Vector calculus MA2VC 2015–16: Assignment 1
SOLUTIONS

(Exercise 1) *Consider the vector field* $\vec{\mathbf{F}} = -x^3y^4\hat{\boldsymbol{\imath}} + 3x^2y^4z\hat{\boldsymbol{k}}$.
Compute the divergence and the curl of $\vec{\mathbf{F}}$.
Is $\vec{\mathbf{F}}$ *solenoidal, irrotational?*
Is $\vec{\mathbf{F}}$ *conservative? If the answer is positive compute a scalar potential.*
Does $\vec{\mathbf{F}}$ *admit a vector potential* $\vec{\mathbf{A}}$*? If the answer is positive compute a potential. (In this case, look for the simplest one!)*

We apply the definitions of divergence and curl:

$$\begin{aligned}
&\vec{\mathbf{F}} = -x^3y^4\hat{\boldsymbol{\imath}} + 3x^2y^4z\hat{\boldsymbol{k}}, && \\
\text{divergence:}\quad &\vec{\nabla}\cdot\vec{\mathbf{F}} = -3x^2y^4 + 3x^2y^4 = 0 &&\Rightarrow \quad \vec{\mathbf{F}} \text{ is solenoidal,}\\
\text{curl:}\quad &\vec{\nabla}\times\vec{\mathbf{F}} = 12x^2y^3z\hat{\boldsymbol{\imath}} - 6xy^4z\hat{\boldsymbol{\jmath}} - 4x^3y^3\hat{\boldsymbol{k}} &&\Rightarrow \quad \vec{\mathbf{F}} \text{ is not irrotational.}
\end{aligned}$$

Since the field is not irrotational, it is not conservative and there exists no scalar potential (recall the box on page 26).

On the other hand, the field is solenoidal, so it may admit a vector potential $\vec{\mathbf{A}}$. (Actually, $\vec{\mathbf{F}}$ is solenoidal and defined on the whole of $\mathbb{R}^3$, so it admits a vector potential by Remark 1.68.) The vector potential $\vec{\mathbf{A}}$ has to satisfy $\vec{\nabla}\times\vec{\mathbf{A}} = \vec{\mathbf{F}}$, which, by definition (23) of curl, is equivalent to the following three conditions:

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = F_1 = -x^3y^4, \qquad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = F_2 = 0, \qquad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = F_3 = 3x^2y^4z.$$

There are many possible vector potentials, we look for the simplest possible. In particular we look for $\vec{\mathbf{A}}$ with only one non-zero component. From the conditions we have just written, we see that the non-zero component must necessarily be A_2, as it is the only one entering the two equations with non-zero right-hand sides. So, setting $A_1 = A_3 = 0$, we have

$$-\frac{\partial A_2}{\partial z} = -x^3y^4, \qquad \frac{\partial A_2}{\partial x} = 3x^2y^4z.$$

Integrating the first equation with respect to z we have $A_2 = x^3y^4z + g(x,y)$ for some g independent of z. We see immediately that $A_2 = x^3y^4z$ with $g = 0$ satisfies the second condition $\frac{\partial A_2}{\partial x} = 3x^2y^4z$, so $\boxed{\vec{\mathbf{A}} = x^3y^4z\hat{\boldsymbol{\jmath}}}$. Recall that there are plenty of other correct vector potentials, your solution might differ from this one.

(Exercise 2) *Let* f *be a smooth scalar field. Prove the following identity:*

$$\vec{\nabla}\cdot\big(\vec{\nabla}f\times(\vec{\mathbf{r}}f)\big) = 0.$$

Hint: use the identities of Section 1.4 and the values of the curl and the divergence of the position vector $\vec{\mathbf{r}}$*. Recall also Exercise 1.15.*

