Vector calculus MA2VC 2016–17: Assignment 1
SOLUTIONS AND FEEDBACK

(Exercise 1) *Prove that if f is a (smooth) scalar field and $\vec{\mathbf{G}}$ is an irrotational vector field, then*

$$(\vec{\nabla} f \times \vec{\mathbf{G}})f$$

is solenoidal.

Hint: *do not expand in coordinates and partial derivatives. Use instead the vector differential identities of §1.4 and the properties of the vector product from §1.1.2 (recall in particular Exercise 1.15).*

(Version 1) We compute the divergence of the vector field $(\vec{\nabla} f \times \vec{\mathbf{G}})f$ using the product rules (29) and (30) for the divergence:

$$\begin{aligned}\vec{\nabla}\cdot\Big((\vec{\nabla} f \times \vec{\mathbf{G}})f\Big) &\overset{(29)}{=} \vec{\nabla} f\cdot(\vec{\nabla} f\times\vec{\mathbf{G}}) + f\vec{\nabla}\cdot(\vec{\nabla} f\times\vec{\mathbf{G}})\\ &\overset{(30)}{=} \vec{\nabla} f\cdot(\vec{\nabla} f\times\vec{\mathbf{G}}) + f(\vec{\nabla}\times\vec{\nabla} f)\cdot\vec{\mathbf{G}} - f\vec{\nabla} f\cdot(\vec{\nabla}\times\vec{\mathbf{G}}).\end{aligned}$$

The first term in this expression vanishes[1] because of the identities $\vec{\mathbf{u}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{w}}) = \vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{u}})$ (by Exercise 1.15) and $\vec{\mathbf{u}}\times\vec{\mathbf{u}} = \vec{\mathbf{0}}$ (by the anticommutativity of the vector product (3)), which hold for all $\vec{\mathbf{u}},\vec{\mathbf{w}}\in\mathbb{R}^3$. Applying these formulas with $\vec{\mathbf{u}} = \vec{\nabla} f$ and $\vec{\mathbf{w}} = \vec{\mathbf{G}}$ we get $\vec{\nabla} f\cdot(\vec{\nabla} f\times\vec{\mathbf{G}}) = \vec{\mathbf{G}}\cdot(\vec{\nabla} f\times\vec{\nabla} f) = \vec{\mathbf{G}}\cdot\vec{\mathbf{0}} = 0$.

The second term vanishes because of the "curl-grad identity" (26): $\vec{\nabla}\times\vec{\nabla} f = \vec{\mathbf{0}}$.

The last term vanishes because $\vec{\mathbf{G}}$ is irrotational: $\vec{\nabla}\times\vec{\mathbf{G}} = \vec{\mathbf{0}}$.

So we conclude that the field $(\vec{\nabla} f\times\vec{\mathbf{G}})f$ is solenoidal because its divergence vanishes:

$$\vec{\nabla}\cdot\Big((\vec{\nabla} f\times\vec{\mathbf{G}})f\Big) = \vec{\mathbf{G}}\cdot(\underbrace{\vec{\nabla} f\times\vec{\nabla} f}_{=\vec{\mathbf{0}}}) + f(\underbrace{\vec{\nabla}\times\vec{\nabla} f}_{=\vec{\mathbf{0}}})\cdot\vec{\mathbf{G}} - f\vec{\nabla} f\cdot(\underbrace{\vec{\nabla}\times\vec{\mathbf{G}}}_{=\vec{\mathbf{0}}}) = 0.$$

(Version 2) Alternatively, one could expand the field in coordinates. This solution is of course much more complicated and prone to errors than the previous one.

