vec{\mathbf{G}})\varphi.$$

Rearranging for $2\vec{\mathbf{F}} \cdot \vec{\mathbf{G}}$ immediately gives the desired result.

(Version ii) If we use the expansion in components, we need to use twice the product rule for partial derivatives (8), together with the definitions of Laplacian (20), divergence (17) and scalar potentials $\vec{\mathbf{F}} = \vec{\nabla}\varphi$, $\vec{\mathbf{G}} = \vec{\nabla}\psi$. The right-hand side of the identity can be expanded as follows:

$$\begin{aligned}
&\Delta(\varphi\psi) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}}\\
&\overset{(17),(20)}{=} \frac{\partial^2(\varphi\psi)}{\partial x^2} + \frac{\partial^2(\varphi\psi)}{\partial y^2} + \frac{\partial^2(\varphi\psi)}{\partial z^2} - \psi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \varphi\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right)\\
&\overset{(8)}{=} \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\psi + \varphi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\psi + \varphi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\psi + \varphi\frac{\partial\psi}{\partial z}\right)\\
&\qquad - \psi\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) - \varphi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)\\
&\overset{(8)}{=} \frac{\partial^2\varphi}{\partial x^2}\psi + 2\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \varphi\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\psi + 2\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \varphi\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\psi + 2\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z} + \varphi\frac{\partial^2\psi}{\partial z^2}
\end{aligned}$$

---

[1] A nice alternative solution I found in some of the assignments is the following (similar to *i* but slightly more complicated):

$$\begin{aligned}
2\vec{\mathbf{F}} \cdot \vec{\mathbf{G}} &= \vec{\mathbf{F}} \cdot \vec{\mathbf{G}} + \vec{\mathbf{F}} \cdot \vec{\mathbf{G}} = \vec{\nabla}\varphi \cdot \vec{\mathbf{G}} + \vec{\mathbf{F}} \cdot \vec{\nabla}\psi \overset{(28)}{=} \vec{\nabla} \cdot (\varphi\vec{\mathbf{G}}) - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}} + \vec{\nabla} \cdot (\psi\vec{\mathbf{F}}) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} = \vec{\nabla} \cdot (\varphi\vec{\mathbf{G}} + \psi\vec{\mathbf{F}}) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}}\\
&= \vec{\nabla} \cdot (\varphi\vec{\nabla}\psi + \psi\vec{\nabla}\varphi) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}} \overset{(26)}{=} \vec{\nabla} \cdot (\vec{\nabla}(\varphi\psi)) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}} \overset{(22)}{=} \Delta(\varphi\psi) - \psi\vec{\nabla} \cdot \vec{\mathbf{F}} - \varphi\vec{\nabla} \cdot \vec{\mathbf{G}}.
\end{aligned}$$

$$
\begin{aligned}
&- \psi\Big(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\Big) - \varphi\Big(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\Big)\\
&= 2\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+2\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}+2\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}\\
&= 2\vec{\nabla}\varphi\cdot\vec{\nabla}\psi\\
&= 2\vec{\mathbf{F}}\cdot\vec{\mathbf{G}}.
\end{aligned}
$$

(Exercise 3) *Demonstrate the identity in Ex. (2) for the vector field* $\vec{\mathbf{F}}$ *in Ex. (1) and the scalar field* $\psi = y^3$*.*

We compute all the terms appearing in the identity (λ can be fixed to 0):