(Version 1.) We use the product rules for divergence and curl (31) and (32), the curl-of-gradient identity (26), the fact that $\vec{\nabla}\times\vec{\mathbf{r}} = \vec{\mathbf{0}}$ (recall e.g. Exercise 1.60) and the properties of the triple and vector products ($\vec{\mathbf{u}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{w}}) = \vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{u}}) = \vec{\mathbf{w}}\cdot\vec{\mathbf{0}} = 0$, by Exercise 1.15)[1]:

$$\vec{\nabla}\cdot\Big((\vec{\nabla}f)\times(\vec{\mathbf{r}}f)\Big) \overset{(31)}{=} \underbrace{(\vec{\nabla}\times\vec{\nabla}f)}_{=\vec{\mathbf{0}},\text{ by (26)}}\cdot\vec{\mathbf{r}}f - \vec{\nabla}f\cdot\big(\vec{\nabla}\times(\vec{\mathbf{r}}f)\big) \overset{(32)}{=} -\vec{\nabla}f\cdot\big(\vec{\nabla}f\times\vec{\mathbf{r}} + f\underbrace{\nabla\times\vec{\mathbf{r}}}_{=\vec{\mathbf{0}}}\big) \overset{1.15}{=} -\vec{\mathbf{r}}\cdot\underbrace{(\vec{\nabla}f)\times(\vec{\nabla}f)}_{=\vec{\mathbf{0}}} = 0.$$

(Version 2.) We can also prove the identity by expanding in coordinates, however this solution is more complicated and more prone to errors. Using the definition of the vector product (2), of the divergence (22), the product rule for partial derivatives (8) and Clairault's theorem (17):

$$\vec{\nabla}\cdot\Big((\vec{\nabla}f)\times(\vec{\mathbf{r}}f)\Big) = \vec{\nabla}\cdot\bigg(\Big(\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}\Big)\times(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})f\bigg)$$

[1] Alternatively, one can see that the argument of the divergence can be bracketed in equivalent way: $\vec{\nabla}f\times(\vec{\mathbf{r}}f) = (\vec{\nabla}f\times\vec{\mathbf{r}})f$. This leads to a slightly different proof:

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}f) &= \vec{\nabla}\cdot((\vec{\nabla}f\times\vec{\mathbf{r}})f) \overset{(30)}{=} (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) + f\vec{\nabla}\cdot((\vec{\nabla}f)\times\vec{\mathbf{r}})\\
&\overset{(31)}{=} (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) + f\big((\underbrace{\vec{\nabla}\times\vec{\nabla}f}_{=\vec{\mathbf{0}},\text{ by (26)}})\cdot\vec{\mathbf{r}} - (\vec{\nabla}f)\cdot\underbrace{(\nabla\times\vec{\mathbf{r}})}_{=\vec{\mathbf{0}}}\big) = (\vec{\nabla}f)\cdot(\vec{\nabla}f\times\vec{\mathbf{r}}) \overset{\text{Ex. 1.15}}{=} \vec{\mathbf{r}}\cdot\underbrace{(\vec{\nabla}f)\times(\vec{\nabla}f)}_{=\vec{\mathbf{0}}} = 0.
\end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} \vec{\nabla}\cdot\bigg(\Big(\frac{\partial f}{\partial y}zf - \frac{\partial f}{\partial z}yf\Big)\hat{\boldsymbol\imath} + \Big(\frac{\partial f}{\partial z}xf - \frac{\partial f}{\partial x}zf\Big)\hat{\boldsymbol\imath} + \Big(\frac{\partial f}{\partial x}yf - \frac{\partial f}{\partial y}xf\Big)\hat{\boldsymbol\imath}\bigg)\\
&\overset{(22)}{=} \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}zf - \frac{\partial f}{\partial z}yf\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial z}xf - \frac{\partial f}{\partial x}zf\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial f}{\partial x}yf - \frac{\partial f}{\partial y}xf\Big)\\
&\overset{(8)}{=} \frac{\partial^2 f}{\partial x\partial y}zf + \frac{\partial f}{\partial y}z\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial x\partial z}yf - \frac{\partial f}{\partial z}y\frac{\partial f}{\partial x}\\
&\qquad + \frac{\partial^2 f}{\partial y\partial z}xf + \frac{\partial f}{\partial z}x\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y\partial x}zf - \frac{\partial f}{\partial x}z\frac{\partial f}{\partial y}\\
&\qquad + \frac{\partial^2 f}{\partial z\partial x}yf + \frac{\partial f}{\partial x}y\frac{\partial f}{\partial z} - \frac{\partial^2 f}{\partial z\partial y}xf - \frac{\partial f}{\partial y}x\frac{\partial f}{\partial z}\\
&= \Big(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\Big)xf + \Big(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\Big)yf + \Big(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\Big)zf \overset{(17)}{=} 0.
\end{aligned}$$