We first write the product rule for products of three scalar fields, which is proved by applying twice the usual product rule for partial derivatives (8):

$$\frac{\partial}{\partial x}(fgh) = \frac{\partial}{\partial x}\big((fg)h\big) \overset{(8)}{=} \frac{\partial(fg)}{\partial x}h + fg\frac{\partial h}{\partial x} \overset{(8)}{=} \Big(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\Big)h + fg\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}gh + f\frac{\partial g}{\partial x}h + fg\frac{\partial h}{\partial x}. \qquad (\spadesuit)$$

Combining this formula with the definitions of gradient (10), vector product (2), divergence (22) and curl (23), using Clairault's theorem (17) (i.e. $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$) and rearranging the terms, we obtain:

$$\begin{aligned}\vec{\nabla}\cdot\Big((\vec{\nabla} f\times\vec{\mathbf{G}})f\Big) &\overset{(10)}{=} \vec{\nabla}\cdot\bigg(\Big(\Big(\frac{\partial f}{\partial x}\hat{\boldsymbol\imath} + \frac{\partial f}{\partial y}\hat{\boldsymbol\jmath} + \frac{\partial f}{\partial z}\hat{\boldsymbol k}\Big)\times\vec{\mathbf{G}}\Big)f\bigg)\\
&\overset{(2)}{=} \vec{\nabla}\cdot\bigg(\Big(\Big(\frac{\partial f}{\partial y}G_3 - \frac{\partial f}{\partial z}G_2\Big)\hat{\boldsymbol\imath} + \Big(\frac{\partial f}{\partial z}G_1 - \frac{\partial f}{\partial x}G_3\Big)\hat{\boldsymbol\jmath} + \Big(\frac{\partial f}{\partial x}G_2 - \frac{\partial f}{\partial y}G_1\Big)\hat{\boldsymbol k}\Big)f\bigg)\\
&= \vec{\nabla}\cdot\bigg(\Big(f\frac{\partial f}{\partial y}G_3 - f\frac{\partial f}{\partial z}G_2\Big)\hat{\boldsymbol\imath} + \Big(f\frac{\partial f}{\partial z}G_1 - f\frac{\partial f}{\partial x}G_3\Big)\hat{\boldsymbol\jmath} + \Big(f\frac{\partial f}{\partial x}G_2 - f\frac{\partial f}{\partial y}G_1\Big)\hat{\boldsymbol k}\bigg)\\
&\overset{(22)}{=} \frac{\partial}{\partial x}\Big(f\frac{\partial f}{\partial y}G_3 - f\frac{\partial f}{\partial z}G_2\Big) + \frac{\partial}{\partial y}\Big(f\frac{\partial f}{\partial z}G_1 - f\frac{\partial f}{\partial x}G_3\Big) + \frac{\partial}{\partial z}\Big(f\frac{\partial f}{\partial x}G_2 - f\frac{\partial f}{\partial y}G_1\Big)\\
&\overset{(\spadesuit)}{=} \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}G_3 + f\frac{\partial^2 f}{\partial x\partial y}G_3 + f\frac{\partial f}{\partial y}\frac{\partial G_3}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial z}G_2 - f\frac{\partial^2 f}{\partial x\partial z}G_2 - f\frac{\partial f}{\partial z}\frac{\partial G_2}{\partial x}\\
&\quad + \frac{\partial f}{\partial y}\frac{\partial f}{\partial z}G_1 + f\frac{\partial^2 f}{\partial y\partial z}G_1 + f\frac{\partial f}{\partial z}\frac{\partial G_1}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial f}{\partial x}G_3 - f\frac{\partial^2 f}{\partial y\partial x}G_3 - f\frac{\partial f}{\partial x}\frac{\partial G_3}{\partial y}\\
&\quad + \frac{\partial f}{\partial z}\frac{\partial f}{\partial x}G_2 + f\frac{\partial^2 f}{\partial z\partial x}G_2 + f\frac{\partial f}{\partial x}\frac{\partial G_2}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial f}{\partial y}G_1 - f\frac{\partial^2 f}{\partial z\partial y}G_1 - f\frac{\partial f}{\partial y}\frac{\partial G_1}{\partial z}\\
&= G_1\underbrace{\Big(\frac{\partial f}{\partial y}\frac{\partial f}{\partial z} + \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial f}{\partial z}\frac{\partial f}{\partial y} - f\frac{\partial^2 f}{\partial z\partial y}\Big)}_{=0,\ (17)}\\
&\quad + G_2\underbrace{\Big(-\frac{\partial f}{\partial x}\frac{\partial f}{\partial z} - f\frac{\partial^2 f}{\partial x\partial z} + \frac{\partial f}{\partial z}\frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial z\partial x}\Big)}_{=0,\ (17)}\end{aligned}$$