$$
\begin{aligned}
\varphi &= \frac{1}{2\pi}\sin 2\pi x + \frac{1}{2}y^2z^2 + \lambda \qquad \text{from Exercise (1)},\\
\psi &= y^3,\\
\vec{\mathbf{F}} &= \cos 2\pi x\hat{\boldsymbol{\imath}} + yz(z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}),\\
\vec{\mathbf{G}} &= \vec{\nabla}\psi = 3y^2\hat{\boldsymbol{\jmath}},\\
\vec{\nabla}\cdot\vec{\mathbf{F}} &= \frac{\partial(\cos 2\pi x)}{\partial x} + \frac{\partial(yz^2)}{\partial y} + \frac{\partial(y^2z)}{\partial z} = -2\pi\sin 2\pi x + z^2 + y^2,\\
\vec{\nabla}\cdot\vec{\mathbf{G}} &= \frac{\partial(3y^2)}{\partial y} = 6y,\\
\psi(\vec{\nabla}\cdot\vec{\mathbf{F}}) &= \psi\Delta\varphi = -2\pi y^3\sin 2\pi x + y^3z^2 + y^5,\\
\varphi(\vec{\nabla}\cdot\vec{\mathbf{G}}) &= \varphi\Delta\psi = \frac{3}{\pi}y\sin 2\pi x + 3y^3z^2 + 6y\lambda,\\
\vec{\mathbf{F}}\cdot\vec{\mathbf{G}} &= 3y^3z^2,\\
\varphi\psi &= \frac{1}{2\pi}y^3\sin 2\pi x + \frac{1}{2}y^5z^2 + y^3\lambda\\
\Delta(\varphi\psi) &= \frac{\partial^2(\frac{1}{2\pi}y^3\sin 2\pi x)}{\partial x^2} + \frac{\partial^2(\frac{1}{2\pi}y^3\sin 2\pi x + \frac{1}{2}y^5z^2 + y^3\lambda)}{\partial y^2} + \frac{\partial^2(\frac{1}{2}y^5z^2)}{\partial z^2}\\
&= -2\pi y^3\sin 2\pi x + \frac{3}{\pi}y\sin 2\pi x + 10y^3z^2 + 6y\lambda + y^5,\\
LHS &= 2\vec{\mathbf{F}}\cdot\vec{\mathbf{G}} = 6y^3z^2,\\
RHS &= \Delta(\varphi\psi) - \psi\vec{\nabla}\cdot\vec{\mathbf{F}} - \varphi\vec{\nabla}\cdot\vec{\mathbf{G}}\\
&= -2\pi y^3\sin 2\pi x + \frac{3}{\pi}y\sin 2\pi x + 10y^3z^2 + 6y\lambda + y^5\\
&\qquad - \Big(-2\pi y^3\sin 2\pi x + y^3z^2 + y^5\Big) - \Big(\frac{3}{\pi}y\sin 2\pi x + 3y^3z^2 + 6y\lambda\Big) = 6y^3z^2.
\end{aligned}
$$

The left-hand side (LHS) and the right-hand side (RHS) of the identity coincide, so the identity is demonstrated.

(Exercise 4) *Consider the cylinder C with radius 2, axis of rotation on the x-axis and flat faces lying in the planes $\{x = 0\}$ and $\{x = 10\}$. Write C in the form $C = \{\vec{\mathbf{r}} \in \mathbb{R}^3$ s.t. $\dots\}$ and compute the outward pointing unit normal vector field $\hat{\boldsymbol{n}}$ defined on the boundary of C.*

The cylinder can be written either as an open set

$$C_o = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ such that } 0 < x < 10, y^2 + z^2 < 4\big\},$$

or as a closed set

$$C_c = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3 \text{ such that } 0 \le x \le 10, y^2 + z^2 \le 4\big\}.$$

The two flat faces lie in the planes $\{x = 0\}$ and $\{x = 10\}$, so their normals must be either $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{\imath}}$. (If this is not clear from geometric intuition, we note that these planes are level sets for the scalar field $f(\vec{\mathbf{r}}) = x$, whose gradient is $\vec{\nabla}f = \hat{\boldsymbol{\imath}}$ and has already unit length.) With simple geometric consideration, since $\hat{\boldsymbol{n}}$ must point outward, it is clear that $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{\imath}}$ on $\{x = 0\}$ and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}}$ on $\{x = 10\}$.

The side has equation $\{y^2 + z^2 = 4\}$, which is a level set of the field $f(\vec{\mathbf{r}}) = y^2 + z^2$. Its gradient is $\vec{\nabla}f = 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}}$, which has length $|\vec{\nabla}f| = 2\sqrt{y^2+z^2}$. Thus, as in the exercises seen in class or in Example 1.33, $\hat{\boldsymbol{n}} = \pm\vec{\nabla}f/|\vec{\nabla}f| = \pm(y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})/\sqrt{y^2+z^2}$. Since $\hat{\boldsymbol{n}}$ is "outward pointing", it must point away from the x-axis, e.g. it must satisfy $\hat{\boldsymbol{n}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = \hat{\boldsymbol{\jmath}}$, so we choose its sign as

$$\hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \frac{y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{\sqrt{y^2+z^2}}.$$