(Exercise 3) *Demonstrate the identity in Exercise 2 for the field* $f = \mathrm{e}^{xy}$.

We compute all the terms in the identity:

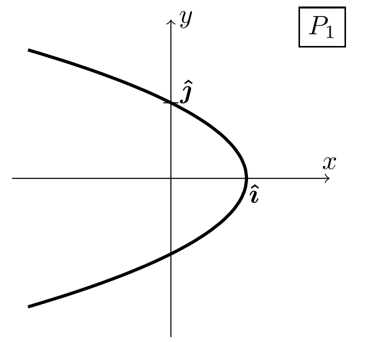
$$f = \mathrm{e}^{xy}, \qquad \vec{\nabla} f = y\mathrm{e}^{xy}\hat{\boldsymbol\imath} + x\mathrm{e}^{xy}\hat{\boldsymbol\jmath}, \qquad \vec{\mathbf{r}}f \overset{(2)}{=} x\mathrm{e}^{xy}\hat{\boldsymbol\imath} + y\mathrm{e}^{xy}\hat{\boldsymbol\jmath} + z\mathrm{e}^{xy}\hat{\boldsymbol k},$$

$$\begin{aligned}
\vec{\nabla} f\times\vec{\mathbf{r}}f &\overset{(2)}{=} xz\mathrm{e}^{2xy}\hat{\boldsymbol\imath} - yz\mathrm{e}^{2xy}\hat{\boldsymbol\jmath} + (y^2-x^2)\mathrm{e}^{2xy}\hat{\boldsymbol k},\\
\vec{\nabla}\cdot(\vec{\nabla} f\times\vec{\mathbf{r}}f) &\overset{(22)}{=} \frac{\partial}{\partial x}(xz\mathrm{e}^{2xy}) + \frac{\partial}{\partial y}(-yz\mathrm{e}^{2xy}) + \frac{\partial}{\partial z}\big((y^2-x^2)\mathrm{e}^{2xy}\big)\\
&= (z\mathrm{e}^{2xy} + 2xyz\mathrm{e}^{2xy}) - (z\mathrm{e}^{2xy} + 2xyz\mathrm{e}^{2xy}) + 0 = 0.
\end{aligned}$$

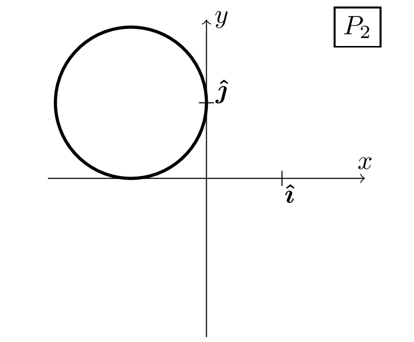
(Exercise 4) (Match plots and formulas)

The correct matches are: $\boxed{P_1 : \vec{\mathbf{a}}_G,\quad P_2 : \vec{\mathbf{a}}_C,\quad P_3 : \vec{\mathbf{a}}_D,\qquad F_1 : \vec{\mathbf{G}}_C,\quad F_2 : \vec{\mathbf{G}}_F,\quad F_3 : \vec{\mathbf{G}}_A.}$