[1]Equivalently $\vec{\nabla} f\cdot(\vec{\nabla} f\times\vec{\mathbf{G}}) = 0$ because the vectors $\vec{\nabla} f$, $\vec{\mathbf{G}}$, $\vec{\nabla} f$ are linearly dependent, or because $\vec{\nabla} f\times\vec{\mathbf{G}}$ is orthogonal to $\vec{\nabla} f$.

$$
\begin{aligned}
&+ G_2 \underbrace{\Big(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} - f\frac{\partial^2 f}{\partial y \partial x}\Big)}_{=0,\ (17)} \\
&+ f\frac{\partial f}{\partial x}\Big(\frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y}\Big) + f\frac{\partial f}{\partial y}\Big(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}\Big) + f\frac{\partial f}{\partial z}\Big(\frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x}\Big) \\
&\overset{(23)}{=} f\frac{\partial f}{\partial x}(-\vec{\nabla} \times \vec{\mathbf{G}})_1 + f\frac{\partial f}{\partial y}(-\vec{\nabla} \times \vec{\mathbf{G}})_2 + f\frac{\partial f}{\partial z}(-\vec{\nabla} \times \vec{\mathbf{G}})_3 \\
&= 0 \qquad \text{since } \vec{\nabla} \times \vec{\mathbf{G}} = \vec{\mathbf{0}}.
\end{aligned}
$$

(Exercise 2) *Demonstrate that the field* $(\vec{\nabla} f \times \vec{\mathbf{G}})f$ *is indeed solenoidal for* $f = xyz$ *and* $\vec{\mathbf{G}} = \vec{\mathbf{r}}$.

We compute all the terms in the expression $(\vec{\nabla} f \times \vec{\mathbf{G}})f$ and its divergence:

$$
\begin{aligned}
\vec{\mathbf{G}} &= \vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}, \qquad f = xyz, \\
\vec{\nabla} f &= yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}, \\
\vec{\nabla} f \times \vec{\mathbf{G}} &= (xz^2 - xy^2)\hat{\boldsymbol{\imath}} + (yx^2 - yz^2)\hat{\boldsymbol{\jmath}} + (zy^2 - zx^2)\hat{\boldsymbol{k}}, \\
(\vec{\nabla} f \times \vec{\mathbf{G}})f &= (x^2yz^3 - x^2y^3z)\hat{\boldsymbol{\imath}} + (x^3y^2z - xy^2z^3)\hat{\boldsymbol{\jmath}} + (xy^3z^2 - x^3yz^2)\hat{\boldsymbol{k}}, \\
\vec{\nabla} \cdot \Big((\vec{\nabla} f \times \vec{\mathbf{G}})f\Big) &= 2xyz^3 - 2xy^3z + 2x^3yz - 2xyz^3 + 2xy^3z - 2x^3yz = 0.
\end{aligned}
$$

(Exercise 3) *We define the following subsets of the xy-plane (recall that for* $\vec{\mathbf{r}} \in \mathbb{R}^2$ *we write* $\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$*)*

$$S_1 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, x + y = 1\}, \qquad S_2 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, x^2 + y^2 = 4\}, \qquad S_3 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, x^2 = y + 1\},$$

the two-dimensional scalar fields

$$f_A(\vec{\mathbf{r}}) = \mathrm{e}^x\mathrm{e}^y, \qquad f_B(\vec{\mathbf{r}}) = x^2 - \vec{\mathbf{r}} \cdot \hat{\boldsymbol{\jmath}}, \qquad f_C(\vec{\mathbf{r}}) = \sin x \cos y + \cos x \sin y, \qquad f_D(\vec{\mathbf{r}}) = \log(|\vec{\mathbf{r}}| + \pi),$$

and the planar curves, defined for all $t \in \mathbb{R}$,

$$\vec{\mathbf{a}}_I(t) = 2\sin t\hat{\boldsymbol{\imath}} - 2\cos t\hat{\boldsymbol{\jmath}}, \quad \vec{\mathbf{a}}_{II}(t) = \cos t\hat{\boldsymbol{\imath}} + (1 - \cos t)\hat{\boldsymbol{\jmath}}, \quad \vec{\mathbf{a}}_{III}(t) = (t-1)\hat{\boldsymbol{\imath}} + (t^2 - 2t)\hat{\boldsymbol{\jmath}}, \quad \vec{\mathbf{a}}_{IV}(t) = \hat{\boldsymbol{\imath}} + t^3(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}).$$

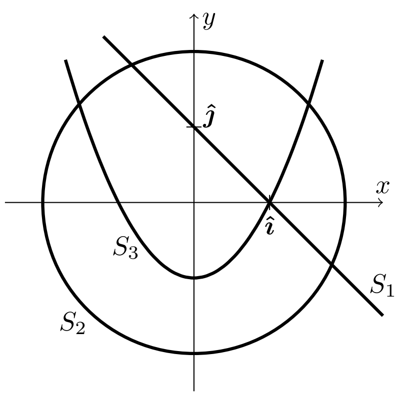
- *Each set* S_1, S_2, S_3 *is the level set of one of the fields* f_A, f_B, f_C, f_D *and path of one of the curves* $\vec{\mathbf{a}}_I, \vec{\mathbf{a}}_{II}, \vec{\mathbf{a}}_{III}, \vec{\mathbf{a}}_{IV}$*: match them.*
- *One of the level sets of the remaining field contains as a subset one of the three sets* S_1, S_2, S_3*, which one?*
- *The path of the remaining curve is a subset of one of the three sets* S_1, S_2, S_3*, which one?*

You do NOT have to justify your answer.
Hint: *Recall §1.2.3 and try to sketch the sets.*

The matches are:

S_1	f_A	$\vec{\mathbf{a}}_{IV}$	straight line through $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$
S_2	f_D	$\vec{\mathbf{a}}_I$	circle of radius 2 centred at the origin
S_3	f_B	$\vec{\mathbf{a}}_{III}$	parabola through $-\hat{\boldsymbol{\imath}}, -\hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{\imath}}$

S_1 is a subset of a level set of f_C,
and the path of $\vec{\mathbf{a}}_{II}$ is a subset of S_1.



We justify these assertions.

- The first field can be written as $f_A(\vec{\mathbf{r}}) = \mathrm{e}^{x+y}$, so $f_A(\vec{\mathbf{r}}) = \mathrm{e}$ if and only if $x + y = 1$. In other words, f_A takes value e in S_1: $S_1 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, f_A(\vec{\mathbf{r}}) = \mathrm{e}\}$.
- The second field can be expanded as $f_B(\vec{\mathbf{r}}) = x^2 - y$, so in S_3 it takes value 1: $S_3 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, f_B(\vec{\mathbf{r}}) = 1\}$.
- From the definition of magnitude, the second set can be written as $S_2 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, |\vec{\mathbf{r}}| = 2\}$, so the fourth field is constant on it: $S_2 = \{\vec{\mathbf{r}} \in \mathbb{R}^2, f_D(\vec{\mathbf{r}}) = \log(2 + \pi)\}$

- The third field can be written as $f_C(\vec{\mathbf{r}}) = \sin(x+y)$ using a trigonometric addition formula, so in S_1 it takes values $\sin 1$. However, f_C takes the same value on all the parallel lines $x+y = 1+2\pi n$ and $x+y = \pi-1+2\pi n$, for any $n \in \mathbb{Z}$. This means that S_1 is a proper subset of a level set of f_C: $S_1 \subsetneqq \{\vec{\mathbf{r}} \in \mathbb{R}^2, f_C(\vec{\mathbf{r}}) = \sin 1\}$.