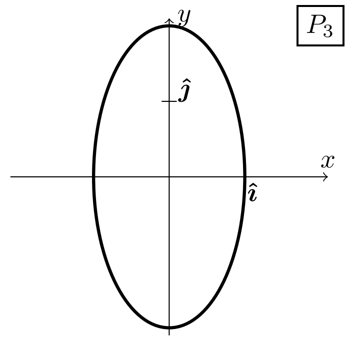
We show here the plots of the 7 curves together with the equations of the corresponding paths. Each path can be drawn by computing the values of the curve for several choices of $t\in\mathbb{R}$ and "joining the points". In most cases it helps to first write the equations of the path, identifying $x = a_1(t)$ and $y = a_2(t)$. The paths of $\vec{\mathbf{a}}_B$ and $\vec{\mathbf{a}}_C$ can be immediately drawn recalling the formula for the parametrisation of a circumference in item 3 of Remark 1.24. It is easy to see that the path of $\vec{\mathbf{a}}_E$ is the graph of the function $y = x^2+1$, and similarly the path of $\vec{\mathbf{a}}_G$ is a "tilted graph", i.e. the coordinate x is function of y.



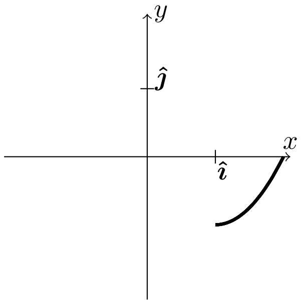
$\vec{\mathbf{a}}_G(t) = (1-t^2)\hat{\boldsymbol\imath} + t\hat{\boldsymbol\jmath}$
$\{x = 1-y^2\}$



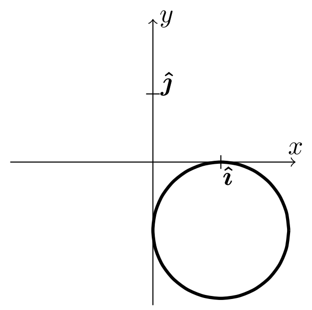
$\vec{\mathbf{a}}_C(t) = (\cos t - 1)\hat{\boldsymbol\imath} + (\sin t + 1)\hat{\boldsymbol\jmath}$
$\{(x+1)^2 + (y-1)^2 = 1\}$



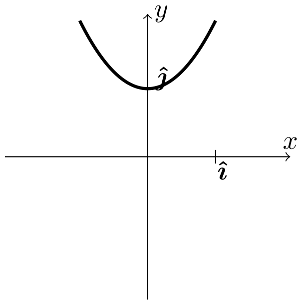
$\vec{\mathbf{a}}_D(t) = \sin t\hat{\boldsymbol\imath} + 2\cos t\hat{\boldsymbol\jmath}$
$\{x^2 + (\frac{y}{2})^2 = 1\}$



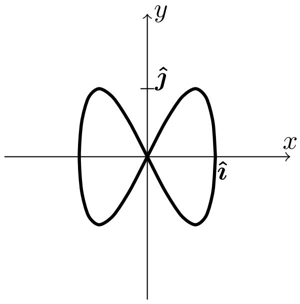
$\vec{\mathbf{a}}_A(t) = (t^2+1)\hat{\boldsymbol\imath} + (t^4-1)\hat{\boldsymbol\jmath}$
$\{y = x^2 - 2x,\ x\ge 1\}$



$\vec{\mathbf{a}}_B(t) = (\cos t + 1)\hat{\boldsymbol\imath}$
$+(\sin t - 1)\hat{\boldsymbol\jmath}$
$\{(x-1)^2 + (y+1)^2 = 1\}$



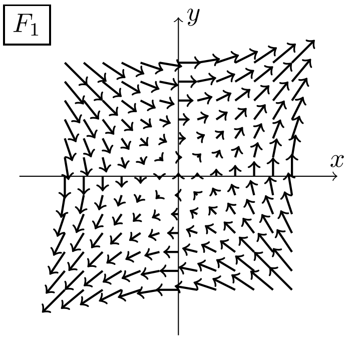
$\vec{\mathbf{a}}_E(t) = t\hat{\boldsymbol\imath} + (t^2+1)\hat{\boldsymbol\jmath}$
$\{y = x^2+1\}$