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- All points in the path of the curve $\vec{\mathbf{a}}_I$ satisfy $x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4$, namely they belong to the circle S_2. All points of the circle S_2 are reached (many times) by the curve $\vec{\mathbf{a}}_I$; recall §1.2.3.
- The points of the path of the curve $\vec{\mathbf{a}}_{III}$ satisfy $x^2 = (t-1)^2 = t^2 - 2t + 1 = y + 1$ thus belong to the parabola S_3. All points of S_3 are taken because for all $x \in \mathbb{R}$ there is a t such that $x = t - 1$ and the parabola $y = x^2 - 1$ takes a single value for every x.
- The points of the path of the curve $\vec{\mathbf{a}}_{IV}$ satisfy $x + y = (1 - t^3) + t^3 = 1$, so they belong to S_1. The whole line is parametrised by $\vec{\mathbf{a}}_{IV}$ because for any $\vec{\mathbf{r}} = x\hat{\imath} + y\hat{\jmath} \in S_1$ we have $\vec{\mathbf{a}}_{IV}(t) = \vec{\mathbf{r}}$ for $t = y^{1/3}$.
- The points of the paths of $\vec{\mathbf{a}}_{II}$ belong to the straight line S_1, since $x+y = a_1(t)+a_2(t) = \cos t+(1-\cos t) = 1$. However, since the cos function takes value in the interval $[-1, 1]$ only, and the line S_1 contains points with any real value of the x coordinate, the curve is a parametrisation of only a segment of S_1, i.e. $\{\vec{\mathbf{r}} \in \mathbb{R}^2, \vec{\mathbf{r}} = \vec{\mathbf{a}}_{II}(t), t \in \mathbb{R}\} \subsetneqq S_1$. (To be more precise, $\vec{\mathbf{a}}_{II}$ runs the segment between $\hat{\imath}$ and $2\hat{\jmath} - \hat{\imath}$ back and forth infinitely many times.)

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MA2VC: Feedback after grading assignment 1

Check carefully the points below and all the corrections in your assignment; even if you got full marks, your solution (and its presentation) can probably be improved. See also page 108 in the notes.

In general, in your coursework, a red check mark ✓ means "correct", a × mark means "error", a check mark in brackets (✓) means "correct step leading to a wrong solution due to previous errors".

If you have any question or comment about the assignment, the solutions or the marking, please do ask me.

Exercise 1:

Here I have found many serious mistakes. If you did not get full marks, please revise carefully the solutions.

- The most common mistake was in the use of the differential operators and the **nabla symbol** "$\vec{\nabla}$". The nabla symbol $\vec{\nabla}$ cannot be treated as a vector! It is just the symbol we use to write gradients $\vec{\nabla} f$, divergences $\vec{\nabla} \cdot \vec{\mathbf{F}}$ and curls $\vec{\nabla} \times \vec{\mathbf{F}}$. The gradient $\vec{\nabla} f$ of a scalar field is a vector field, while $\vec{\nabla}$ alone is not. This is similar to the fact that "$\cos \pi$" is a number, but "cos" alone is not.
Many frequent errors are consequences of this.
  - $\vec{\nabla}$ is not "multiplied" but "applied": $\vec{\nabla} \cdot \vec{\mathbf{F}}$ and $\vec{\nabla} \times \vec{\mathbf{F}}$ are divergence and curl of a field $\vec{\mathbf{F}}$. If you call them "dot product" and "cross product" you do a mistake.
  - The nabla symbol can appear in a formula *only* as part of a gradient, a divergence or a curl. Otherwise its use is wrong. For example, terms like $(\vec{\nabla} \times \vec{\mathbf{F}})$ and $(\vec{\nabla} f)$ are correct, while $(\vec{\mathbf{F}} \times \vec{\nabla})$ and $(f\vec{\nabla})$ are meaningless (exactly as $\sin \pi$ is 0, while $\pi \sin$ is meaningless).
  - Identities and properties that are true for vectors are not necessarily true for $\vec{\nabla}$.
For example, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$ holds for any three vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$. But in general $\vec{\nabla} \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \neq \vec{\mathbf{G}} \cdot (\vec{\nabla} \times \vec{\mathbf{F}})$. The expression $\vec{\nabla} \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}})$ is the divergence of the vector product between two fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, to compute it you need to use the appropriate product rule (i.e. equation (30)). Many people committed this mistake.
  - You cannot change the position of the symbol $\vec{\nabla}$ in a formula using some sort of "commutativity". For example, the products $f\vec{\nabla} \cdot \vec{\mathbf{G}}$ and $\vec{\nabla} f \cdot \vec{\mathbf{G}}$ are different: the former contains some partial derivatives of $\vec{\mathbf{G}}$ (its divergence), while the latter contains the partial derivatives of f (its gradient). Thus you cannot equate them using some commutativity of products, as $\vec{\nabla}$ is not "multiplied" but "applied".