$\vec{\mathbf{a}}_F(t) = \cos t\hat{\boldsymbol\imath} + \sin 2t\hat{\boldsymbol\jmath}$
$\{y^2 = 4x^2(1-x^2)\}$

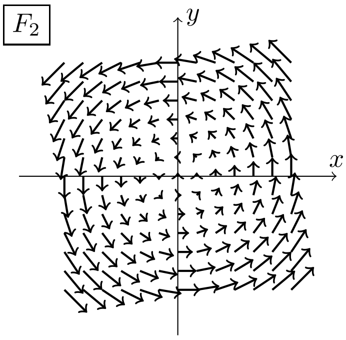
In order to draw the vector fields, we sample them at several points in the plane and we draw the values of the fields as arrows placed at those points. The signs of the components of the fields tell us whether they point left or right and up or down. Under the picture below you can see also the divergence and the curl of the fields; compare them with the interpretation given in Sections 1.3.5–6 in the notes.

F_1



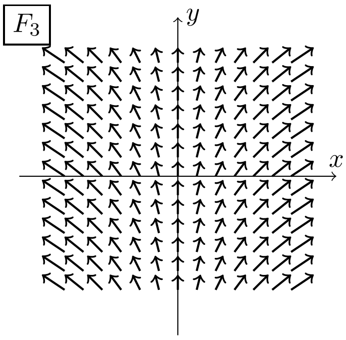
$\vec{\mathbf{G}}_C = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_C = 0, \quad \vec{\nabla}\times\vec{\mathbf{G}}_C = \vec{\mathbf{0}}$

F_2

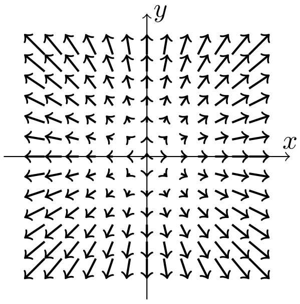


$\vec{\mathbf{G}}_F = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_F = 0, \quad \vec{\nabla}\times\vec{\mathbf{G}}_F = 2\hat{\boldsymbol{k}}$

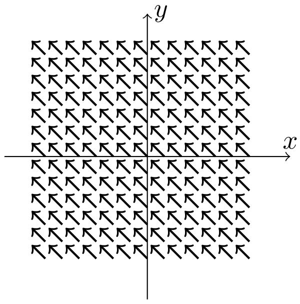
F_3



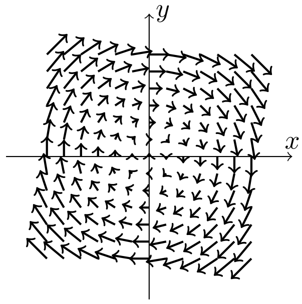
$\vec{\mathbf{G}}_A = x\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_A = 1, \quad \vec{\nabla}\times\vec{\mathbf{G}}_A = \vec{\mathbf{0}}$



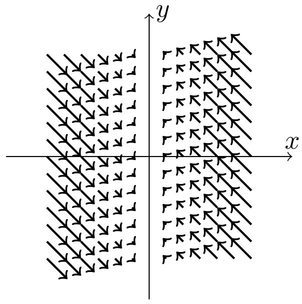
$\vec{\mathbf{G}}_B = \vec{\mathbf{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_B = 2, \vec{\nabla}\times\vec{\mathbf{G}}_B = \vec{\mathbf{0}}$



$\vec{\mathbf{G}}_D = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_D = 0, \vec{\nabla}\times\vec{\mathbf{G}}_D = \vec{\mathbf{0}}$



$\vec{\mathbf{G}}_E = y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_E = 0, \vec{\nabla}\times\vec{\mathbf{G}}_E = -2\hat{\boldsymbol{k}}$



$\vec{\mathbf{G}}_G = -x\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$
$\vec{\nabla}\cdot\vec{\mathbf{G}}_G = -1, \vec{\nabla}\times\vec{\mathbf{G}}_G = \hat{\boldsymbol{k}}$