Tip: the use of the notation $\vec{grad} f$, $div\, \vec{\mathbf{F}}$ and $\vec{curl}\vec{\mathbf{F}}$ instead of $\vec{\nabla} f$, $\vec{\nabla} \cdot \vec{\mathbf{F}}$ and $\vec{\nabla} \times \vec{\mathbf{F}}$ can prevent many of these errors. E.g. the two terms in the last example would read $f\, div\, \vec{\mathbf{G}}$ and $\vec{\mathbf{G}} \cdot \vec{grad} f$, which are evidently different.

- Many students computed $[\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\mathbf{G}})]f$ instead of $\vec{\nabla}\cdot[(\vec{\nabla}f\times\vec{\mathbf{G}})f]$. In order to prove that the field $(\vec{\nabla}f\times\vec{\mathbf{G}})f$ is solenoidal, f cannot be moved out of the divergence freely, the product rule (29) for the divergence has to be used.
- Many students used some sort of distributivity property in a completely wrong way, rewriting $(\vec{\nabla}f\times\vec{\mathbf{G}})f$ as $\vec{\nabla}ff\times\vec{\mathbf{G}}f$. The distributivity property relates addition and multiplication ($(a+b)c = ac+bc$ for $a,b,c\in\mathbb{R}$), while in $(\vec{\nabla}f\times\vec{\mathbf{G}})f$ only products are present (a vector product and a vector-times-scalar product). This would be the same as saying that $(ab)c = acbc$.
- Writing $\vec{\nabla}ff$ is slightly ambiguous. All students who wrote this, treated it first as $(\vec{\nabla}f)f$ and then as $\vec{\nabla}(ff)$, which are not the same. Use brackets to avoid mistakes!
- Some students tried to solve exercise 1 by expanding the field in coordinates (version 2 above). Nobody got close to the solution and all did mistakes in the use of partial derivatives. The symbol $\frac{\partial}{\partial x}$ can not be moved around in an expression freely and fields cannot be moved in and out the derivative.
  For example, $\frac{\partial f}{\partial y}G_3$ and $\frac{\partial(fG_3)}{\partial y}$ are not the same: the former is the product of two scalar fields (the partial derivative of f in y and the third component of $\vec{\mathbf{G}}$), while the latter is the partial derivative of the product of f and G_3. On the other hand, writing things like $\frac{\partial fG_3}{\partial y}$ is ambiguous, thus incorrect, and led many to errors.
- In exercises where you have to "prove" something, write down explicitly which tool you use at each step (e.g. product rule, this or that identity...).
- Some students wrote that "$\vec{\nabla}f = \vec{\mathbf{0}}$ because f is smooth". This is not correct, smooth fields are not necessarily constant.

Exercise 2:

- As in the first exercise, many students computed $[\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\mathbf{G}})]f$ instead of $\vec{\nabla}\cdot[(\vec{\nabla}f\times\vec{\mathbf{G}})f]$.
  Some others computed the scalar field $(\vec{\nabla}f\times\vec{\mathbf{G}})\cdot\vec{\mathbf{G}}$ (which is zero) instead of the correct vector field.
  Check carefully if what you are computing is what you are asked to.
- When you are asked to **demonstrate** an identity for a certain choice of fields, you have to compute both its left-hand side and its right-hand side, and verify that the two results coincide. This exercise was even easier, as the right-hand side is zero. The best strategy is simply to compute and list all the terms appearing in the final identity. In this exercise one can write four steps: (i) $\vec{\nabla}f$, (ii) $\vec{\nabla}f\times\vec{\mathbf{G}}$, (iii) $(\vec{\nabla}f\times\vec{\mathbf{G}})f$ and (iv) $\vec{\nabla}\cdot((\vec{\nabla}f\times\vec{\mathbf{G}})f)$.
  You cannot use the identity itself to deduce that the left-hand side is zero (or prove again the identity), this is a circular argument.
- The position vector is $\vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$; its components are the independent variables x, y and z. You will not be able to solve this exercise and many others if you ignore this.

Exercise 3:

Here points were given only for the correct matching between sets and fields or curves, thus many errors were not penalised. For example, sentences like "f_c is a subset of S_1" are completely wrong. A field cannot be a subset of a subset of $\mathbb{R}^3$. A correct statement is: "one of the level sets of f_c contains S_1 as a subset".